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# The Schwinger mass in the massive Schwinger model

Christoph Adam  
Institut für theoretische Physik, Universität Bern  
Sidlerstraße 5, CH-3012 Bern, Switzerland\*)

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## Abstract

We derive a systematic procedure to compute Green functions for the massive Schwinger model via a perturbation expansion in the fermion mass. The known exact solution of the massless Schwinger model is used as a starting point.

We compute the corrections to the Schwinger mass up to second order.

\*)permanent address: Institut für theoretische Physik, Universität Wien  
Boltzmannngasse 5, 1090 Wien, Austria  
email address: adam@pap.univie.ac.at

## 1 Introduction

The massless Schwinger model is wellknown to be equivalent to a theory of a free massive boson with Schwinger mass  $\mu_0^2 = \frac{e^2}{\pi}$  ([1]). This massive field is formed via the chiral anomaly and may be interpreted as a fermion–antifermion bound state ([6], [5], [10], [9], [19]). Besides, the massless Schwinger model shows other nontrivial features like fermion condensate, instantons and nontrivial vacuum structure ( $\theta$  vacuum) ([2]–[5], [7]–[9], [11]–[13], [23]). In the massless Schwinger model physical quantities do not depend on the vacuum angle  $\theta$ .

The massive Schwinger model is different in some respects. First, it is no longer exactly solvable ([13]–[16], [10], [9]). The physical state – the massive boson – is no longer free, and its mass acquires corrections due to the interaction. Instanton–like gauge fields and a nontrivial vacuum structure persist to be present, and, in addition, physical quantities now depend on the vacuum angle  $\theta$ . The fermion condensate, too, acquires corrections due to the fermion mass ([21]).

Here we show how to compute Green functions for the massive Schwinger model within the Euclidean path integral formalism, using a perturbation expansion in the fermion mass. (The existence and finiteness of the mass perturbation theory for the massive Schwinger model was proven in [17]). For this purpose we use the known exact solution of the massless Schwinger model as a starting point. From the perturbation expansion we explicitly calculate corrections to the Schwinger mass up to second order in the fermion mass  $m$ , for a general vacuum angle  $\theta$ .

## 2 Exact solution of the massless case

The vacuum functional (and Green functions) of the massive Schwinger model may be inferred from  $n$ –point functions of the massless Schwinger model via an expansion in the fermion mass. Indeed, we may write for the Euclidean vacuum functional ( $k \dots$  instanton number)

$$Z(m, \theta) = \sum_{k=-\infty}^{\infty} e^{ik\theta} Z_k(m) \quad (1)$$

where

$$\begin{aligned} Z_k(m) &= N \int D\bar{\Psi} D\Psi D A_k^\mu e^{\int dx \left[ \bar{\Psi}(i\partial - e A_k + m)\Psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right]} \\ &= N \int D\bar{\Psi} D\Psi D\beta_k \sum_{n=0}^{\infty} \frac{m^n}{n!} \prod_{i=1}^n \int dx_i \bar{\Psi}(x_i) \Psi(x_i) \cdot \\ &\quad \cdot \exp\left\{ \int dx \left[ \bar{\Psi}(i\partial - \epsilon_{\mu\nu} \gamma^\mu \partial^\nu \beta_k) \Psi + \frac{1}{2} \beta_k \square^2 \beta_k \right] \right\} \end{aligned} \quad (2)$$

( $A_\mu = \epsilon_{\mu\nu} \partial^\nu \beta$  corresponding to Lorentz gauge). Therefore the perturbative computation of  $Z(m, \theta)$  is traced back to the computation of scalar VEVs ( $\langle \prod_i S(x_i) \rangle_0$ ,  $S(x) \equiv \bar{\Psi}(x)\Psi(x)$ ) for the massless Schwinger model and some space time integrations. It is useful to rewrite the scalar densities in terms of chiral ones,  $S(x) = S_+(x) + S_-(x)$ ,  $S_\pm \equiv \bar{\Psi} P_\pm \Psi$ , because then only a definite instanton sector  $k = n_+ - n_-$  contributes to the VEV  $\langle \prod_{i=1}^{n_+} S_+(x_i) \prod_{j=1}^{n_-} S_-(x_j) \rangle_0$ . A general VEV may be computed exactly (see e.g. [7]–[9], [22], [21]),

