Algebraic aspect and construction of Lax operators in quantum integrable systems

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Abstract

An algebraic construction more general and intimately connected with that of Faddeev[1], along with its application for generating different classes of quantum integrable models are summarised to complement the recent results of ref. 1 (L.D. Faddeev, *Int. J. Mod. Phys.* **A10**, 1845 (1995)).

A significant portion of an excellent recent review by Faddeev[1] is devoted to an important result showing deep relation between the Yang–Baxter equation and the quantum group. We are delighted to note that a scheme for systematic construction of the Lax operators through Yang–Baxterisation of the Faddeev–Reshetikhin–Takhtajan (FRT) algebra persued by us over the last few years[2,3] has an intimate connection with this approach. In fact our formulation gives a more general framework allowing construction of a wide range of Lax operators of lattice models and recovers the corresponding result of ref 1 related to $U_q sl(2)$ as a particular case. Another convincing demonstration of the usefulness of our approach is that, the well known Lax operator of the lattice Liouville model[4] as well as the recently discovered[5] nontrivial spectral parameter dependent variant of it can be constructed directly and easily from our general form, instead of going through the involved limiting procedure and latice gauge transformation, as is the standard practice[4,5].

We also consider the *twisted* generalisation of our Lax operator as well as the $q \to 1$ limit of it and apply them for generating different classes of integrable models.

Therefore we hope that, the present letter would complement the relevant results of ref. 1. and would serve as a guide to understand and formulate the related approach of Faddeev in a more general form and apply it for constructing wider class of quantum integrable models.

For generating the class of models associated with the trigonometric R-matrix of spin-$\frac{1}{2}$ XXZ chain:

$$R_{trig}(\lambda) = \begin{pmatrix} \sin\alpha(\lambda+1) & & & \\ & \sin\alpha\lambda & \sin\alpha & \\ & \sin\alpha & \sin\alpha\lambda & \\ & & & \sin\alpha(\lambda+1) \end{pmatrix} . \qquad (1)$$

we start with a general Lax operator of the form[2]

$$L(\lambda) \; = \; \begin{pmatrix} \frac{1}{\xi}\tau_1^+ + \xi\tau_1^- & \tau_{21} \\ \tau_{12} & \frac{1}{\xi}\tau_2^+ + \xi\tau_2^- \end{pmatrix} . \qquad (2)$$

Here $\xi = e^{-i\alpha\lambda}$ is the spectral parameter and the operator valued elements in $L(\lambda)$ satisfy the algebra

$$[\tau_{12}, \tau_{21}] \; = \; -2i\sin\alpha\left(\tau_1^+\tau_2^- - \tau_1^-\tau_2^+\right) \; , \; \tau_i^\pm \tau_{ij} \; = \; e^{\pm i\alpha}\tau_{ij}\tau_i^\pm \; , \; \tau_i^\pm \tau_{ji} \; = \; e^{\mp i\alpha}\tau_{ji}\tau_i^\pm \; , \qquad (3)$$

($i, j \in [1,2]$) with all $\tau_i^\pm$ commuting among themselves. The Hopf algebra structure of (3) including the coproduct, unity, antipode etc. can be shown to exist in the usual way. It is also worthnoting that the above quadratic algebra is an extension of the trigonometric Sklyanin algebra (TSA)[6] and reduces to it in the particular symmetric case $\tau_2^- = -\tau_1^+, \ \tau_2^+ = -\tau_1^-$. Moreover, in analogy with the well known relation[7] between TSA and $U_q sl(2)$, one can express the elements of extended TSA (3) as[3,8,9]

$$\tau_1^\pm = \left(\tau_2^\pm\right)^{-1} = q^{\pm S_3} \ , \quad \tau_{12} = -(q-q^{-1})S_+ \ , \quad \tau_{21} = (q-q^{-1})S_- \ , \tag{4}$$

where $q = e^{i\alpha}$ and $S_3, \ S_\pm$ are the generators of the quantised algebra $U_q sl(2)$. For the above realisation of its operator elements, our Lax operator (2) coincides with the Lax operator of q-deformed XXX-model given by eqn. (126) in ref. 1. Furthermore, by using algebra (3), it is easy to check that the $L(\lambda)$ operator (2) satisfies the quantum Yang–Baxter equation (QYBE)

$$R(\lambda-\mu)\ L_1(\lambda)\ L_2(\mu) \ = \ L_2(\mu)\ L_1(\lambda)\ R(\lambda-\mu); \quad L_1 \equiv L \otimes \mathbf{1}, \ L_2 \equiv \mathbf{1} \otimes L \tag{5}$$

for the trigonometric R-matrix (1) and thus may be considered as the Lax operator of some abstract integrable model, concrete realisations of which should yield different physical models.

