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Wave Propagation in Gravitational Systems: Late Time Behavior

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Abstract

It is well-known that the dominant late time behavior of waves propagating on a Schwarzschild spacetime is a power-law tail; tails for other spacetimes have also been studied. This paper presents a systematic treatment of the tail phenomenon for a broad class of models via a Green's function formalism and establishes the following. (i) The tail is governed by a cut of the frequency Green's function $\tilde{G}(\omega)$ along the $-$ Im ω axis, generalizing the Schwarzschild result. (ii) The ω dependence of the cut is determined by the asymptotic but not the local structure of space. In particular it is independent of the presence of a horizon, and has the same form for the case of a star as well. (iii) Depending on the spatial asymptotics, the late time decay is not necessarily a power law in time. The Schwarzschild case with a power-law tail is exceptional among the class of the potentials having a logarithmic spatial dependence. (iv) Both the amplitude and the time dependence of the tail for a broad class of models are obtained analytically. (v) The analytical results are in perfect agreement with numerical calculations.

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I. INTRODUCTION

In this paper we are concerned with the systematics of the propagation of linearized waves on curved background, in particular, the late time behavior [1]. The problem is of interest for understanding wave phenomena in relativity [2–6], and possibly in other areas of physics. The propagation of linearized gravitational, electromagnetic and scalar waves is often modeled by the Klein-Gordon (KG) equation with an effective potential

$$D\phi(x,t) \equiv \left[\partial_t^2 - \partial_x^2 + V(x)\right] \phi(x,t) = 0, \tag{1.1}$$

which includes the Zerilli and Regge-Wheeler equations as special examples [7]. The effective potential $V(x)$ describes the scattering of ϕ by the background curvature. We are here interested in the evolution of ϕ in the presence of this scattering.

On flat space ($V = 0$), waves travel along the light cone: $\phi(x,t) = \phi_+(x-t) + \phi_-(x+t)$, but the situation is much more interesting in the presence of a nontrivial background. Fig. 1 shows the result of a typical numerical experiment evolving (1.1) on $0 \le x < \infty$ (so that x may be thought of as a radial variable, see below), with the boundary condition $\phi(x = 0, t) = 0$, and the condition of outgoing waves for $x \to \infty$. An initial gaussian ϕ centered at y_o with initial $\dot{\phi} = 0$ is allowed to evolve, and is observed at $x_1 > y_o$. The figure shows $\phi(x_1, t)$ versus t. Our interest in this paper is the behavior at large t, but it is useful to first place the time dependence in an overall context.

The generic time dependence of waves observed at a fixed spatial location x_1 consists of three components, as illustrated in Fig. 1. (i) The prompt contribution (peaks A and C in Fig. 1a) depends strongly on the initial conditions. The first peak A is due to the initial pulse propagating from y_o directly to the right, while the later peak C is due to the initial pulse propagating from y_o to the left, being reflected and inverted at the origin, and then propagating to the observation point x_1. (ii) Each of these pulses is followed by a trailing signal (respectively B and D) caused by the potential V (in the sense that B and D vanish when $V = 0$). The latter pulse D has traversed regions of larger V, and as a result is both more pronounced in amplitude and also more compressed in time. On a longer time scale (Fig. 1b), it is seen that scattering by the potential (e.g. peak D) merges into a train of quasinormal ringing, decaying in an exponential manner. (iii) Finally, there is a tail, which decays approximately as a power in t (a straight line in this log-log plot), and it dominates at late times.

The prompt contribution is most intuitive, being the obvious counterpart of the light-cone propagation in the $V = 0$ case, and is in any event expected in the semi-classical limit. The ringing is due to a superposition of quasinormal modes [which are solutions to (1.1) with the time dependence $e^{-i\omega t}$ for some complex ω]. These are particularly interesting in that the time dependence (both of the quasinormal modes and of the trailing signals B and D) observed far away carries information about the structure of the intervening space. In the mathematical sense, this is because the quasinormal mode frequencies ω are determined by $V(x)$. In a more physical sense, it is useful to consider the analogy of electromagnetic waves emitted by a source in an optical cavity; the cavity, in providing a nontrivial environment, is analogous to the presence of a potential $V(x)$ in (1.1). [In fact, a one-dimensional scalar model of electromagnetic waves would be described by a wave equation with a nontrivial dielectric constant distribution $n^2(x)$ [8]; such a wave equation would be very similar to (1.1) and in fact can be transformed into it [9].] For a broad band source in a laser cavity of length L, the spectrum of electromagnetic waves observed far away would have frequency components $\omega \approx j\pi c/L$, where j is an integer. In this sense, these quasinormal modes carry information about the cavity. In much the same way, the ringing component of the solution in (1.1), if observed, would carry significant information about the background curvature of the intervening space. For this reason, there have been extensive numerical studies of various models. In addition, one particularly intriguing possibility is that the sum of quasinormal modes may be complete in a certain sense, as the normal modes of a conservative system are complete. If this were the case, one would have a discrete spectral representation of the dynamics, with obvious conceptual and computational advantages. The completeness has been demonstrated in the analogous case of the wave equation [10], and its generalization to the KG equation (1.1) will be reported elsewhere [11].

This paper develops a general treatment for the late time behavior of a field satisfying (1.1) with a broad class of time-independent potentials $V(x)$ and the outgoing wave boundary condition. Such a system could describe linearized waves evolving in stationary spacetimes with or without horizons, and in vacuum or nonvacuum spacetimes. For example, for a general static spherically symmetric spacetime,

$$ds^2 = g_{tt}(r)dt^2 + g_{rr}(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \quad , \tag{1.2}$$

a Klein-Gordon scalar field Φ can be expressed as

$$\Phi = \sum_{lm} \frac{1}{r}\phi_{lm}(x,t)Y_{lm}(\theta,\varphi). \tag{1.3}$$

The evolution of $\phi_{lm}(x,t)$ is given by (1.1) with the effective potential

$$V(x) = -g_{tt}\frac{l(l+1)}{r^2} - \frac{1}{2r}\frac{g_{tt}}{g_{rr}}\left[\frac{\partial_r g_{tt}}{g_{tt}} - \frac{\partial_r g_{rr}}{g_{rr}}\right] \quad , \tag{1.4}$$

where x is related to the circumferential radius r by

$$x = \int \sqrt{-\frac{g_{rr}}{g_{tt}}}dr \tag{1.5}$$

and extends over a half-line if the metric is non-singular, but extends over the full line when there is a horizon.

The special case of linearized perturbations of a black hole background has received much attention [4–7,12,13]. In the simplest case of the Schwarzschild spacetime

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \quad , \tag{1.6}$$

the transformation (1.5) leads to

$$x = r + 2M\log\left(\frac{r}{2M} - 1\right). \tag{1.7}$$

For a Klein-Gordon scalar field, V is defined by (1.4) and takes the explicit form

$$V(x) = \left(1-\frac{2M}{r}\right)\left[\frac{l(l+1)}{r^2} + (1-S^2)\frac{2M}{r^3}\right] \quad , \tag{1.8}$$

with $S = 0$. Electromagnetic and gravitational waves also obey (1.1) and (1.8), but with $S = 1, 2$ respectively [7]. The boundary condition is outgoing waves at $r = 2M$ ($x \to -\infty$, into the black hole) and as $r \to \infty$ ($x \to \infty$). With the effective potential (1.8), it is well known that the tail decays as $t^{-(2l+3)}$ independent of the spin S of the field [4]. For Reissner-Nordström black holes, the evolution equation can again be put in form of (1.1), but with an effective potential slightly different from (1.8). The tail is again found to decay as $t^{-(2l+3)}$ [5]. Although a similar tail should exist for Kerr black holes, we are not aware of the corresponding analysis in that case. Most of these previous analyses (except Ref. 3) are specific to a particular form of the potential.

Several recent developments make a more general study of the late time tail phenomenon of gravitational systems particularly interesting. As studies of the Schwarzschild spacetime show that the tail comes from scattering at large radius [4], it has been suggested [5] that a power-law tail would develop even when there is no horizon in the background, implying that such tails should be present in perturbations of stars, or after the collapse of a massless field which does not result in black hole formation. In Ref. 6, the late time behavior of scalar waves evolving in their own gravitational field or in gravitational fields generated by other scalar field sources was studied numerically, and power-law tails have been reported (though with exponents different from the Schwarzchild case). These interesting results call for a systematic analysis of the tail phenomenon in (i) nonvacuum, (ii) time-dependent, and (iii) nonlinear spacetimes, for which the present work will lay a useful foundation.

In Section II, we formulate the problem in terms of the Green's function, leading to a precise definition of the prompt, quasinormal mode and tail components in terms of contributions from different parts of the complex frequency plane. In Section III, numerical experiments are presented. The analytic treatment is then given in Section IV (potentials decreasing faster than x^{-2}), Section V (inverse square potentials) and Section VI (composite potentials consisting of a centrifugal barrier plus a subsidiary term). The last case, though technically most complicated, is also the most interesting physically, since it includes, for example, the potential in (1.8). Section VII contains a discussion and the conclusion.

The highlights of this paper are as follows. Firstly, the analytic results agree perfectly with the numerical experiments, including both the time dependence and the magnitude; this agreement indicates that the phenomenon of the late time tail has now been quite fully understood for a large class of models. Secondly, our work extends, places in context, and provides insight into the known results for the Schwarzschild case. For the Schwarzschild spacetime, the late time tail has previously been related to either reflection from the asymptotic region of the potential [4], or to the Green's function having a branch cut along the negative imaginary axis in the complex frequency plane [13]. In this paper, we demonstrate that the cut is a general feature, if $V(x)$ asymptotically ($x \to \infty$) tends to zero less rapidly than an exponential, except for the particular case of the "centrifugal barrier" $V(x) = l(l+1)/x^2$, $l = 1, 2....$ Furthermore, we determine the strength of the cut in terms of the asymptotic form of the potential for a broad class of potentials. These results then relate the two different explanations of the late time tail in the literature.