$$\langle S_{H_1}(x_1) \cdots S_{H_n}(x_n) \rangle_0 = \left( \frac{\Sigma}{2} \right)^n \exp \left[ \sum_{i < j} (-)^{\sigma_i \sigma_j} 4\pi D_{\mu_0}(x_i - x_j) \right] \quad (3)$$

where  $\sigma_i = \pm 1$  for  $H_i = \pm$ ,  $D_{\mu_0}$  is the massive scalar propagator,

$$D_{\mu_0}(x) = -\frac{1}{2\pi}K_0(\mu_0|x|), \quad \tilde{D}_{\mu_0}(p) = \frac{-1}{p^2 + \mu_0^2}, \quad (4)$$

( $K_0 \dots$  McDonald function) and  $\Sigma$  is the fermion condensate of the massless Schwinger model,

$$\Sigma = \langle \bar{\Psi}\Psi \rangle_0 = \frac{e^\gamma}{2\pi}\mu_0 \quad (5)$$

( $\gamma \dots$  Euler constant). The index 0 for  $\mu_0$  indicates that it is the order zero result, the index 0 for the VEVs means that they are with respect to the massless Schwinger model. From this  $Z(m, \theta)$  may be computed (see [21] for details, [24] for its physical implications),

$$Z(m, \theta) = e^{V\alpha(m, \theta)},$$

$$\alpha(m, \theta) = m \frac{\Sigma}{2} 2 \cos \theta + m^2 \left(\frac{\Sigma}{2}\right)^2 (E \cos 2\theta + F) + o(m^3) \quad (6)$$

( $V \dots$  space time volume) where

$$E = \int d^2x E(x) \equiv \int d^2x (e^{-2K_0(\mu_0|x|)} - 1) = -8.9139 \cdot \frac{1}{\mu_0^2}$$

$$F = \int d^2x F(x) \equiv \int d^2x (e^{2K_0(\mu_0|x|)} - 1) = 9.7384 \cdot \frac{1}{\mu_0^2} \quad (7)$$

and for  $F$  a subtraction of a free field singularity is necessary.

In order to compute VEVs for the massive Schwinger model one has to insert the corresponding operators into the path integral (1), (2) and divide by the vacuum functional  $Z(m, \theta)$ :

$$\langle \hat{O} \rangle_m = \frac{1}{Z(m, \theta)} \langle \hat{O} \sum_{n=0}^{\infty} \frac{m^n}{n!} \prod_{i=1}^n \int dx_i \bar{\Psi}(x_i) \Psi(x_i) \rangle_0 \quad (8)$$

Via the normalization all volume factors cancel completely, as it certainly has to be (we will explicitly see this in the computations). For the computation of VEVs of scalar ( $S$ ) and chiral ( $S_{\pm}$ ) densities formula (3) suffices and could be used e.g. for the computation of the fermion condensate  $\langle \bar{\Psi}\Psi \rangle_m$  (this however may be derived at once from (6), see [21]).

For the computation of the Schwinger mass additional Green functions of the massless Schwinger model are needed. It is wellknown that the vector current correlator is the free massive propagator in the massless Schwinger model:

$$\langle J_{\mu}(x) J_{\nu}(y) \rangle_0 = \frac{1}{\pi} \epsilon_{\mu\mu'} \partial_x^{\mu'} \epsilon_{\nu\nu'} \partial_y^{\nu'} D_{\mu_0}(x - y) \quad (9)$$

Therefore, for a perturbative calculation we need the  $n$ -point functions

$$\langle J_{\mu}(y_2) J_{\nu}(y_1) \prod_{i=1}^n S_{H_i}(x_i) \rangle_0 = \frac{1}{\pi} \epsilon_{\mu\mu'} \partial_x^{\mu'} \epsilon_{\nu\nu'} \partial_y^{\nu'} \cdot$$

$$\cdot \left(\frac{\Sigma}{2}\right)^n \left[ D_{\mu_0}(y_1 - y_2) + 4\pi \left( \sum_{i=1}^n (-)^{\sigma_i} D_{\mu_0}(x_i - y_2) \right) \left( \sum_{j=1}^n (-)^{\sigma_j} D_{\mu_0}(x_j - y_1) \right) \right] \cdot$$

$$\cdot e^{4\pi \sum_{k < l} (-)^{\sigma_k \sigma_l} D_{\mu_0}(x_k - x_l)} \quad (10)$$

which may be computed from the Euclidean path integral or from bosonization (the latter method is easier).

