Title: Critical curvature of large-N nonlinear $O(N)$ sigma model on S^2

Short title: Critical curvature on S^2

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Abstract

We study the nonlinear $O(N)$ sigma model on S^2 with the gravitational coupling term, by evaluating the effective potential in the large-N limit. It is shown that there is a critical curvature R_c of S^2 for any positive gravitational coupling constant ξ, and the dynamical mass generation takes place only when $R < R_c$. The critical curvature is analytically found as a function of ξ (> 0), which leads us to define a function looking like a natural generalization of Euler-Mascheroni constant.

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1 Introduction

For a model described by boson fields n_i ($n = 1, 2, \cdots, N$) on a curved spacetime, a gravitational interaction term $\xi_i R n_i n_i$ may be added to the Lagrangian [1], where ξ_i is a gravitational coupling constant and R is the scalar curvature of the spacetime (In this paper, we consider only Euclidean spacetime.). Since this interaction term is a bilinear term of the fields, the change of mass through quantum loop corrections may be of interest.

Recently one of us [2] studied the three dimensional nonlinear $O(N)$ sigma model on $S^2 \times R^1$ in the leading order of $1/N$ expansion, where S^2 is the two sphere with the canonical metric. For the $O(N)$ symmetry we assume that ξ_i is the same ξ for all i. On the three dimensional flat spacetime, there exists a critical coupling constant of the model and the dynamical mass generation takes place only when the coupling constant is larger than the critical one (strong coupling regime) [3]. In the strong coupling regime, we showed that there is a critical curvature (radius) of S^2 for some value of ξ. Though the $O(N)$ model can be treated by the $1/N$ expansion in both two and three dimensions, the properties of the models are substantially different. One of them is that the dynamical mass generation takes place in two dimension no matter how small the coupling constant is, as it should be.

In this paper, we will study the two-dimensional $O(N)$ model [4] on S^2 with the gravitational coupling term in the large-N limit, by evaluating the effective potential [5]. We will show that there is a critical curvature $R_c(\xi)$ for every ξ (> 0). As ξ approaches 0, the R_c goes to infinity and decreases monotonically as ξ increases. By introducing a function which looks like a functional generalization of Euler-Mascheroni constant, the R_c can be neatly expressed by the function.

2 Effective potential and critical curvature

We assume that the metric of S^2 is the canonical one so that the curvature R is $2/\rho^2$, where ρ is the radius of S^2. Then the Lagrangian of our model is written as:

$$L = \int d\theta d\varphi \ \rho^2 \sin\theta + (\rho^2 \sin^2\theta)^{-1}\partial_\varphi n^i \partial_\varphi n^i$$
$$+ \xi R n^2 + \sigma(n^2 - Ng^{-2}) \ ], \quad (1)$$

where g_0 is the bare coupling constant. For the evaluation of the effective potential we can write the Lagrangian density as

$$\mathcal{L} = n^i D n_i - N\sigma/g_0^2. \qquad (2)$$

In (2) D is defined as $\rho^{-2}(\mathbf{L}^2 + 2\xi) + \sigma$, where $\mathbf{L}$ is the quantum mechanical angular momentum operator. By performing the Gaussian functional integral [6], the effective potential per unit volume is given as

$$\frac{V}{N} = -\frac{\sigma}{g_0^2} + \frac{1}{4\pi\rho^2}(\mathrm{Tr}\ln D + C), \qquad (3)$$

where C is a constant which may be determined by requiring $V\mid_{\sigma=0}= 0$. As is well-known, in the leading order of $1/N$ expansion the effective potential in (3) is written in terms of spacetime independent σ:

$$\frac{V}{N} = -\frac{\sigma}{g_0} + \frac{1}{4\pi\rho^2}\sum_{l=0}^{I}(2l+1)\ln\left(1 + \frac{\sigma}{\frac{l(l+1)}{\rho^2} + \frac{2\xi}{\rho^2}}\right), \qquad (4)$$

where we introduce an integer cut-off I for the quantum number of the operator $\mathbf{L}$.

In order to compare effective potential in (4) with that on R^2, as in the previous paper [2], one can use the following formula:

$$\sum_{l=0}^{N} f(l) = \frac{1}{2}f(0) + \int_0^{N+1} f(x)dx + \sum_{l=0}^{N}\int_0^1 f'(x+l)(x-\frac{1}{2})dx - \frac{1}{2}f(N+1),$$

which gives

$$\frac{V}{N} = -\frac{\sigma}{g_0^2} + \frac{1}{2\pi\rho}\int_0^I (x+\frac{1}{2})\ln\left(1 + \frac{\sigma\rho^2}{(x+\frac{1}{2})^2 + 2\xi - \frac{1}{4}}\right)dx + \frac{1}{8\pi\rho^2}\ln\left(1 + \frac{\sigma\rho^2}{2\xi}\right)$$
$$+ \frac{1}{2\pi\rho^2}\sum_{l=0}^{I}\int_0^1 dx(x-\frac{1}{2})\left[\begin{array}{c} \ln\left(1 + \frac{\sigma\rho^2}{(x+l+\frac{1}{2})^2 + 2\xi - \frac{1}{4}}\right) \\ + \frac{2(x+l+\frac{1}{2})^2}{(x+l+\frac{1}{2})^2 + 2\xi - \frac{1}{4} + \sigma\rho^2} - \frac{2(x+l+\frac{1}{2})^2}{(x+l+\frac{1}{2})^2 + 2\xi - \frac{1}{4}} \end{array}\right]. \qquad (5)$$