Some specific late time behavior we discovered in this paper are of particular interest. Potentials going as a centrifugal barrier of angular momentum l (l is an integer) plus $\sim x^{-\alpha}(\log x)^{\beta}$ ($\beta = 0, 1$) are of interest, because these include the Schwarzschild case ($\alpha = 3, \beta = 1$). For such potentials, the generic late time behavior is $\sim t^{-(2l+\alpha)}(\log t)^{\beta}$. Thus, for $\beta = 1$, the late time tail is not a simple power law, but exhibits an additional $\log t$ factor. However, when α is an odd integer less than $2l + 3$, the generic leading behavior vanishes. For $\beta = 0$, the next leading term is expected to be $t^{-(2l+2\alpha-2)}$, while for $\beta = 1$, the next leading term is $t^{-(2l+\alpha)}$ without a $\log t$ factor. Interestingly, the well-known Schwarzschild spacetime belongs to such exceptions.

As a collorary, we establish that the form of the late time tail is independent of the potential at finite x. In particular, whether the background metric describes a star (so that x is itself the radial variable $0 \le x < \infty$), or a black hole [so that x is the variable defined in (1.5), $-\infty < x < \infty$], the late time tail is the same as long as the asymptotic potential $V(x \to \infty)$ is the same, thus confirming the suggestion of Ref. 5.

As a by-product, we point out that numerical calculations are sometimes plagued by a "ghost" potential arising out of spatial discretization. The effects of this "ghost" potential may in some cases mask the genuine tail behavior — which is often numerically very small (see Fig. 1b). This problem could have led to errors, or at least ambiguities, in some numerical studies, in particular numerical studies of tail phenomena. This subtlety with numerical calculations is discussed in Appendix A.

II. FORMALISM

The time evolution of a field $\phi(x,t)$ described by (1.1) with given initial data can be written as [14]

$$\phi(x,t) = \int dy\ G(x,y;t)\ \partial_t\phi(y,0) + \int dy\ \partial_t G(x,y;t)\ \phi(y,0) \tag{2.1}$$

for $t > 0$, where the retarded Green's function G is defined by

$$DG(x,y;t) = \delta(t)\delta(x-y)\ , \tag{2.2}$$

and the initial condition $G(x,y;t) = 0$ for $t < 0$. The Fourier transform

$$\tilde{G}(x,y;\omega) = \int_0^\infty dt G(x,y;t) e^{i\omega t} \tag{2.3}$$

satisfies

$$\tilde{D}(\omega)\tilde{G} \equiv \left[-\omega^2 - \partial_x^2 + V(x)\right] \tilde{G}(x,y;\omega) = \delta(x-y) \tag{2.4}$$

and is analytic in the upper half ω plane.

Waves propagating on curved space and observed by a distant observer should satisfy the outgoing wave condition. In terms of this, there are two classes of systems.

(a) For non-singular systems, e.g., an oscillating relativistic star generating gravitational waves, the variable x, as defined by (1.5), is the radial variable of a three-dimensional problem, $0 \le x < \infty$. The left boundary condition is taken to be

$$\tilde{\phi}(x=0,\omega) = 0\ , \tag{2.5a}$$

while the right boundary condition is one of outgoing waves

$$\tilde{\phi}(x,\omega) \propto e^{i\omega x} \quad , \qquad x \to +\infty \quad , \tag{2.5b}$$

where $\tilde{\phi}$ is the wavefunction in the frequency domain.

(b) For the case of black holes, x given by (1.5) is a full line variable, $-\infty < x < \infty$, where the left boundary $x \to -\infty$ is the event horizon. We have outgoing waves at both ends, i.e.,

$$\tilde{\phi}(x,\omega) \propto e^{-i\omega x} \quad , \qquad x \to -\infty \quad , \tag{2.6}$$

together with (2.5b).

For simplicity, we shall write the formalism in terms of the half-line problem, and indicate briefly the simple changes necessary for the full-line case. We shall assume $V \to 0$ as $|x| \to \infty$.

Define two auxiliary functions $f(\omega, x)$ and $g(\omega, x)$ as solutions to the homogeneous equation $\tilde{D}(\omega)f(\omega,x) = \tilde{D}(\omega)g(\omega,x) = 0$, where f satisfies the left boundary condition and g satisfies the right boundary condition. To be definite, we adopt the following normalization conventions:

$$\lim_{x\to\infty} [e^{-i\omega x} g(\omega, x)] = 1; \tag{2.7}$$

and for the half-line problem $f(\omega, x = 0) = 0, f'(\omega, x = 0) = 1$ [15], while for the full-line problem $\lim_{x\to-\infty}[e^{i\omega x} f(\omega, x)] = 1$. Let the Wronskian be $W(\omega) = W(g, f) = g\partial_x f - f\partial_x g$, which is independent of x. It is then straightforward to see that

$$\tilde{G}(x,y;\omega) = \begin{cases} \dfrac{f(\omega,x)g(\omega,y)}{W(\omega)} \ , & x < y \\ \\ \dfrac{f(\omega,y)g(\omega,x)}{W(\omega)} \ , & y < x \end{cases} \quad . \tag{2.8}$$

The essence of the problem is, in one guise or another, how to integrate f from the left and g from the right, until they can be compared at a common value of x, at which the Wronskian can then be evaluated. The above choice of normalization is for convenience only, since a change of normalization, $f \to Af$ and $g \to Bg$, would result in $W \to AB\,W$ with no effect on $\tilde{G}$.

Consider the inverse transform of (2.3), and for $t > 0$ attempt to close the contour in ω by a semicircle of radius C in the lower half-plane, where eventually one anticipates taking $C \to \infty$. One is then led to consider the singularities of $\tilde{G}$ for Im $\omega < 0$ (Fig. 2), from which different contributions to G can be identified as follows.

(a) Tail contribution If the potential $V(x)$ has a finite range, say $V(x) = 0$ for $x > a$, then one can impose the condition (2.7) at $x = a^+$, and integrate the equation through a *finite* distance to obtain $g(\omega, x)$ for all $x < a$. Such an integration over a finite range using an ordinary differential equation cannot lead to any singularity in ω [16]. It is not surprising that if $V(x)$ does not have finite support, but nevertheless vanishes sufficiently rapidly as $x \to +\infty$, $g(\omega, x)$ is still regular. The necessary condition is that $V(x)$ must vanish faster than any exponential [17], and this is shown in Section IV. However, if $V(x)$ has a tail that does not decrease rapidly enough, then $g(\omega, x)$ will have singularities in ω. A tail that is exponential at large x, $V(x) \sim e^{-\lambda x}$, would in general lead to singularities in $g(\omega, x)$ for $-$ Im $\omega \geq \lambda/2$. In so far as these singularities stay away from $\omega = 0$, they have no effect on the late time behavior. However, if $V(x \to \infty) \sim x^{-\alpha}(\log x)^{\beta}$ $(\beta = 0, 1)$, then $g(\omega, x)$ will have singularities that lie on the $-$ Im ω axis, and take the form of a branch cut going all the way to the origin [17], as in the Schwarzschild case [13]. This contribution to G will be denoted as G_L. It is the purpose of this paper to evaluate G_L for a broad class of $V(x)$.

For the half-line problem, $f(\omega, x)$ is integrated from $x = 0$ through a finite distance, and hence does not have any singularities in ω. For the full-line problem, $f(\omega, x)$ is dealt with in the same manner. In all cases of physical interest, $V(x)$ vanishes on the left either faster than an exponential, or precisely as an exponential [e.g. Schwarzschild spacetime under the transformation (1.5)]. For the former, $f(\omega, x)$ has no singularities in ω, while for the latter, there will be a series of poles, but at a finite distance from $\omega = 0$. In both cases, there is no contribution to the late time behavior.

(b) Quasinormal modes Secondly, the Wronskian $W(\omega)$ may have zeros at $\omega = \omega_j, j =$

$\pm 1, \pm 2, \ldots$, on the complex ω plane [18]. At these frequencies, f and g are linearly dependent, i.e., one can find a solution that satisfies *both* the left and the right boundary condition. Such solutions are by definition quasinormal modes (QNM's). The collective contribution of the QNM's is denoted as $G_Q(x, y; t)$. In the case of a Schwarzschild black hole, G_Q has been evaluated semi-analytically by using the phase-integral method [19]. Since each $\text{Im } \omega_j < 0, G_Q$ decays exponentially as $t \to \infty$, and is, therefore, irrelevant for the late time behavior. In some cases, $G_Q(x, y; t)$ is the sole contribution to the Green's function; this would of course be important, since the quasinormal modes would then be complete. This case will be discussed elsewhere [11].

(c) Prompt contribution Finally there is the contribution from the large semicircle at $|\omega| = C, C \to \infty$. Since $|\omega|$ is large, this term contributes to the short time or prompt response, and the corresponding part of the Green function will be denoted as $G_P(x, y; t)$. This term can be shown to be zero beyond a certain time; thus it does not affect the late time behavior of the system.