### 3 Computation of the Schwinger mass

For the perturbative computation of the massive propagator we simply have to insert successive orders of equ. (10) into (8). The factor  $(\frac{1}{\pi}\epsilon_{\mu\mu'}\partial_{y_2}^{\mu'}\epsilon_{\nu\nu'}\partial_{y_1}^{\nu'})$  is the same for all orders, therefore we will ignore it in the sequel.

There are two terms from (10) to be inserted in first order, namely  $S_H, H = \pm$ , and four terms,  $S_{H_1}S_{H_2}, H_i = \pm$ , in second order. For the moment we consider the case where all  $H_i = +$ ; the other contributions can be inferred from this one by a rearrangement of some signs. Up to second order, we find  $(\langle J_\mu(y_2)J_\nu(y_1)\rangle_m \equiv \frac{1}{\pi}\epsilon_{\mu\mu'}\partial_{y_2}^{\mu'}\epsilon_{\nu\nu'}\partial_{y_1}^{\nu'}T(y_1, y_2))$

$$T(y_1, y_2) = \frac{1}{Z(m, \theta)} \left[ D_{\mu_0}(y_1 - y_2) + m \frac{\Sigma}{2} V D_{\mu_0}(y_1 - y_2) + 4\pi m \frac{\Sigma}{2} \int dx D_{\mu_0}(x - y_2) D_{\mu_0}(x - y_1) + \right. \\ \left. \frac{m^2}{2!} \left( \frac{\Sigma}{2} \right)^2 \int dx_1 dx_2 [D_{\mu_0}(y_1 - y_2) + 4\pi(D_{\mu_0}(x_1 - y_2) + D_{\mu_0}(x_2 - y_2)) \cdot \right. \\ \left. \cdot (D_{\mu_0}(x_1 - y_1) + D_{\mu_0}(x_2 - y_1))] e^{4\pi D_{\mu_0}(x_1 - x_2)} \right] \quad (11)$$

Inserting equ. (6) for the vacuum functional  $Z(m, \theta)$  and using the perturbation formula of second order,

$$\frac{a_0 + a_1 x + a_2 x^2 + o(x^3)}{1 + b_1 x + b_2 x^2 + o(x^3)} = \\ a_0 + a_1 x + a_2 x^2 - a_0 b_1 x - a_0 b_2 x^2 - a_1 b_1 x^2 + a_0 b_1^2 x^2 + o(x^3) \quad (12)$$

we arrive at

$$T(y_1, y_2) = D_{\mu_0}(y_1 - y_2) + 4\pi m \frac{\Sigma}{2} \int dx D_{\mu_0}(x - y_2) D_{\mu_0}(x - y_1) + \\ \frac{m^2}{2!} \left( \frac{\Sigma}{2} \right)^2 4\pi \int dx_1 dx_2 [2D_{\mu_0}(x_1 - y_2) D_{\mu_0}(x_1 - y_1) + 2D_{\mu_0}(x_1 - y_1) D_{\mu_0}(x_2 - y_2)] \cdot \\ \cdot (E(x_1 - x_2) + 1) - 4\pi m^2 \left( \frac{\Sigma}{2} \right)^2 V \int dx D_{\mu_0}(x - y_2) D_{\mu_0}(x - y_1) \\ = D_{\mu_0}(y_1 - y_2) + 4\pi m \frac{\Sigma}{2} \int dx D_{\mu_0}(x - y_2) D_{\mu_0}(x - y_1) + \\ m^2 \left( \frac{\Sigma}{2} \right)^2 4\pi E \int dx D_{\mu_0}(x - y_2) D_{\mu_0}(x - y_1) + \\ m^2 \left( \frac{\Sigma}{2} \right)^2 4\pi \int dx_1 dx_2 D_{\mu_0}(x_1 - y_1) D_{\mu_0}(x_2 - y_2) (E(x_1 - x_2) + 1) \quad (13)$$

where  $E, E(x)$  are given in (7) and we used the  $x \rightarrow -x$  symmetry of all occurring functions. Observe that, as claimed, all volume factors  $V$  have dropped out.

