

Evidence for Complex Subleading Exponents from the High-Temperature Expansion of the Hierarchical Ising Model

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Abstract

Using a renormalization group method, we calculate 800 high-temperature coefficients of the magnetic susceptibility of the hierarchical Ising model. The conventional quantities obtained from differences of ratios of coefficients show unexpected smooth oscillations with a period growing logarithmically and can be fitted assuming corrections to the scaling laws with complex exponents.

The renormalization group method [1] has enhanced considerably our understanding of elementary processes and critical phenomena. In particular, it has allowed the computation of the critical exponents of lattice models in various dimensions. On the other hand, the critical exponents can be estimated from the analysis of high-temperature series [2]. Showing that the two methods give precisely the same answers is a challenging problem [3]. More generally, much could be gained if we could combine these two approaches, in particular, in the context of lattice gauge theories.

As far as the numerical values of the critical exponents are concerned, there are two difficulties. The first one [4] is that one needs *much* longer high-temperature series than the ones available [5] (which do not go beyond order 25 in most of the cases) in order to make precise estimates. The second is that the practical implementation of the renormalization group usually requires projections into a manageable subset of parameters characterizing the interactions. It is nevertheless possible to design lattice models [6], called hierarchical models, which can be seen as approximate version of nearest neighbor models, for which such projections are unnecessary. For the hierarchical models, the renormalization group transformation reduces to a recursion formula which is a simple integral equation involving only the local measure. This simplicity allows one to control rigorously [7] the renormalization group transformation and to obtain accurate estimates of the critical exponents [8]. As explained in the next paragraph, the recursion formula also allows to calculate the high-temperature expansion to very high order. Consequently, the hierarchical model is well suited to study the questions addressed above.

In this letter, we report the results of a large scale calculation which performs the high-temperature expansion of the magnetic susceptibility of the hierarchical Ising model up to order 800. The method of calculation uses directly the renormalization group transformation with a rescaling of the spin variable appropriate to the study of the high-temperature fixed point. This method has been presented in Ref. [9] and checked using results obtained with conventional graphical methods [10]. For the sake of brevity, we shall follow exactly the set-up and the notations of Refs. [9,11] where the basic facts concerning the hierarchical

model were reviewed. The hierarchical models considered here have 2^n sites, and the free parameter c which controls the strength of the interactions is set equal to $2^{1-2/D}$ in order to approximate a nearest neighbor model in D -dimensions. We define the finite volume magnetic susceptibility as

$$\chi_n(\beta) = 1 + b_{1,n}\beta + b_{2,n}\beta^2 + \dots \quad (1)$$

In most of the calculations presented below, we have used $n=100$, which corresponds to a number of sites larger than 10^{30} . From a mathematical point of view, the calculation of χ_n amounts to repeating n times the Fourier transform of the recursion formula [9]

$$R_{n+1}(k) = C_{n+1} \exp\left(-\frac{1}{2}\beta\left(\frac{c}{2}\right)^{n+1} \frac{\partial^2}{\partial k^2}\right) \left(R_n\left(\frac{k}{\sqrt{2}}\right)\right)^2, \quad (2)$$

expanded to the desired order in β , with the initial condition $R_0 = \cos(k)$. The constant C_{n+1} can be adjusted in such a way that the Taylor expansion of $R_{n+1}(k)$ reads $1 - (1/2)k^2\chi_{n+1} + \dots$. This calculation has been implemented with a C program and ran for 6 weeks on a DEC-alpha 3000 in order to obtain 800 coefficients for $D = 3$.

For the discussion which follows, it is crucial to estimate precisely the errors made in the calculation of the coefficients. There are two sources of errors: the numerical round-offs and the finite number of sites. We claim that with 2^{100} sites and $3 \leq D \leq 4$, the finite volume effects are several order of magnitude smaller than the round-off errors. From Eq. (2), one sees that the leading volume dependence will decay like $(c/2)^n$. This observation can be substantiated by using exact results at finite volume [10] for low order coefficients, or by displaying the values of higher order coefficients at successive iterations as in Figure 1 of Ref. [9]. In both cases, we observed that the $(c/2)^n$ law worked remarkably well. For the main calculation presented below, we have used $c = 2^{\frac{1}{3}}$ (i.e. $D = 3$) and $n = 100$, which gives volume effects of the order of 10^{-20} . On the other hand, the round-off errors are expected to grow like the square root of the number of arithmetical operations. In Ref. [9], we have estimated that this number was approximately $n.m^2$ for a calculation up to order m in the high-temperature expansion with 2^n sites. Putting this together, we estimated that