The three contributions G_L, G_Q and G_P can be understood heuristically using a space-time diagram of wave propagation, as illustrated in Fig. 3, in which free propagation is represented by 45° lines, and the action of $V(x)$ is represented by scattering vertices. The initial wavefunction is assumed to have compact support, so the propagation starts at some initial y, and one imagines observation at some x. Rays 1a and 1b show "direct" propagation without scattering. These rays arrive promptly at x, and correspond to G_P. The ray labelled as 2 suffers repeated scatterings at finite x. For "correct" frequencies, the multiple reflections add coherently, so that retention of the wave is maximum. Moreover, the amplitude decreases in a geometric progression with the number of scatterings, which is in turn proportional to the time t. Thus ϕ decreases exponentially in t; these contributions correspond to G_Q.

Lastly, for a wave from a source point y reaching a distant observer at x, there could be reflections at very large x', such as indicated by ray 3. The ray first propagates to a point $x' \gg x$, is scattered by $V(x')$, and returns to the observation point x, arriving at a time $t \simeq (x' - y) + (x' - x) \simeq 2x'$. If the potential $V(x) \sim x^{-\alpha} (\log x)^\beta$ with $\alpha \geq 2$, then the scattering amplitude goes as $V(x') \simeq V(t/2) \sim t^{-\alpha}(\log t)^{\beta}$. Thus this contribution leads to G_L. In Section V, we shall see that this heuristic argument does not work for $\alpha = 2$. In particular, a centrifugal barrier of angular momentum l corresponds to free propagation in three-dimensional space, and does not contribute to the late time tail.

This heuristic picture requires two modifications, which we shall demonstrate both numerically and analytically in the following Sections. First, for $V(x) = l(l+1)/x^2 + \overline{V}(x)$ as $x \to \infty$, where $\overline{V}(x) \sim x^{-\alpha}(\log x)^{\beta}$, the late time tail is due to $\overline{V}(x)$ and turns out to be $t^{-(2l+\alpha)}(\log t)^{\beta}$ generically. The suppression by a factor t^{-2l}, at least in the case $\alpha = 3$, is known for specific black hole geometries [4,5]. Second, if α is an odd integer less than $2l+3$, the leading term in the late time tail vanishes. For $\beta = 0$, the next leading term is expected to be $t^{-(2l+2\alpha-2)}$, while for $\beta = 1$, the next leading term is $t^{-(2l+\alpha)}$ without a $\log t$ factor.

A minor technical complication should be mentioned at the outset. We close the contour by a semicircle of radius C (or other large contour). For any finite C, one can define the contributions from (a) the cut, (b) the poles and (c) the large circle. When $C \to \infty$, the sum of these must tend to the finite limit $G(x, y; t)$; however, there is no *a priori* guarantee that each of the three terms would individually converge to some limit. It can be shown [11] that provided a regulation is introduced, then each term would individually approach a finite limit, and hence the tail, QNM and prompt contributions would each be well defined. The regulator could be $e^{-\omega\tau}, \tau \to 0^+$ (for the part coming from $\text{Re}\ \omega \geq 0$). The need for a regulator comes from the region $|\text{Im}\ \omega| = O(\log\ |\text{Re}\ \omega|)$, $|\text{Re}\ \omega| \to \infty$, and has no bearing on the tip of the cut, near the origin, which controls the late time behavior. Therefore, the regulating factor will not be written out in the rest of this paper.

To study the late time behavior, it then remains to study the spatial asymptotics of the potential and the consequent singularities of g. Before carrying out these analyses for various potentials in Sections IV to VI, we first explore the late time behavior numerically.

III. NUMERICAL CALCULATIONS

In this Section, we present numerical investigations of the late time behavior. Eq. (1.1) is integrated numerically in time using standard techniques. Instead of the usual second-order differencing for the spatial derivative term, which could lead to erroneous results on account of a subtlety to be discussed in Appendix A, we have used a higher-order differencing scheme.

We evolve $\phi(x,t)$ for the half line $x \in [0,\infty)$, with the boundary conditions $\phi = 0$ at $x = 0$ and outgoing waves for $x \to \infty$. The full line case [$x \in (-\infty,\infty)$, with outgoing wave boundary conditions for $|x| \to \infty$] is basically the same. From (2.1), the generic leading late time behavior is given by the integral $\int dy G(x,y;t)\partial_t\phi(y,0)$, so we take $\phi(y,t=0) = 0$ and use gaussian initial data: $\partial_t\phi(y,t=0) = e^{-(y-y_o)^2/\eta^2}$.

We study a broad class of potentials:

$$V(x) = \frac{\nu(\nu+1)}{x^2} + \overline{V}(x) \quad , \tag{3.1}$$

with

$$\overline{V}(x) = \frac{x_o^{\alpha-2}}{x^\alpha}\left[\log\left(\frac{x}{x_o}\right)\right]^\beta \quad , \qquad\qquad x > x_c \quad , \tag{3.2}$$

for some $\alpha > 2$, and some x_o and x_c, which includes (i) power-law potentials ($\beta = 0$) and (ii) logarithmic potentials ($\beta = 1$). We study both integral and non-integral ν. The logarithmic potential is interesting as it includes the Schwarzschild metric as a special case. For $x \le x_c$, we round off the potential by taking

$$V(x) = \frac{2\nu(\nu+1)}{(x^2+x_c^2)} + \frac{2x_o^{\alpha-2}}{(x^\alpha + x_c^\alpha)}\left[\log\left(\frac{x+x_c}{2x_o}\right)\right]^\beta \quad , \qquad\qquad x \le x_c \quad . \tag{3.3}$$

We have checked that the form of the late time behavior does not depend on the round-off. For all the numerical calculations shown below, we take $x_c = 100$, $y_o = 500$ and $\eta = 50$.

We first consider power-law potentials $\overline{V}$ with $\nu = 0$ and study the dependence of the late time tails on the parameters α and x_o. Fig. 4 shows $\log|\phi(x_1,t)|$ versus $\log t$ at a fixed point $x_1 > x_c$ for different values of α and x_o. The solid lines are numerical results while the dashed lines are analytical results, which will be derived in the following Sections. For all cases studied, the tails are of the form $t^{-\mu}$ with $\mu \simeq \alpha$. For a given α, the magnitude of the tails is larger when x_o is larger.

Next, we look at the case when $\overline{V}(x)$ is zero and investigate how the late time tail depends on ν. It turns out that the result depends significantly on whether ν is an integer. For non-integral ν, the tail is again of the form $t^{-\mu}$ but with $\mu \simeq 2+2\nu$. On the other hand, when ν is an integer, we see only quasinormal ringing and no power-law late time tail. This qualitative difference in the tail is observed even when ν is altered by a small value, and this is demonstrated in Fig. 5, which shows $\log|\phi(x_1,t)|$ versus $\log t$ for $\nu = 1$ and 1.01. This result suggests that the magnitude of the tail is probably proportional to $\sin \nu\pi$, and therefore vanishes when ν is an integer.

Then, we study power-law and logarithmic potentials with nonzero integral values of ν, i.e., $\nu = l = 1, 2, \ldots$. Two interesting results are found. First, logarithmic potentials often lead to logarithmic tails of the form $t^{-\mu}\log t$, with $\mu \simeq 2l+\alpha$. Thus, the late time tail is not necessarily an inverse power of t. To exhibit the logarithmic factor, we plot $|\phi|t^{2l+\alpha}$ versus $\log t$ for several logarithmic potentials with $l=1$ in Fig. 6. The existence of a $\log t$ factor is indicated clearly by the sloping straight lines at late times [see below for case (iii), which is an exception]. However, when we vary the parameter α continuously from 2.9 to 3.1, the behavior of the late time tail changes discontinuously. This is seen from case (iii) in Fig. 6. Case (iii) is asymptotically flat, showing that there is no $\log t$ factor, and that the tail becomes a simple power law $\sim t^{-(2l+\alpha)}$. Such discontinuous jumps are observed whenever α goes through an odd integer less than $2l+3$. Interestingly, the Schwarzchild metric is such an exception, and the late time tail decays as $t^{-(2l+3)}$, with no $\log t$ factor, as is well known.

Discontinuous jumps are also observed for the power-law potentials at these values of α, and the change in behavior is more pronounced. The generic late time behavior for the power-law potentials is $t^{-\mu}$ with $\mu \simeq 2l+\alpha$, but when α goes through an odd integer less than $2l+3$, μ jumps to a larger value $\simeq 2l+2\alpha-2$. One such discontinuous jump is demonstrated in Fig. 7 with $l=1$ and $\alpha = 2.9, 3.0$ and 3.1.

These results and other cases studied but not shown here are summarized in Table 1. In the following sections, we present an analytic treatment for the various different potentials. The results so obtained will be shown to agree very well with the numerical evolutions plotted here.

IV. POTENTIAL DECREASING FASTER THAN X^{-2}

The differential equation $\tilde{D}(\omega)g(\omega,x) = 0$, together with the boundary condition at $x = +\infty$, leads to the exact integral equation

$$g(\omega,x) = e^{i\omega x} - \int_x^\infty dx' \, \frac{\sin\omega(x-x')}{\omega} V(x')g(\omega,x') \ . \tag{4.1}$$

In Born approximation, the factor $g(\omega,x')$ in the integrand may be replaced by $e^{i\omega x'}$, so (4.1) can be reduced to

$$g(\omega,x) = e^{i\omega x} - I(\omega,x) \ , \tag{4.2}$$

where

$$I(\omega,x) = \int_x^\infty dx' \, \frac{\sin\omega(x-x')}{\omega} V(x')e^{i\omega x'} \ . \tag{4.3}$$

Note that if $V(x)$ is of finite range, then there can be no singularities in ω.