However, expression (13) is not yet the desired result for further computation, there is still one unwanted term. The last line of expression (13) consists of two terms, due to the factor  $(E(x_1 - x_2) + 1)$ . The second one, proportional to 1, reads

$$m^2 \left( \frac{\Sigma}{2} \right)^2 4\pi \int dx_1 D_{\mu_0}(x_1 - y_1) \int dx_2 D_{\mu_0}(x_2 - y_2) \quad (14)$$

and obviously does not contribute to the  $y_1 - y_2$  correlation. It is a disconnected part stemming from the VEVs

$$\langle \mathbf{J}_\mu(y_i) S_+(x_j) \rangle_0 \sim \epsilon_{\mu\mu'} \partial_{y_i}^{\mu'} D_{\mu_0}(x_j - y_i) \quad (15)$$

and must be subtracted from (13).

Looking for the moment at the first order correction only, it is very easy to find the Schwinger mass correction:

$$\begin{aligned} \square_{y_1} T^{(1)}(y_1, y_2) &= \delta(y_1 - y_2) + \mu_0^2 D_{\mu_0}(y_1 - y_2) + \\ &4\pi m \frac{\Sigma}{2} (D_{\mu_0}(y_1 - y_2) + \mu_0^2 \int dx D_{\mu_0}(x - y_2) D_{\mu_0}(x - y_1)) \\ &\equiv \delta(y_1 - y_2) + \mu_1^2 D_{\mu_1}(y_1 - y_2) + o(m^2) \end{aligned} \quad (16)$$

where

$$\mu_1^2 = \mu_0^2 + 4\pi m \frac{\Sigma}{2} \quad (17)$$

To obtain the second order result, we rewrite expression (13), without the disconnected part, in momentum space and substitute all functions by their Fourier transforms (thereby the convolutions turn into products):

$$\begin{aligned} \tilde{T}(p) &= \frac{-1}{p^2 + \mu_0^2} + 4\pi m \frac{\Sigma}{2} \frac{1}{(p^2 + \mu_0^2)^2} + 4\pi m^2 \left(\frac{\Sigma}{2}\right)^2 \frac{1}{(p^2 + \mu_0^2)^2} (E + \tilde{E}(p)) = \\ &\frac{-1}{p^2 + \mu_0^2} \left(1 - 4\pi m \frac{\Sigma}{2} \frac{1}{p^2 + \mu_0^2} - 4\pi m^2 \left(\frac{\Sigma}{2}\right)^2 (E + \tilde{E}(p)) \frac{1}{p^2 + \mu_0^2}\right) = \\ &\frac{-1}{p^2 + \mu_0^2 + 4\pi m \frac{\Sigma}{2} + 4\pi m^2 \left(\frac{\Sigma}{2}\right)^2 (E + \tilde{E}(p)) + (4\pi m \frac{\Sigma}{2})^2 \frac{1}{p^2 + \mu_0^2}} + o(m^3) \end{aligned} \quad (18)$$

Therefore, for finding the mass pole,  $p^2$  has to obey the self consistency equation (after a rescaling  $p'^2 = \frac{p^2}{\mu_0^2}$ ,  $E' = E(\mu_0^2 \equiv 1) = \mu_0^2 E$  etc.)

$$p'^2 = -1 - 4\pi \frac{m}{\mu_0} \frac{\Sigma}{2\mu_0} - 4\pi \frac{m^2}{\mu_0^2} \left(\frac{\Sigma}{2\mu_0}\right)^2 \left[E' + \tilde{E}'(p') + \frac{4\pi}{p'^2 + 1}\right] \quad (19)$$