for $n = 100$, the error on the m -th coefficient will be of order $m \times 10^{-16}$. We have verified this approximate law by calculating the coefficients using a rescaled temperature and undoing this rescaling after the calculation. We have chosen the rescaling factor to be 0.8482 and the rescaled critical temperature is then approximately 1. This prevents the appearance of small numbers in the calculation. If all the calculations could be performed exactly, we would obtain the same results as with the original method. However, for calculations with finite precision, the two calculations have independent round-off errors. Comparing the results obtained with the two methods for the coefficients up to order 200 shows that the numerical fluctuations of b_m grow approximately like $m \times 10^{-16}$. More conservatively, we can say that the numerical errors are bounded by $m \times 10^{-15}$. We conclude that for the calculations reported below, the errors on the coefficients are dominated by the numerical round-offs and we estimate that they do not exceed 10^{-12} .

In order to estimate γ , we used standard methods described in Refs. [2,4]. For the sake of definiteness, we recall a few definitions. First, we define $r_m = b_m/b_{m-1}$, the ratio of two successive coefficients. We then define the normalized slope S_m and the extrapolated slope \hat{S}_m as

$$S_m = -m(m-1)(r_m - r_{m-1})/(mr_m - (m-1)r_{m-1}) ; \tag{3}$$

$$\hat{S}_m = mS_m - (m-1)S_{m-1} .$$

The extrapolated slope, which is free of order n^{-1} corrections [4], is displayed in Fig.1 for $m \leq 200$. For comparison, we have also displayed the results for $D = 3.5$ and 4. A surprising feature is the clear appearance of large oscillations for $D = 3$. When D is increased, the amplitude of these oscillations diminishes. They are still present at $D = 4$ and can be seen better by plotting $\hat{S}_{m+1} - \hat{S}_m$. One important point of this letter is to establish that these oscillations are not due to the errors discussed above. As a consequence of the multiplications by m appearing in the definition of the the extrapolated slope, the errors are amplified by a factor which can be as large as 10^5 for m near 100 and 10^7 for m near 500. However, even

when multiplied by such a large factors our most conservative estimate of the numerical errors gives errors on the extrapolated slope which are several order of magnitude smaller than the amplitude of the oscillations. We have made independent checks of this statement for $D = 3$ by calculating directly the extrapolated slope for $n = 100$ and 200 and by using an intermediate temperature rescaling as explained above. The smoothness of the oscillations appears clearly in Fig. 2, where \widehat{S}_m is displayed for $50 \leq m \leq 800$. This smoothness rules out large numerical fluctuations. In conclusion, we have established that the oscillations in the extrapolated slope are a genuine feature of the model considered. Fig. 2 also shows that the extrema are not equally spaced. Instead, the location of one extremum can be approximately found by multiplying the location of the previous extremum by 1.19. In other words, the extrema of Fig. 2 would look equally spaced if the abscissa variable had been $\ln(m)$ instead of m . This of course suggests the use of a complex exponents since $Re(m^{i\sigma}) = \cos(\sigma \ln(m))$.

In the conventional description [12] of the renormalization group flow near a fixed point with only one eigenvalue $\lambda_1 > 1$, one expects that the magnetic susceptibility can be expressed as

$$\chi = (\beta_c - \beta)^{-\gamma} (A_0 + A_1(\beta_c - \beta)^\Delta + \dots) , \quad (4)$$

with $\Delta = |\ln(\lambda_2)|/\ln(\lambda_1)$ and λ_2 being the largest of the remaining eigenvalues. It is usually assumed that these eigenvalues are real. This implies [4] that

$$\widehat{S}_m = \gamma - 1 + Bm^{-\Delta} + O(m^{-2}) . \quad (5)$$

If Δ is real, there is no room for the oscillations in this description. Nevertheless, the fact that the period of oscillation increases logarithmically with m suggests that one could modify slightly Eq. (5) by allowing B and Δ to be complex and selecting the real part of the modified expression. This introduces two new parameters and we have chosen to use the following modified parametrization of the extrapolated slope:

$$\widehat{S}_m = \gamma - 1 + Km^{-\rho} \cos\left(2\pi \frac{\ln(m/m_0)}{\ln(\mu)}\right) + O(m^{-2}) . \quad (6)$$

This parametric expression allows good quality fits for m large enough. For instance, a least square fit for the $m \geq 300$ data, yields $\mu = 1.412$, $m_0 = 512$, $\rho = 0.67$, $\gamma = 1.310$ and $K = 2.53$. The fit is displayed on Fig. 2. More accurate results could presumably be obtained if we had a consistent description of the oscillations involving definite relations among the parameters of Eq. (6). The value of γ is in good agreement with the result [7,8] obtained with the ϵ -expansion, namely 1.300. The value of ρ is not far from $|\ln(\lambda_2)|/|\ln(\lambda_1)$ which is approximately 0.46 according to Ref. [7,8].