Suppose the potential decreases like an exponential, i.e., $V(x) \propto e^{-\lambda x}$, then we have

$$I(\omega,x) \propto \frac{e^{(-\lambda+i\omega)x}}{\lambda(\lambda-2i\omega)} \ , \tag{4.4}$$

implying a pole at $\omega = -i\lambda/2$ on the $-$ Im ω axis. Heuristically, if the potential decreases faster than any exponential, then $\lambda \to \infty$ and the pole effectively disappears, resulting in an analytic $g(\omega,x)$. As a result, there is no tail contribution to the Green's function. For the exponential potential, $g(\omega,x)$ can actually be solved exactly without using the Born approximation [17]. The pole structure and the outgoing wave condition can then be seen explicitly [20]. In any event, since these singularities stay away from $\omega = 0$, they will not contribute to the late time behavior.

Next, we consider potentials decreasing less rapidly than an exponential, and in particular $\nu = 0$ and a power-law $\overline{V}$, viz.,

$$V(x) = \frac{K}{x_o^2}\left(\frac{x_o}{x}\right)^{\alpha} \quad . \tag{4.5}$$

We take $\alpha > 2$ and the coefficient K will be absorbed into the scale parameter x_o. When α equals an integer n, we have (Appendix B)

$$\begin{aligned} I(\omega, x) = e^{-i\omega x}\Bigg\{ & \frac{(2i\omega x_o)^{n-2}}{(n-1)!}\left[\gamma - \sum_{m=1}^{n-2}\frac{1}{m} + \log(-2i\omega x)\right] \\ & + \frac{1}{n-1}\left(\frac{x_o}{x}\right)^{n-2} \sum_{m=0, m\neq n-2}^{\infty} \frac{(2i\omega x)^m}{m!(m+2-n)}\Bigg\} \ , \end{aligned} \tag{4.6}$$

where γ is Euler's constant. The appearance of $e^{-i\omega x}$ in I and thus in $g(\omega, x)$ indicates that a right-propagating wave at $x = +\infty$, when extrapolated to finite x, contains a small admixture of left-propagating wave. With reference to ray 3 in Fig. 3, it is clear that this is the term responsible for the late time behavior. The logarithm in (4.6) causes a branch cut along the $-$ Im ω axis (see Fig. 2). To describe the cut, let

$$g_{\pm}(-i\sigma, x) = \lim_{\epsilon \to o^+} g(-i\sigma \pm \epsilon, x)$$

for σ real and positive, and

$$\begin{aligned} \Delta(\sigma, x) &= g_+(-i\sigma, x) - g_-(-i\sigma, x) \\ &= -\lim_{\epsilon \to 0^+}\left[I(-i\sigma + \epsilon, x) - I(-i\sigma - \epsilon, x)\right] \ . \end{aligned} \tag{4.7}$$

Then from (4.2) and (4.6), the strength of the cut is

$$\Delta(\sigma, x) \simeq 2\pi i \frac{(2\sigma x_o)^{n-2}}{(n-1)!} e^{-\sigma x} \quad . \tag{4.8}$$

For non-integral α, we have (Appendix B)

$$I(\omega, x) = \frac{-e^{-i\omega x}}{(\alpha - 1)}\left[(-2i\omega x_o)^{\alpha-2}\Gamma(2-\alpha) - \left(\frac{x_o}{x}\right)^{\alpha-2}\sum_{m=0}^{\infty}\frac{(2i\omega x)^m}{m!(m+2-\alpha)}\right] \ . \tag{4.9}$$

In this case, the term $(-2i\omega x_o)^{\alpha-2}$ causes a cut, again along the $-$ Im ω axis. The strength of the cut is

$$\Delta(\sigma,x) \simeq 2\pi i \frac{(2\sigma x_o)^{\alpha-2}}{\Gamma(\alpha)} e^{-\sigma x} \quad , \tag{4.10}$$

which reduces to (4.8) when $\alpha \to n$. In this sense, the result for non-integral α is more general.

The Green's function $\tilde{G}(x,y;\omega)$ as given by (2.8) will likewise have a cut along the $-\operatorname{Im}\omega$ axis. The values on either side are given by

$$\tilde{G}_\pm(x,y;\omega) = \frac{f(\omega,y)g_\pm(\omega,x)}{W(g_\pm,f)} \quad , \tag{4.11}$$

where henceforth we take $y < x$. A little manipulation gives the discontinuity

$$\begin{aligned} \Delta\tilde{G}(x,y;\sigma) &\equiv \tilde{G}_+(x,y;\omega) - \tilde{G}_-(x,y;\omega) \,|_{\omega=-i\sigma} \\ &= \frac{W(g_-,g_+)}{W(g_+,f)W(g_-,f)} f(\omega,x)f(\omega,y) \,|_{\omega=-i\sigma} \quad . \end{aligned} \tag{4.12}$$

From (4.2) and (4.9)

$$W(g_-,g_+)|_{\omega=-i\sigma} \simeq -\frac{2\pi i}{x_o}\frac{(2\sigma x_o)^{\alpha-1}}{\Gamma(\alpha)} \ . \tag{4.13}$$

Since $W(g_-,g_+)$ is independent of x, in order to derive (4.13) it is only necessary to know $g_\pm$ at *one* value of x. For a potential decreasing faster than x^{-2}, the Born approximation is valid for very large x, and we have used the resultant (4.9) only in that region. Nowhere else do we rely on the Born approximation. Therefore, these results are *exact*, and do not suffer from any inaccuracies arising from the use of the first Born approximation. In other words, the $t \to \infty$ behavior relates only to scattering at $x' \to \infty$, for which the first Born approximation is accurate, provided $\alpha > 2$. It remains to evaluate $W(g_\pm, f)$. To be specific, consider the half-line problem and evaluate the Wronskian at $x = 0$,

$$W(g_\pm,f) = g_\pm(\omega, x=0) \ . \tag{4.14}$$

Substituting (4.13) and (4.14) in (4.12), we have

$$\Delta\tilde{G}(x,y;\sigma) \simeq -\frac{2\pi i}{x_o}\frac{(2\sigma x_o)^{\alpha-1}}{\Gamma(\alpha)}\frac{f(-i\sigma,x)f(-i\sigma,y)}{g_+(-i\sigma,0)g_-(-i\sigma,0)} \ . \tag{4.15}$$

The contribution arising from the cut along the $-\ \mathrm{Im}\ \omega$ axis is

$$G_L(x,y;t) = -i\int_0^\infty \frac{d\sigma}{2\pi}\Delta\tilde{G}(x,y;-i\sigma)e^{-\sigma t}\ . \tag{4.16}$$

The $t\to\infty$ behavior is controlled by the $\sigma\to 0$ behavior of (4.15). A little arithmetic then leads to

$$G_L(x,y;t) \simeq -\frac{2^{\alpha-1}x_o^{\alpha-2}}{t^\alpha}\frac{f(0,x)f(0,y)}{g(0,0)^2}\ . \tag{4.17}$$

Let the initial conditions be $\phi(y,0) = \phi_o(y)$, $\partial_t\phi(y,0) = \phi_1(y)$, and for simplicity we assume ϕ_o, ϕ_1 to have compact support, or at least to vanish at infinity faster than any exponential. Then the late time behavior is

$$\phi(x,t) \simeq -\frac{2^{\alpha-1}x_o^{\alpha-2}}{t^\alpha}\left[\int_0^\infty dy f(0,y)\phi_1(y)\right]\frac{f(0,x)}{g(0,0)^2}\ . \tag{4.18}$$

The term due to ϕ_o is proportional to $\partial_t G$; hence it goes as $t^{-(\alpha+1)}$, and becomes subdominant when $t\to\infty$.

To compare the theoretical result (4.18) with the numerical evolutions in Section III, we obtain the constant $g(0,0)$ and the spatial function $f(0,x)$ by integrating the time-independent equation $\tilde{D}(\omega\to 0)f = \tilde{D}(\omega\to 0)g = 0$. The two results agree very well when $\log t$ is large (see Fig. 4). In Fig. 8, we plot ϕ versus x at a time t_L when the $t^{-\alpha}$ behavior is observed, showing excellent agreement also in the x-dependence of ϕ at late times.

V. INVERSE SQUARE LAW POTENTIAL

In this Section we consider inverse-square-law potentials ($\overline{V} = 0$):

$$V(x) = \frac{\nu(\nu+1)}{x^2} \quad , \quad x > x_c\ . \tag{5.1}$$

We assume the potential is repulsive and take $\nu > 0$, and define $\rho \equiv \nu + 1/2$. For $V \propto x^{-2}$, the Born approximation is not accurate at *any* x (unless the coefficient of the potential is small), so the derivation in Section IV is not valid. However, in this case, an explicit solution can be written down. The solution satisfying the outgoing wave boundary condition as $x \to +\infty$ is exactly

$$g(\omega, x) = e^{i(\rho+\frac{1}{2})\frac{\pi}{2}}\sqrt{\frac{\pi\omega x}{2}}H_\rho^{(1)}(\omega x) \ . \tag{5.2}$$