The second order part (the term in square brackets) may be rewritten like

$$\begin{aligned} [\dots] &= \int d^2x [e^{-2K_0(|x|)} - 1 + e^{ip'x} (e^{-2K_0(|x|)} - 1 + 2K_0(|x|))] = \\ &\int_0^\infty dr r [2\pi (e^{-2K_0(r)} - 1) + \int_0^{2\pi} d\theta e^{i|p'|r \cos \theta} (e^{-2K_0(r)} - 1 + 2K_0(r))] = \\ &2\pi \int_0^\infty dr r [e^{-2K_0(r)} - 1 + J_0(|p'|r) (e^{-2K_0(r)} - 1 + 2K_0(r))] \end{aligned} \quad (20)$$

where  $J_0$  is the Bessel function of the first kind. This expression behaves well around  $|p'| = i$  and therefore we may set  $|p'| = i$  because deviations from this value are of higher order in  $m$ . Using  $I_0(r) = J_0(ir)$  we find

$$p'^2 = -1 - 4\pi \frac{m}{\mu_0} \frac{\Sigma}{2\mu_0} - 8\pi^2 \frac{m^2}{\mu_0^2} \left(\frac{\Sigma}{2\mu_0}\right)^2 \cdot A \quad (21)$$

$$A := \int_0^\infty dr r [e^{-2K_0(r)} - 1 + I_0(r)(e^{-2K_0(r)} - 1 + 2K_0(r))] \\ A = -0.6599 \quad (22)$$

This result was computed for the positive chirality densities  $S_+$ . For the pure negative chirality densities  $S_-$  the result is completely identical. However in second order there are mixed products  $S_+S_-$ ,  $S_-S_+$ , too. Both of them lead to the same result, where the above integral  $A$  is substituted by a similar expression  $B$ ,

$$B := \int_0^\infty dr r [e^{+2K_0(r)} - 1 + I_0(r)(-e^{+2K_0(r)} + 1 + 2K_0(r))] \\ B = 1.7277 \quad (23)$$

In this expression the nice feature of cancellation of UV divergencies occurs. Indeed, both  $e^{2K_0(r)}$  and  $-I_0(r)e^{2K_0(r)}$  diverge like  $\frac{1}{r^2}$  for small  $r$  (this divergency corresponds to the free fermion field divergency of the underlying theory), but obviously the divergencies cancel each other. In fact, this cancellation was already observed twenty years ago in [13] within a bosonization approach, and on general grounds it should continue to hold for higher orders.

Collecting all results and multiplying each contribution with its corresponding  $\theta$  factor ( $S_\pm \rightarrow e^{\pm i\theta}$ , see (1-3)), we find for the Schwinger mass in second order

$$-p'^2 \equiv \frac{\mu_2^2}{\mu_0^2} = 1 + 8\pi \frac{m}{\mu_0} \frac{\Sigma}{2\mu_0} \cos \theta + 16\pi^2 \frac{m^2}{\mu_0^2} \left(\frac{\Sigma}{2\mu_0}\right)^2 (A \cos 2\theta + B) \quad (24)$$

or, inserting all numbers (remember  $\frac{\Sigma}{2\mu_0} = \frac{e^\gamma}{4\pi}$ , equ. (5))

$$\mu_2^2 = \mu_0^2 (1 + 3.5621 \cdot \frac{m}{\mu_0} \cos \theta + 5.4807 \cdot \frac{m^2}{\mu_0^2} - 2.0933 \cdot \frac{m^2}{\mu_0^2} \cos 2\theta) \quad (25)$$

which is our final result.

For the special case  $\theta = 0$  our result (24) precisely coincides with the result in [25], where the second order correction for  $\theta = 0$  was computed within bosonization and using near light cone coordinates. In the same article this result was compared with a lattice calculation ([26]), and a good agreement is obtained within the range of the expansion parameter  $\frac{m}{\mu_0}$  where the lattice calculations were performed.

## 4 Summary

We have demonstrated a general method of computing  $n$ -point functions for the massive Schwinger model in mass perturbation theory within the Euclidean path integral formalism, using the known exact solution of the massless Schwinger model as a starting point.

All  $n$ -point functions exist perturbatively and are finite in the infinite volume limit.

Using this approach we were able to compute the mass perturbation corrections to the Schwinger mass up to second order, and almost all calculations could be done analytically. This feature does not persist to hold for higher orders. Already in third order an equation analogous to (19) will be a true self consistency equation that can be solved only numerically.

For  $\theta = 0$ , the result (24) is well reproduced by lattice calculations ([25]).

The numerical calculations in this article were done with Mathematica 2.2.

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