We have considered two possible explanations of the oscillatory behavior. The first one is that one could replace λ_2 by a couple of complex conjugated eigenvalues. This possibility is not realized in any perturbative calculation we know. For instance, the gaussian spectrum is real and its largest eigenvalues are widely separated. One could imagine that when D is continuously evolved from 4 to 3 for the hierarchical model, two real eigenvalues merge into each other and subsequently evolve as complex conjugate of each other. However, the fact that the oscillations persists at $D = 4$, as indicated in Fig. 1, goes against such an explanation. Conformal theories in two dimensions provide examples of calculations of the renormalization group eigenvalues as the eigenvalues of the matrix of derivative of the beta functions. For low order calculations, this matrix is symmetric [13] which implies real eigenvalues. A more attractive possibility, is that the susceptibility of the hierarchical model satisfies a renormalization group equation of the type discussed in section II of Ref. [14]. This equation allows corrections to the scaling law of the type $(1 + (A_1(\beta_c - \beta)^{\frac{i2\pi}{\ln(\lambda)}} + c.c) + ..)$, in the simplified case where only one eigenvalue λ is considered. Considering the values obtained with the fit we see that $\mu = 1.412$ is close to the the value of the largest eigenvalue [7,8] $\lambda_1=1.427$. This result suggests that one should try to find an equation for the susceptibility of the hierarchical model related to the one discussed in Ref. [14].

In conclusion, we have shown that a calculational method of the high-temperature expansion based on the renormalization group method can be a very powerful tool when the hierarchical approximation is used. It would be worth trying to improve this method beyond this approximation. Our analysis of the magnetic susceptibility has shown that unexpected

oscillations appear in the extrapolated slope. A detailed understanding of these oscillations is required in order to allow a precise comparison between the results obtained from the high-temperature expansion and the ϵ -expansion .

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REFERENCES

- [1] K. Wilson, Phys. Rev. B. **4** 3185 (1971) ; K. Wilson and J. Kogut, Phys. Rep. **12** 75 (1974).
- [2] D. S. Gaunt and A. J. Guttmann, in *Phase Transitions and Critical Phenomena* vol. 3, C. Domb and M. S. Green, eds., (Academic Press, New York, 1974) ; A. J. Guttmann, in *Phase Transitions and Critical Phenomena* vol. 13, C. Domb and Lebowitz, eds., (Academic Press, New York, 1989).
- [3] There is a large amount of literature on this subject. References can be found, e.g., in *Phase Transitions, Cargese 1980*, M. Levy, J.C. Le Guillou and J. Zinn-Justin eds., (Plenum Press, New York, 1982).
- [4] B. Nickel, Lecture Notes published in Ref. [3].
- [5] For recent calculations see e.g: G. Bhanot, M. Creutz, U. Glasser and K. Schilling, Phys. Rev. B **49** , 12909 (1994) .
- [6] F. Dyson, Comm. Math. Phys. **12**, 91 (1969) ; G. Baker, Phys. Rev. B**5**, 2622 (1972).
- [7] P. Bleher and Y. Sinai, Comm. Math. Phys. **45**, 247 (1975) ; P. Collet and J. P. Eckmann, Comm. Math. Phys. **55**, 67 (1977) and *Lecture Notes in Physics* **74** (1978) ; H. Koch and P. Wittwer, Comm. Math. Phys. **106** 495 (1986) , **138** (1991) 537 , **164** (1994) 627 .
- [8] P. Collet, J.-P. Eckmann, and B. Hirsbrunner, Phys. Lett. **71B**, 385 (1977).
- [9] Y. Meurice and G. Ordaz, J. Stat. Phys. **80** (in press), hep-lat 9409021.
- [10] Y. Meurice, J. Math. Phys. **36** 1812 (1995).
- [11] Y. Meurice, G. Ordaz and V. G. J. Rodgers, J. Stat. Phys. **77**, 607 (1994).
- [12] G. Parisi, *Statistical Field Theory* (Addison Wesley, New-York, 1988).

[13] A. Zamolodchikov, *Sov. J. Nucl. Phys.* **46**, 1090 (1987).

[14] Th. Niemeijer and J. van Leeuwen, in *Phase Transitions and Critical Phenomena*, vol. 6, C. Domb and M. S. Green, eds., (Academic Press, New York, 1976).

Figure Captions

Fig. 1: The extrapolated slope \hat{S}_m for $m \leq 200$ and $D = 3, 3.5$ and 4 .

Fig.2: The dots are the extrapolated slope \hat{S}_m for $50 \leq m \leq 800$ and $D = 3$. The continuous curve is the fit described in the text.