Since $H_\rho^{(1)}(z) = J_\rho(z) + iN_\rho(z)$ and

$$J_\rho(z) \sim \frac{1}{\Gamma(\rho+1)}(\frac{z}{2})^\rho + \dots \quad , \tag{5.3a}$$

$$N_\rho(z) \sim -\frac{\Gamma(\rho)}{\pi}(\frac{z}{2})^{-\rho} + \dots \quad , \tag{5.3b}$$

the origin is a branch point for non-integral ν. The symmetry in the present problem dictates that the cut should be placed along the $-$ Im ω axis, and some algebra leads to

$$g_+(-i\sigma, x) = \sqrt{\frac{i\pi\sigma x}{2}}e^{i\nu\pi/2}[2\cos\rho\pi H_\rho^{(1)}(i\sigma x) + e^{-i\rho\pi}H_\rho^{(2)}(i\sigma x)] \ , \tag{5.4a}$$

$$g_-(-i\sigma, x) = \sqrt{\frac{i\pi\sigma x}{2}}e^{i\nu\pi/2}[e^{-i\rho\pi}H_\rho^{(2)}(i\sigma x)] \ , \tag{5.4b}$$

where the functions $H_\rho^{(1,2)}(i\sigma x)$ have no discontinuity for $\sigma > 0$. Then

$$W(g_-, g_+) \mid_{\omega=-i\sigma} = -4i\sigma \sin\nu\pi \ , \tag{5.5}$$

which vanishes for integral ν. This is readily understood since $V(x) = l(l+1)/x^2$ represents a pure centrifugal barrier, which corresponds to free propagation in three-dimensional space and does not cause any scattering. Note that for $\nu \to 0$, the potential is weak and the Born approximation derived in Section IV should be valid. It can be readily checked that the $\nu \to 0$ limit of (5.5) indeed agrees with the $\alpha \to 2$ limit of (4.13). [For $\alpha = 2$, it is no longer possible to absorb the magnitude K of the potential into the definition of x_o, so (4.13) should be multiplied by $K = \nu(\nu+1) \simeq \nu$ in this case.] This agreement may be regarded as a check on the formalism in Section IV. Thus the dependence $W(g_-, g_+) \propto \sigma^{n-1}$ remains valid for $n = 2$, but the amplitude vanishes.

For $\omega = -i\sigma \to 0$,

$$g_+(-i\sigma,x) \sim -\frac{1}{\sqrt{\pi}}e^{i(3\rho/2+1/4)\pi}\Gamma(\rho)\left(\frac{i\sigma x}{2}\right)^{-\nu} \quad , \tag{5.6a}$$

$$g_-(-i\sigma,x) \sim \frac{1}{\sqrt{\pi}}e^{i(-\rho/2+1/4)\pi}\Gamma(\rho)\left(\frac{i\sigma x}{2}\right)^{-\nu} \quad , \tag{5.6b}$$

for $x > x_c$. Since $g_\pm \propto x^{-\nu}$,

$$W(g_\pm,f) = \left[f'(x_c) + \frac{\nu}{x_c}f(x_c)\right]g_\pm(x_c) \quad , \tag{5.7}$$

and we have evaluated the Wronskian at $x = x_c$. As discussed in Section II, $f(\omega,x)$ is analytic in ω, which implies $f(\omega,x) = \sum_{n=0}^{\infty} f_n(x)\omega^n$. From the Wronskian, it can be seen that $f_0(x) \neq 0$. Thus,

$$\lim_{\omega=-i\sigma\to 0} W(g_+,f)W(g_-,f) = -\left[f_0'(x_c) + \frac{\nu}{x_c}f_0(x_c)\right]^2 \frac{\Gamma(\rho)^2}{\pi}\left(\frac{\sigma a}{2}\right)^{-2\nu} \quad . \tag{5.8}$$

Using (4.12), we have

$$\Delta\tilde{G}(x,y;\sigma) \sim \frac{4i\pi\sin\nu\pi}{\left[f_0'(x_c) + (\nu/x_c)f_0(x_c)\right]^2}\left(\frac{x_c}{2}\right)^{2\nu}\frac{\sigma^{2\nu+1}}{\Gamma(\nu+\frac{1}{2})^2}f_0(x)f_0(y) \quad , \tag{5.9}$$

and the late time behavior is

$$G_L(x,y;t) \simeq \frac{2\sin\nu\pi}{\left[f_0'(x_c) + (\nu/x_c)f_0(x_c)\right]^2}\frac{\Gamma(2\nu+2)}{\Gamma(\nu+\frac{1}{2})^2}\left(\frac{x_c}{2}\right)^{2\nu} t^{-(2+2\nu)}f_0(x)f_0(y) \quad . \tag{5.10}$$

Two features are worthy of remark. First, whereas the lowest order Born approximation in Section IV would have predicted a time-dependence of t^{-2} for a potential $V \sim x^{-2}$, the power is in fact $t^{-(2+2\nu)}$ for non-integral ν. For $\nu \ll 1$, the two agree, whereas for $\nu = O(1)$, the difference arises from higher-order Born approximation. Secondly, the power-law dependence in t vanishes for integral ν and the late time behavior is then dominated by the exponentially decaying QNM's.

The late time behavior is

$$\phi(x,t) \sim \frac{2\sin(\nu\pi)}{\left[f_0'(x_c) + \nu f_0(x_c)/x_c\right]^2}\frac{\Gamma(2\nu+2)}{\Gamma(\nu+\frac{1}{2})^2}\left(\frac{x_c}{2}\right)^{2\nu} t^{-(2+2\nu)}f_0(x)\int_0^\infty dy f_0(y)\phi_1(y) \quad . \tag{5.11}$$

This analytical result for $\phi(x,t)$ is evaluated and plotted as the dashed line for $\nu = 1.01$ in Fig. 5. The dashed line is indistinguishable from the solid line (obtained from numerical evolution) when $\log t \geq 8$. On the other hand, for $\nu = 1$, ϕ decreases much faster than $t^{-(2+2\nu)}$, in fact faster than any power law, as derived above.

VI. COMPOSITE POTENTIAL

We now consider potentials of the following form:

$$V(x) = \frac{l(l+1)}{x^2} + \overline{V}(x), \qquad x > x_c \ , \qquad\qquad l = 1, 2, \ldots \tag{6.1}$$

with $\overline{V}(x)$ given in (3.2). We first take $\alpha > 2$ to be non-integer, and results for integral α will be obtained by taking the limit $\alpha \to n$, where n is a integer. For $l = 0$ and $\beta = 0$, $V(x)$ reduces to the case discussed in Section IV.

We see from Section V that if V is treated as a perturbation, the lowest order Born approximation no longer gives correct results. Rather, we should consider $\overline{V}(x)$ as the perturbation. The solution for the inverse square potential, $l(l+1)/x^2$, is given exactly by

$$g^{(0)}(\omega, x) \equiv i^{l+1}(\omega x) h_l^{(1)}(\omega x) \ \ , \tag{6.2}$$

where $h_l^{(1)}$ is the spherical Hankel function of the first kind. Applying the Born approximation now gives, in analogy to (4.2),

$$g(\omega, x) \simeq g^{(0)}(\omega, x) - I(\omega, x) \ \ , \tag{6.3}$$

where

$$I(\omega, x) \equiv \int_x^\infty dx' M(x, x'; \omega) \overline{V}(x') g^{(0)}(\omega, x') \ \ , \tag{6.4}$$

with the zeroth-order Green's function being

$$M(x, x'; \omega) = \frac{i\omega}{2} x x' \left[h_l^{(1)}(\omega x') h_l^{(2)}(\omega x) - h_l^{(1)}(\omega x) h_l^{(2)}(\omega x') \right] \ \ , \tag{6.5}$$

and $h_l^{(2)}$ is the spherical Hankel function of the second kind.

First consider a power-law $\overline{V}$. After some algebra, we get

$$I = \frac{-i^l \omega x}{2} \left[I_1(\omega, x) h_l^{(1)}(\omega x) - I_2(\omega, x) h_l^{(2)}(\omega x) \right] \ \ , \tag{6.6}$$

where

$$I_1(\omega,x) = (\omega x_o)^{\alpha-2} \int_{\omega x}^{\infty} du \ \frac{h_l^{(1)}(u) h_l^{(2)}(u)}{u^{\alpha-2}} \quad , \tag{6.7}$$

$$I_2(\omega,x) = (\omega x_o)^{\alpha-2} \int_{\omega x}^{\infty} du \ \frac{h_l^{(1)}(u)^2}{u^{\alpha-2}} \quad . \tag{6.8}$$

While I_1 is straightforwardly evaluated to be

$$I_1(\omega,x) = 2^{-2l} (\omega x)^{-2l-1} \left(\frac{x_o}{x}\right)^{\alpha-2} \sum_{m=0}^{l} a_m(l,\alpha)(\omega x)^{2m} \ , \tag{6.9}$$

I_2 is somewhat more difficult to obtain. We find

$$\begin{aligned} I_2(\omega,x) = & (-1)^{l+1} 2iC(l,\alpha) \frac{\Gamma(2-\alpha)}{(\alpha-1)} (-2i\omega x_o)^{\alpha-2} \\ & +(-1)^l (\omega x)^{-2l-1} \left(\frac{x_o}{x}\right)^{\alpha-2} \sum_{m=0}^{\infty} b_m(l,\alpha)(\omega x)^m \quad , \end{aligned} \tag{6.10}$$

where

$$C(l,\alpha) = \prod_{j=0}^{l-1} \frac{\alpha-2j-3}{\alpha+1+2j} \quad , \qquad l = 1,2,\ldots \qquad , \tag{6.11}$$

and $C(l,\alpha) = 1$ for $l = 0$. [See Appendix C for the derivation of (6.9) and (6.10) and the definitions of a_m and b_m.] The term $(-2i\omega x_o)^{\alpha-2}$ causes a cut in $g(\omega,x)$, which lies along the negative imaginary axis, with a strength

$$\Delta(\sigma,x) \simeq \pi(-i)^{l+1} (2\sigma x_o)^{\alpha-2} \frac{C(l,\alpha)}{\Gamma(\alpha)} (2\sigma x) h_l^{(2)}(-i\sigma x) \quad . \tag{6.12}$$