To extract the spectral parameterless limit of $L(\lambda)$ and the corresponding $R(\lambda)$ given by (2) and (1), it is helpful to make a 'gauge transformation' on them (see eqns. (2.5)-(2.7) of ref.2 or, eqns. (131), (132) of ref. 1), which allows us to write them as

$$R(\lambda) \ = \ \frac{1}{\xi}R^+ \ - \ \xi R^- \ , \quad L(\lambda) \ = \ \frac{1}{\xi}L^{(+)} \ + \ \xi L^{(-)} \ , \tag{6}$$

where

$$L^{(+)} \ = \ \begin{pmatrix} \tau_1^+ & \tau_{21} \\ 0 & \tau_2^+ \end{pmatrix} \ , \quad L^{(-)} \ = \ \begin{pmatrix} \tau_1^- & 0 \\ \tau_{12} & \tau_2^- \end{pmatrix} \ . \tag{7}$$

Again, for realisation (4), the above $L^{(\pm)}$-matrices coincide with their counterparts given through eqns. (139) and (140) of ref. 1.

Subsequently, we may insert, as shown in ref. 2, the 'gauge transformed' $L(\lambda)$ and $R(\lambda)$ matrices (6) in the QYBE (5) and compare the coefficients of various powers of spectral parameters from its both sides and arrive at a set of seven relations

$$R^{\pm}\, L_1^{(\pm)}\, L_2^{(\pm)} \;=\; L_2^{(\pm)}\, L_1^{(\pm)}\, R^{\pm}\;, \quad R^{\pm}\, L_1^{(\pm)}\, L_2^{(\mp)} \;=\; L_2^{(\mp)}\, L_1^{(\pm)}\, R^{\pm}\;, \tag{8}$$

and

$$R^{+}L_1^{(-)}L_2^{(+)} \;-\; L_2^{(+)}L_1^{(-)}R^{+} \;=\; R^{-}L_1^{(+)}L_2^{(-)} \;-\; L_2^{(-)}L_1^{(+)}R^{-}\;. \tag{9}$$

which are evidently similar to eqns. (144)-(147) of ref. 1. By using once again realisation (4), it is easy to see that the relations (8) are essentially same as the well known FRT relations for the quantised algebra $U_q sl(2)$. Moreover, the remaining eqn. (9) can also be reduced to these FRT relations with the help of crucial Hecke condition

$$R^{+} \;-\; R^{-} \;=\; c\,\mathcal{P}, \tag{10}$$

($\mathcal{P}$ being the permutation operator) satisfied by $R^{\pm}$ matrices[2,1]. Thus, as we have shown in ref. 2, all spectral parameter independent FRT relations corresponding to the quantised algebra can in fact be derived from a single, but spectral parameter dependent relation QYBE.

We shall stress again that by using different realisations of extended TSA (3) in the general Lax operator (2), one can easily construct the representative Lax operators of a large set of quantum integrable models, all of them associated with the same trigonometric R-matrix (1). Examples of such models are the lattice sine-Gordon model (for the symmetric reduction (4)), lattice Liouville model, an ultralocal derivative nonlinear Schrödinger (NLS) equation, massive Thirring model etc. including some new ones[3]. Moreover, reverting the above outlined procedure, it is possible to Yang–Baxterise the FRT algebra in a straightforward way and construct other solutions of QYBE including those with higher dimensional Lax operators[2,10].

The following is the reason, why it is often easier to construct different types of Lax operators of quantum integrable models from $L(\lambda)$ operator (2), in comparison with the Lax operator of q-deformed XXX-model given by eqn. (126) in ref. 1. The extended TSA (3) yields a realisation like (4) through the generators of the quantised algebra $U_q sl(2)$, only when all $\tau_i^{\pm}$ are invertible nonsingular operators. However, for singular values of some $\tau_i^{\pm}$, there exist other realisations of extended TSA which cannot be connected with $U_q sl(2)$ algebra, without making any complicated limiting transition. To illustrate this important point, we may consider the example of lattice Liouville model. Recall that for constructing the associated well known Lax operator[4] as well as the Lax operator with nontrivial spectral parameter dependence discovered very recently[5], one starts usually with an infinite dimensional representation of $U_q sl(2)$ quantum algebra, which maps the q-deformed XXX-spin Lax operator to the lattice sine-Gordon Lax operator. Subsequently one has to perform a rescaling of the field and then take the massless limit for obtaining the standard Lax operator[4]. The construction of the recent nontrivial variant of it is even more involved. Along with the above rescaling of the field and the massless limiting procedure, it requires also a renormalisation of the spectral parameter in addition to a lattice gauge transformation on the sine-Gordon Lax operator[5]. It is however interesting to observe that, one can use simple realisations of extended TSA (3) to get in a straightforward way, both the standard and the nontrivial Lax operators of the lattice Liouville model from our $L(\lambda)$ operator (2), without involving any rescaling, gauge transformation or limiting procedure. It is easy to check that the realisation of (3) through canonical operators $\Phi,\ \Pi$ (with $[\Phi, \Pi] = i\alpha$) as

$$\tau_1^+ = \tau_2^- = -i\, e^{-i\Phi} \ , \ \ \tau_2^+ \ = \ \tau_1^- \ = 0, \ \tau_{12} \ = \ e^{i\Pi} \ h^{\frac{1}{2}}(\Phi) \ \ , \ \ \tau_{21} \ = \ h^{\frac{1}{2}}(\Phi) \ e^{-i\Pi}, \tag{11}$$