Thus the factor $W(g_-,g_+)$ is given by

$$W(g_-,g_+)|_{\omega=-i\sigma} \simeq W(g^{(0)},\Delta) = -4\pi i\sigma (2\sigma x_o)^{\alpha-2} \frac{C(l,\alpha)}{\Gamma(\alpha)} \quad . \tag{6.13}$$

In the case of α being an integer $n \geq 3$, we get

$$W(g_-,g_+) = -4\pi\sigma (2\sigma x_o)^{n-2} \frac{C(l,n)}{(n-1)!} \quad . \tag{6.14}$$

Approximate $W(g_\pm, f)$ by $W(g^{(0)}, f)$, and we have

$$\lim_{\sigma\to 0} W(g_+,f) W(g_-,f) = \sigma^{-2l} g_o^2 \quad ,$$

where $g_o \equiv \lim_{\omega\to 0}[(i\omega)^l W(g^0, f)]$ is a non-vanishing constant and reduces to $g(0,0)$ for $l = 0$. Thus,

$$\Delta\tilde{G}(x,y;-i\sigma) \simeq -\pi i 2^\alpha x_o^{\alpha-2}\sigma^{2l+\alpha-1}\frac{C(l,\alpha)}{\Gamma(\alpha)}\frac{f(-i\sigma,x)f(-i\sigma,y)}{g_o^2} \quad , \tag{6.15}$$

and as a result,

$$\begin{aligned} G_L(x,y;t) &\simeq -\frac{f(0,x)f(0,y)}{g_o^2}C(l,\alpha)2^\alpha x_o^{\alpha-2}\frac{\Gamma(2l+\alpha)}{\Gamma(\alpha)}t^{-(2l+\alpha)} \\ &\equiv -\frac{f(0,x)f(0,y)}{g_o^2}C(l,\alpha)\frac{F(\alpha)}{t^{2l+\alpha}} \end{aligned} \tag{6.16}$$

for both integral and non-integral α.

We note the interesting result that $C(l,\alpha) = 0$ when α equals an odd integer less than $2l+3$, in which case the late time tail vanishes in first Born approximation, and higher order corrections have to be considered. This implies multiple scatterings from asymptotically far regions are important in such cases. We shall not go into the details of the higher order calculations, but from dimensional analysis, we expect

$$G_L(x,y;t) \sim t^{-(2l+\alpha)}\left[A_1 + A_2\left(\frac{x_o}{t}\right)^{\alpha-2} + A_3\left(\frac{x_o}{t}\right)^{2(\alpha-2)} + \cdots\right] \quad , \tag{6.17}$$

where A_i 's are constants proportional to the $i^{\rm th}$-order scattering amplitude.

Generically the late time behavior arising from the first Born approximation is linear in the potential. The potential with a logarithmic $\overline{V}$ is obtained by taking $(-\partial/\partial\alpha)$ on that with a power-law $\overline{V}$. So we immediately obtain from (6.16) that for logarithmic $\overline{V}$

$$G_L(x,y;t) = \left\{C(l,\alpha)F(\alpha)\frac{\log t}{t^{2l+\alpha}} - \frac{\partial}{\partial\alpha}\left[C(l,\alpha)F(\alpha)\right]\frac{1}{t^{2l+\alpha}}\right\}\frac{f(0,x)f(0,y)}{g_o^2} \quad . \tag{6.18}$$

Hence, the leading terms are $t^{-(2l+\alpha)}(c\log t + d)$, except that $c \propto C(l,\alpha)$ vanishes when α is an odd integer less than $2l+3$.

Analytical results obtained from (6.16) and (6.18) with *no* adjustable parameters [except in case (ii) in Fig. 7; see below] are plotted as dashed lines in Figs. 6 and 7. The agreement between numerical evolutions and analytical results is perfect. For case (iii) in Fig. 6, the vanishing of the leading term implies that the asymptotic slope should be zero (i.e., no log t, but only a pure power, whose magnitude is determined), and this indeed agrees with the numerical results, with no adjustable parameters. For case (ii) in Fig. 7, the leading term vanishes, and the dashed line represents the next leading term arising from multiple scatterings, whose time dependence is determined, but whose magnitude has been left as an adjustable normalization. In Fig. 9, we show the dependence of ϕ on x at late times; perfect agreement is again found.

VII. CONCLUSION

In this paper we have studied the dynamical evolution of waves on a curved background that can be described by the Klein-Gordon equation (1.1) with an effective potential. This includes linearized scalar, electromagnetic and gravitational waves on a Schwarzschild background as special cases. Our work extends, places in context and provides understanding for the known results for the Schwarzschild spacetime.

We have given a systematic treatment of waves described by (1.1) using a Green's function formulation. The Green's function consists of three distinct components, G_L, G_Q, and G_P, coming from the three contributions to the contour integral in frequency space. Each component leads to a distinct wave phenomenon in physical spacetime. G_Q leads to quasinormal ringing, G_P leads to prompt responses, and G_L leads to the late time tail phenomenon. Fig. 1 shows clearly these three contributions, while Fig. 3 provides a heuristic understanding in terms of wave propagation in a spacetime diagram.

We further focus on the late time tail phenomenon. Both numerical and analytic results, which agree perfectly with each other, are presented. For a broad class of potentials, we have demonstrated that the tail is related to the Green's function having a branch cut along the $-\,\mathrm{Im}\,\omega$ axis, with the asymptotic late time behavior controlled by the tip of the cut. Using a Born analysis, we determine the strength of the cut in terms of the spatial asymptotics of the potential. This gives analytic understanding of both the magnitude and the time dependence of the late time tail, not just for the Schwarzschild case, but also for a wider class of models. We have applied our formulation to a few interesting cases, including the inverse-square-law potential, other power-law potentials decaying faster than inverse square, potentials containing a logarithmic term, and composite potentials with a centrifugal barrier.

We find that the well-known late time power law behavior of waves on a Schwarzschild spacetime turns out to be an exceptional case. In general, for potentials going as a centrifugal barrier of angular momentum l plus $\overline{V}(x) \sim x^{-\alpha} \log x$, the generic late time behavior is $\sim t^{-(2l+\alpha)} \log t$, which is not a simple power law. However, when α is an odd integer less than $2l+3$, as in the case of Schwarzschild, the coefficient of the leading term vanishes, and the next leading term is $t^{-(2l+\alpha)}$ without a $\log t$ factor. This discontinuous dependence of the time dependence on α is verified numerically and shown in Fig. 6.

There are various implications, colloraries and interesting future extensions discussed throughout the paper. Appendix A outlines a subtlety in the numerical calculations of the late time tail behavior (or more generally speaking the accurate time dependence of solutions of partial differential equations) in the timelike direction.

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APPENDIX A: GHOST POTENTIAL

In this Appendix we consider a subtlety in the numerical solution of (1.1), which we write as

$$\partial_t^2 \phi = L[\phi] \equiv \partial_x^2 \phi - V\phi \,. \tag{A1}$$

The numerical solution does *not* satisfy (A1), but instead satisfies a modified equation with L replaced by

$$\widetilde{L}[\phi] \equiv D_x^2 \phi - V\phi \,, \tag{A2}$$

where D_x^2 is a finite-difference approximation to ∂_x^2, e.g., in a second-order scheme,

$$\begin{aligned} D_x^2 \phi(x,t) &= \frac{[\phi(x+\Delta x,t) - 2\phi(x,t) + \phi(x-\Delta x,t)]}{(\Delta x)^2} \\ &\simeq \partial_x^2 \phi + \frac{1}{12}(\Delta x)^2 \partial_x^4 \phi + \cdots \,. \end{aligned} \tag{A3}$$

For the class of problems at hand, the solution takes the form

$$\phi(x,t) = t^{-\mu} f(0,x) + \cdots \,, \tag{A4}$$

where μ is some exponent, $f(\omega, x)$ is the solution regular at the origin, and the omitted terms are subasymptotic as $t \to \infty$ at fixed x; see e.g. (6.16). (The convergence is not uniform in x, and moreover, a $\log t$ factor can be incorporated without changing the argument below.) To lowest order in Δx, this satisfies (A1), and noticing that $\partial_t^2 \phi$ is down by a factor t^{-2}, we see that $\partial_x^2 \phi \simeq V\phi + \cdots$. This allows us to eliminate all spatial derivatives on ϕ higher than second order, and a little arithmetic leads to

$$\widetilde{L}[\phi] \;\simeq\; L[\phi] + \frac{1}{12}(\Delta x)^2 \left[(V'' + V^2)\phi + 2V'\phi' \right] + \cdots \,, \tag{A5}$$

where $'$ denotes ∂_x.

Two further simplifications are possible. First, because of (A4), the $\partial_t^2 \phi$ term is subasymptotic as $t \to \infty$, and consequently the error caused by finite differencing in t can be ignored, at least to order $(\Delta t)^2$. Secondly, assume that the potential is dominated at large x by the centrifugal barrier, then

$$f(0,x) \simeq K x^{l+1} \tag{A6}$$

[The function $f(0,x)$ corresponds to the radial solution to Laplace's equation, and is regular at the origin. In general, it contains both the decreasing and the increasing component. At large x, the latter dominates, and (A6) then follows. The behavior (A6) has been verified numerically.] As a result, $\phi' \simeq [(l+1)/x]\phi$, and it is possible to write the extra terms in (A5) purely in terms of ϕ without ϕ':

$$\tilde{L}\,[\phi] \;\simeq\; L\,[\phi] + V_{gh}(x)\phi + \cdots\ , \tag{A7}$$

where the ghost potential is

$$V_{gh}(x) = \frac{1}{12}(\Delta x)^2\left[V'' + V^2 + \frac{2(l+1)}{x}V'\right] \propto (\Delta x)^2 x^{-4}\,. \tag{A8}$$

Thus if $\overline{V}$ vanishes at infinity faster than x^{-4}, say as $x^{-\alpha}$ $(\alpha > 4)$, then instead of seeing the true late time behavior $t^{-(2l+\alpha)}$, the numerical solution based on second-order spatial differencing would display a late time behavior $t^{-(2l+4)}$ due to the ghost potential. It is easy to be misled by the ghost behavior unless one has analytic expectations to compare with.