where $h(\Phi) = 1 - e^{-2i\Phi + i\alpha}$, is consistent with algebra (3) and gives directly from (2) the well known Lax operator[4] of the lattice Liouville model. Moreover, in an analogous way through another simple and similar realisation

$$\tau_1^+ = \tau_2^- = (\tau_1^-)^{-1} \ = \ -i\, e^{-i\Phi} \ , \ \ \tau_2^+ = 0 \ , \ \ \tau_{12} \ = \ e^{i\Pi} \ , \ \ \tau_{21} \ = \ h(\Phi)\ e^{-i\Pi}, \tag{12}$$

we obtain readily from (2) the nontrivial spectral parameter dependent Liouville Lax operator of ref.5 without going through any other intermediate steps.

Apart from the construction related to the quantum R-matrix (1) described above, there exist also other interesting possibilities to cover wider range of integrable models. If we consider a *twisting* transformation of the R-matrix (1):

$$R \rightarrow R_\theta = B(\theta) R B(\theta), \qquad B(\theta) = e^{i\theta(\sigma_3 \otimes 1 - 1 \otimes \sigma_3)}, \tag{13}$$

the $L(\lambda)$ operator as a corresponding solution of the QYBE (5) may again be given in the form (2), where the generators $\{\tau\}$ now satisfy a θ-deformed extension of the quadratic algebra (3)

$$\begin{aligned} e^{i\theta}\ \tau_{12}\tau_{21} \ - \ e^{-i\theta}\ \tau_{21}\tau_{12} \ &= \ -2i \sin\alpha \left(\tau_1^+\tau_2^- - \tau_1^-\tau_2^+\right) \ , \\ \tau_i^\pm \tau_{ij} \ = \ e^{i(\pm\alpha+\theta)}\tau_{ij}\tau_i^\pm \ , \quad \tau_i^\pm \tau_{ji} \ &= \ e^{i(\mp\alpha-\theta)}\tau_{ji}\tau_i^\pm \ . \end{aligned} \tag{14}$$

Note that the decomposition (6) and the relations (8-10) also hold good in this case. Symmetric reduction like (4) relates this algebra to the two-parameter algebra $U_{q,p}gl(2)$ and may generate θ-deformed lattice sine-Gordon and Liouville model. However, other possible realisations yield from (2) the Lax operators of the well known Ablowitz–Ladik model and a family of discrete-time models related to the relativistic quantum Toda chain[3,11].

Likewise, using slightly different type of twisting one can also Yang-Baxterise the FRT algebra related to coloured braid group representation and generate integrable Lax operators associated with non-additive type spectral parameter dependent quantum R-matrix[9]. A general formalism in this direction starting from an universal R-matrix of reductive Lie algebras is also presented in a recent work[10].

It is remarkable that most of these structures survives nicely in the $q \rightarrow 1$ or $\alpha \rightarrow 0$ limit, when the trigonometric R-matrix reduces to its rational form

$$R_{rat}(\lambda) = \begin{pmatrix} \alpha(\lambda+1) & & & \\ & \alpha\lambda & \alpha & \\ & \alpha & \alpha\lambda & \\ & & & \alpha(\lambda+1) \end{pmatrix}, \tag{15}$$

and allows us to generate another class of integrable quantum models associated with this rational R-matrix solution. At this undeformed limit the $L(\lambda)$ operator (2) also undergoes a smooth transition to the form[3]

$$L(\lambda) = \begin{pmatrix} K_1^0 + i\frac{\lambda}{\eta}K_1^1 & K_{21} \\ K_{12} & K_2^0 + i\frac{\lambda}{\eta}K_2^1 \end{pmatrix}, \tag{16}$$

where $\mathbf{K}$ operators now satisfy another quadratic algebra being the $q \to 1$ limit of (3):

$$[K_{12}, K_{21}] = (K_1^0 K_2^1 - K_1^1 K_2^0), \quad [K_1^0, K_2^0] = 0$$
$$[K_1^0, K_{12}] = K_{12}K_1^1,\ [K_1^0, K_{21}] = -K_{21}K_1^1,\ \ [K_2^0, K_{12}] = -K_{12}K_2^1,\ [K_2^0, K_{21}] = K_{21}K_2^1, \tag{17}$$

with K_1^1, K_2^1 serving as Casimir operators. A particular symmetric reduction $K_1^1 = K_2^1 = 1$ and $K_1^0 = -K_2^0$ yields the standard $sl(2)$ algebra, which for proper realisation gives XXX spin-$\frac{1}{2}$ model and the lattice NLS model. However using the freedom of other realisations of (17), from L-operator (16) one may obtain the nonrelativistic Toda chain and a simple version of lattice NLS[12] and through twisting transformation as before the θ-deformed models.

The classical aspect of our approach and its application are discussed in ref. 13.

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