As an even clearer example, consider a pure centrifugal barrier, say with $l = 1$. The late time behavior should be exponential (since there should be no cuts along the $-$ Im ω axis, and the leading large t contribution comes from the quasinormal modes). The top line in Fig. 10 shows the result of a numerical calculation with second-order spatial differencing; the straight line portion gives a dependence t^{-6}, which is the ghost behavior as explained above. The lower line in the same figure shows the result when Δx is halved; the t dependence remains unchanged, but the magnitude is decreased, roughly by a factor of 4, as expected. The change with Δx shows that this tail is a numerical artifact.

In short, k-th order spatial differencing leads to

$$V_{gh}(x) \propto (\Delta x)^k x^{-(2+k)}\quad , \tag{A9}$$

which could mask the effect of $\overline{V}$ if the latter decreases as $x \to \infty$ more rapidly. This problem could have led to errors, or at least ambiguities, in some numerical experiments. In all the numerical results presented in this paper, we have taken care to use spatial differencing of a sufficiently high order that V_{gh} will not dominate over $\overline{V}$, and have moreover checked that the $t \to \infty$ asymptotics claimed are stable against (a) decreasing Δx, and (b) increasing the order of spatial differencing.

APPENDIX B

For a power-law potential of the form (4.5) with $K = 1$, using (4.3), we have

$$I(\omega, x) = \int_x^\infty dx' \frac{\sin \omega(x - x')}{\omega} \frac{x_o^{\alpha-2}}{x^\alpha} e^{-i\omega x'} \quad , \tag{B1}$$

which is easily shown to be:

$$I(\omega, x) = \frac{(\omega x_o)^{\alpha-2}}{2i} \left\{ \left[\int_{\omega x}^\infty \frac{du}{u^\alpha} \right] e^{i\omega x} - \left[\int_{\omega x}^\infty \frac{du e^{2iu}}{u^\alpha} \right] e^{-i\omega x} \right\} \quad . \tag{B2}$$

After evaluating the first integral and integrating the second one by parts, we get

$$I(\omega, x) = -e^{-i\omega x} \frac{(-2i\omega x_o)^{\alpha-2}}{(\alpha - 1)} J(\omega x) \quad , \tag{B3}$$

where

$$J(\omega x) \equiv \int_{-2i\omega x}^\infty du \ \frac{e^{-u}}{u^{\alpha-1}} \ .$$

If α is an integer n, we have [21]:

$$J(\omega x) = \frac{(-1)^{n-1}}{(n-2)!} \left[\gamma - \sum_{m=1}^{n-2} \frac{1}{m} + \log(-2i\omega x) \right] + \frac{(-1)^{n-1}}{(2i\omega x)^{n-2}} \sum_{m=0\, m \neq n-2}^{\infty} \frac{(2i\omega x)^m}{m!(m+2-n)!} \ , \tag{B4}$$

where γ is the Euler's constant.

On the other hand, if α is non-integer, we have [21]

$$J(\omega x) = \Gamma(2 - \alpha) + \frac{(-1)^{\alpha-1}}{(2i\omega x)^{\alpha-2}} \sum_{m=0}^{\infty} \frac{(2i\omega x)^m}{m!(m+2-\alpha)} \quad . \tag{B5}$$

Substituting (B4) and (B5) into (B3) then gives us (4.6) and (4.9) respectively.

APPENDIX C

The spherical Hankel functions of the first and second kinds can be written as [22]:

$$h_l^{(1)}(u) = i^{-l-1}u^{-1}e^{iu}\sum_{m=0}^{l}\frac{(l+m)!}{m!(l-m)!}(-2iu)^{-m} \quad , \tag{C1}$$

and

$$h_l^{(2)}(u) = i^{l+1}u^{-1}e^{-iu}\sum_{m=0}^{l}\frac{(l+m)!}{m!(l-m)!}(2iu)^{-m} \quad . \tag{C2}$$

Using (C1) and (C2), I_1 as defined in (6.9) is easily evaluated to be:

$$\begin{aligned} &I_1(\omega, x) \\ &= \frac{2^{-2l}}{(\omega x)^{2l+1}}\left(\frac{x_o}{x}\right)^{\alpha-2}\sum_{m_1,m_2=0}^{l}\frac{(2i)^{m_1+m_2}\left[(-1)^{m_1}+(-1)^{m_2}\right](2l-m_1)!(2l-m_2)!}{2(2l+\alpha-1-m_1-m_2)(l-m_1)!(l-m_2)!m_1!m_2!}(\omega x)^{m_1+m_2} \quad . \end{aligned}$$

Now since $(-1)^{m_1}+(-1)^{m_2}$ vanishes when m_1 and m_2 are not both odd or both even, we have only even powers of ωx in the double sum. Hence, we can define the double sum as

$$\begin{aligned} &\sum_{m_1=0}^{l}\sum_{m_2=0}^{l}\frac{(2i)^{m_1+m_2}\left[(-1)^{m_1}+(-1)^{m_2}\right]2(2l-m_1)!(2l-m_2)!}{(2l+\alpha-1-m_1-m_2)(l-m_1)!(l-m_2)!m_1!m_2!}(\omega x)^{m_1+m_2} \\ &\equiv \sum_{m=0}^{l} a_m(l,\alpha)(\omega x)^{2m} \quad , \end{aligned}$$

and get (6.9).

To evaluate I_2, we consider more generally the following integral:

$$\mathcal{I}(\beta, z) \equiv \int_z^\infty \frac{v(t)}{t^\beta}dt \quad , \tag{C3}$$

where $v(t)$ is an analytic function and $\lim_{t\to\infty} v(t)t^n = 0$ for any positive number n. Assume that $v(t=0) \neq 0$, so that the integrand is singular at $t=0$ for $\beta \geq 0$. Let the Taylor expansion of $v(t)$ be $\sum_{n=0}^{\infty} c_n t^n$, which converges for $t \in [0, \infty)$.

Rewrite $\mathcal{I}(\beta, z)$ for $\beta \geq 1$ as follows:

$$\mathcal{I}(\beta, z) = \int_z^\infty \frac{v(t)-\sum_{n=0}^{N-1}c_n t^n}{t^\beta}dt + \int_z^\infty \frac{\sum_{n=0}^{N-1}c_n t^n}{t^\beta} \quad , \tag{C4}$$

where N is the largest positive integer less than β. It is then straightforward to show that

$$\mathcal{I}(\beta, z) = \int_0^\infty \frac{v(t) - \sum_{n=0}^{N-1} c_n t^n}{t^\beta} dt - \sum_{n=0}^{\infty} \frac{c_n z^{n+1-\beta}}{n+1-\beta} \quad , \tag{C5}$$

Note that the integrand on RHS is well behaved at $t = 0$. We shall show that this integral is, in fact, equal to

$$\mathcal{I}_A(\beta) \equiv \int_0^\infty \frac{v(t)}{t^\beta} dt \quad ,$$

where the method of analytic continuation has applied to define it for $\beta \geq 1$.

First, by considering the muliti-valued property of the function t^α across the branch cut going from $t = 0$ to $t = \infty$, one can argue that for noninteger β

$$\mathcal{I}_A(\beta) = \frac{e^{\pi\beta i}}{2i \sin \beta\pi} \int_C \frac{v(t)}{t^\beta} dt \quad . \tag{C6}$$

The integral goes along a contour C enclosing the positive real t-axis in the clockwise direction as shown in Fig. 11. This representation has the advantage of being analytic in β for both positive and negative values of β and is readily applicable as an analytic continuation of the original integral.

Second, by using a similar trick, one can also prove that

$$\int_0^\infty \frac{v(t) - \sum_{n=0}^{N-1} c_n t^n}{t^\beta} dt = \frac{e^{\pi\beta i}}{2i \sin \beta\pi} \int_C \frac{v(t) - \sum_{n=0}^{N-1} c_n t^n}{t^\beta} dt \quad . \tag{C7}$$

Moreover, the integral

$$\int_C \frac{\sum_{n=0}^{N-1} c_n t^n}{t^\beta} dt$$

vanishes, which can be seen by deforming the contour C into a infinitely large circle enclosing the origin, as shown in Fig. 11. This then completes our proof for the equality

$$\mathcal{I}_A(\beta) = \int_0^\infty \frac{v(t) - \sum_{n=0}^{N-1} c_n t^n}{t^\beta} dt \quad , \tag{C8}$$

and hence

$$\mathcal{I}(\beta, z) = \mathcal{I}_A(\beta) - \sum_{n=0}^{\infty} \frac{c_n z^{n+1-\beta}}{n+1-\beta} \quad . \tag{C9}$$

From (6.8),

$$I_2(\omega, x) = (\omega x_o)^{\alpha-2} J_2(\omega x) \quad , \tag{C10}$$

where

$$J_2(\omega x) \equiv \int_{\omega x}^{\infty} dt \frac{\left[h_l^{(1)}(t)\right]^2}{t^{\alpha-2}} \quad . \tag{C11}$$

We let

$$w(t) = \left[h_l^{(1)}(t)\right]^2 t^{2l+2} \quad , \tag{C12}$$

so that $w(t)$ is analytic in t. Using (C1), we have

$$\begin{aligned} w(t) &= (-1)^{l+1} \sum_{k_1=0}^{l} \sum_{k_2=0}^{l} \sum_{n=0}^{\infty} (-1)^{k_1+k_2} (2i)^{n-k_1-k_2} \frac{(l+k_1)!(l+k_2)!(-1)^{k_1+k_2}}{k_1!k_2!(l-k_1)!(l-k_2)!} t^{n+2l-k_1-k_2} \\ &\equiv (-1)^{l+1} \sum_{m=0}^{\infty} d_m t^m \quad . \end{aligned} \tag{C13}$$

Then

$$J_2(\omega x) = \int_{\omega x}^{\infty} dt \frac{w(t)}{t^{2l+\alpha}} \quad , \tag{C14}$$

which is of the form of (C3) with $v(t) = w(t)$, $\beta = 2l + \alpha$ and $z = \omega x$. Hence using (C9), we get

$$J_2(\omega x) = \mathcal{I}_A + (-1)^l \sum_{m=0}^{\infty} \frac{d_m (\omega x)^{m+1-2l-\alpha}}{m+1-2l-\alpha} \quad , \tag{C15}$$

Since $\mathcal{I}_A$ is the analytic continuation for the case $\alpha \leq -2l$, it can be looked up in tables [22]:

$$\mathcal{I}_A = \frac{(-1)^{l+1}(-i)^{\alpha-1}}{2^\alpha \pi} \frac{\Gamma(\frac{3-\alpha+2l}{2})\Gamma(\frac{2-\alpha}{2})^2\Gamma(\frac{1-\alpha-2l}{2})}{\Gamma(2-\alpha)} \quad , \tag{C16}$$

After some algebra, we get

$$\mathcal{I}_A = (-1)^l (-2i)^{\alpha-1} C(l,\alpha) \frac{\Gamma(2-\alpha)}{(\alpha-1)} \tag{C17}$$

with $C(l,\alpha)$ defined in (6.11). Substitute (C15) and (C17) into (C10) and after some simplification, we obtain (6.10) with

$$b_m \equiv \frac{d_m}{m+1-2l-\alpha} \quad . \tag{C18}$$

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where Γ and J are the gamma and Bessel functions respectively. The pole at $\omega = -i\lambda/2$ in the Born approximation [see (4.4)] corresponds to the first pole of the gamma function when the argument equals zero. When $x \to \infty$, the argument of the Bessel function becomes small and we have $\Gamma(1-2i\omega/\lambda)J_{-2i\omega/\lambda}(2V_o^{1/2}e^{-\lambda x/2}/\lambda) \simeq (V_o^{1/2}e^{-\lambda x/2}/\lambda)^{-2i\omega/\lambda} = (V_o/\lambda^2)^{-i\omega/\lambda}e^{i\omega x}$. Therefore $g(\omega, x) \sim e^{i\omega x}$ when $x \to \infty$, satisfying the required outgoing-wave condition.

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Table 1

$V(x), \quad x \to \infty$		$\phi(t), \quad t \to \infty$
$(K/x_o^2)(x_o/x)^\alpha$	all real $\alpha > 2$	$t^{-\alpha}$
$\nu(\nu+1)/x^2$	integer ν	$e^{-t/\tau}$
	all non-integer ν	$t^{-(2\nu+2)}$
$l(l+1)/x^2 + (K/x_o^2)(x_o/x)^\alpha$	odd integer $\alpha < 2l+3$	$t^{-\mu}, \quad \mu > 2l+\alpha$
l: integer	all other real α	$t^{-(2l+\alpha)}$
$l(l+1)/x^2 + (K/x_o^2)(x_o/x)^\alpha \log(x/x_o)$	odd integer $< 2l+3$	$t^{-(2l+\alpha)}$
l: integer	all other real α	$t^{-(2l+\alpha)} \log t$

Table 1. Behavior of late time tails for various potentials

FIGURE CAPTIONS

Fig. 1 Generic time dependence of $\phi(x_1, t)$ at a fixed spatial point x_1, evolving from an initial gaussian ϕ and initial $\dot{\phi} = 0$. The time evolution is governed by the Klein-Gordon equation (1.1). The potential in this case is $V(x) \sim l(l+1)/x^2 + (\log x)/x^3$ with $l = 1$. (a) Shorter time scale to show the prompt contributions; (b) longer time scale to show the late time behavior.

Fig. 2 Singularity structure of $\tilde{G}(x, y; \omega)$ in the lower-half ω-plane, and the various contributions to the Green's function $G(x, y; t)$.

Fig. 3 A spacetime diagram illustrating heuristically the different contributions G_L, G_Q and G_P (see text for definition) to the Green's function.

Fig. 4 $\log |\phi(x_1, t)|$ versus $\log t$ for potentials with $\nu = 0$ and a power-law $\overline{V}$, which go as $x_o^{\alpha-2}/x^\alpha$ as $x \to \infty$. (a) (i) $x_o = 1$, $\alpha = 3$; (ii) $x_o = 100$, $\alpha = 3$ and (b) (i) $x_o = 1$, $\alpha = 2.5$; (ii) $x_o = 1$, $\alpha = 5.7$. Solid lines are numerical results, which agree very well with the analytical results (dashed lines).

Fig. 5 $\log |\phi(x_1, t)|$ versus $\log t$ for an inverse-square-law potential $V(x) \sim \nu(\nu+1)/x^2$. (i) $\nu = 1.01$ and (ii) $\nu = 1$. Solid lines are numerical results. For $\nu = 1.01$, ϕ decreases as $t^{-\mu}$ with $\mu \simeq 4.03$ which is in excellent agreement with the analytical result of $t^{-(2+2\nu)}$ (dashed line), derived in Section V. For $\nu = 1$, ϕ decreases faster than any power law, again agreeing with the theoretical prediction. For clarity, case (ii) is shifted downwards by 8.0.

Fig. 6 $A|\phi(x_1, t)|t^{2l+\alpha}$ versus $\log t$ for several potentials $V(x) \sim l(l+1)/x^2 + \log x/x^\alpha$. (i) $l = 0$, $\alpha = 3$; (ii) $l = 1$, $\alpha = 2.9$; (iii) $l = 1$, $\alpha = 3$; (iv) $l = 1$, $\alpha = 3.1$. For clarity, the data are multiplied by a constant A with (i) $A = 10^{-9}$; (ii) and (iii) $A = 5.6 \times 10^{-10}$; (iv) $A = 5.86 \times 10^{-10}$. The sloping straight lines observed at late times in cases (i), (ii) and (iii) indicate the existence of a $\log t$ factor. Case (iv) is an exception, in which the nearly zero asymptotic slope shows that the tail is a simple power law. The numerical evolutions (solid lines) are indistinguishable from the analytical results (dashed lines) for $\log t > 9.5$.

Fig. 7 $\log|\phi(x_1,t)|$ versus $\log t$ for various potentials of the form $2/x^2 + 1/x^\alpha$ as $x \to \infty$. (i) $\alpha = 2.9$; (ii) $\alpha = 3.0$; (iii) $\alpha = 3.1$. To make the three sets of lines stagger, vertical shifts have been applied. The generic late time behavior is $t^{-\mu}$ and $\mu \simeq 2+\alpha$ except in case (ii) in which μ jumps discontinuously to 2α. Solid lines are numerical evolutions while dashed lines are analytical results, derived in Section VI. In cases (i) and (iii), the analytical results are completely specified both in form and magnitude; in case (ii), only the form is known and the magnitude is fitted to the numerical result.

Fig. 8 $A\phi(x,t_L)$ versus x at a fixed time t_L for the same potentials plotted in Fig. 4. The numerical evolutions (solid lines) agree very well with the theoretical estimates (pluses). (i) $x_o = 1$, $\alpha = 3$; (ii) $x_o = 100$, $\alpha = 3$; (iii) $x_o = 1$, $\alpha = 2.5$; (iv) $x_o = 1$, $\alpha = 5.7$. The data are multiplied by a constant A for clarity. (i) $A = 5 \times 10^4$; (ii) $A = 10^6$; (iii) $A = 10^3$; (iv) $A = 2 \times 10^{15}$.

Fig. 9 $A\phi(x,t_L)$ versus x for various potentials of the form $2/x^2 + (c_1 \log x + c_2)/x^\alpha$ as $x \to \infty$. Solid lines are numerical results while the pluses are theoretical estimates. (i) $c_1 = 1, c_2 = 0, \alpha = 3$; (ii) $c_1 = 0, c_2 = 1, \alpha = 3.3$; (iii) $c_1 = 0, c_2 = 1, \alpha = 4$; (iv) $c_1 = 0, c_2 = 1, \alpha = 5$. The data are multiplied by a constant A for clarity. (i) $A = 10^8$; (ii) $A = 10^{10}$; (iii) $A = 10^{12}$; (iv) $A = 10^{15}$.

Fig. 10 Ghost behavior observed in the numerical evolution of $\phi(x_1,t)$ using a second-order spatial differencing scheme. The potential used in this case is $V(x) \sim 2/x^2$ as $x \to \infty$ and the expected late time behavior is an exponential decay as shown in Fig. 5 (which uses a higher-order spatial differencing scheme) and Section V. In the present numerical evolution, however, a straight line portion indicating a time dependence of t^{-6} is observed. The magnitude of this "ghost" tail decreases by a factor of 4 when the spatial interval Δx is halved (see Appendix A), showing that this tail is a numerical artifact.

Fig. 11 Contour C for evaluating the integral $\mathcal{I}_A$ defined in (C6).