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Since the integrals in (5) are well defined without infra-red problem for $\xi > 0$, we will restrict our attention to these cases. By defining a cut-off $\Lambda = (I + \frac{1}{2})/\rho$ whose dimension is that of a momentum, we can write the potential as

$$
\begin{aligned}
\frac{V}{N} &= -\frac{\sigma}{g_0^2} + \frac{1}{2\pi} \int_{1/2\rho}^{\Lambda} y \ln(1 + \frac{\sigma}{y^2 + \frac{2\xi - \frac{1}{4}}{\rho^2}}) dy + \frac{1}{8\pi\rho^2} \ln(1 + \frac{\sigma\rho^2}{2\xi}) \\
&\quad + \frac{1}{2\pi\rho^2} \sum_{l=1}^{\infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} dt\, t \left[\begin{array}{l} \ln(1 + \frac{\sigma\rho^2}{(t+l)^2 + 2\xi - \frac{1}{4}}) \\ + \frac{2(t+l)^2}{(t+l)^2 + 2\xi - \frac{1}{4} + \sigma\rho^2} - \frac{2(t+l)^2}{(t+l)^2 + 2\xi - \frac{1}{4}} \end{array} \right] \\
&\quad + O(1/\Lambda) \\
&= -\frac{\sigma}{g_0^2} - \frac{\sigma}{4\pi}(\ln\frac{2\xi + \sigma\rho^2}{\rho^2\Lambda^2} - 1) + \frac{1 - 4\xi}{8\pi\rho^2}\ln(1 + \frac{\sigma\rho^2}{2\xi}) \\
&\quad + \frac{1}{2\pi\rho^2} \sum_{l=1}^{\infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} dt\, t \left[\begin{array}{l} \ln(1 + \frac{\sigma\rho^2}{(t+l)^2 + 2\xi - \frac{1}{4}}) \\ + \frac{2(t+l)^2}{(t+l)^2 + 2\xi - \frac{1}{4} + \sigma\rho^2} - \frac{2(t+l)^2}{(t+l)^2 + 2\xi - \frac{1}{4}} \end{array} \right] \\
&\quad + O(1/\Lambda).
\end{aligned}
\tag{6}
$$

The effective potential in (6) once again confirms the well-known fact that topological change of spacetime does not give rise to new counterterms, and thus we can use the renormalization relation of the model on R^2 [5]

$$
-\frac{1}{g_0^2} = -\frac{1}{g^2} + \frac{1}{4\pi}\ln\frac{M^2}{\Lambda^2},
\tag{7}
$$

where M is the renormalization mass. It is convenient to define σ_0

$$
\sigma_0 = M^2 e^{-4\pi/g^2}
\tag{8}
$$

which denotes the square of the dynamically generated mass on R^2. Making use of the relations (7) and (8), one can find the renormalized effective potential

$$
\begin{aligned}
\frac{V}{N} &= -\frac{\sigma}{4\pi}(\ln\frac{2\xi + \sigma\rho^2}{\sigma_0\rho^2} - 1) + \frac{1 - 4\xi}{8\pi\rho^2}\ln(1 + \frac{\sigma\rho^2}{2\xi}) \\
&\quad + \frac{1}{2\pi\rho^2} \sum_{l=1}^{\infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} dt\, t \left[\begin{array}{l} \ln(1 + \frac{\sigma\rho^2}{(t+l)^2 + 2\xi - \frac{1}{4}}) \\ + \frac{2(t+l)^2}{(t+l)^2 + 2\xi - \frac{1}{4} + \sigma\rho^2} - \frac{2(t+l)^2}{(t+l)^2 + 2\xi - \frac{1}{4}} \end{array} \right].
\end{aligned}
\tag{9}
$$

To have a better understanding of the shape of V in (9), we evaluate the first derivative of V with respect to σ:

$$
\begin{aligned}
\frac{1}{N}\frac{\partial V}{\partial \sigma} \equiv \frac{V'(\sigma)}{N} &= -\frac{1}{4\pi}\ln\frac{2\xi + \sigma\rho^2}{\sigma_0\rho^2} + \frac{1}{8\pi}\frac{1}{2\xi + \sigma\rho^2} \\
&\quad + \frac{1}{2\pi}\sum_{l=1}^{\infty}\int_{-\frac{1}{2}}^{\frac{1}{2}} dt\, t\frac{-(t+l)^2 + 2\xi - \frac{1}{4} + \sigma\rho^2}{[(t+l)^2 + 2\xi - \frac{1}{4} + \sigma\rho^2]^2}.
\end{aligned}
\tag{10}
$$

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As shown in the appendix in detail, one can find a simpler form of the first derivative in (10):

$$\frac{1}{N}\frac{\partial V}{\partial \sigma} = \frac{1}{4\pi}[\ln(\sigma_0\rho^2) + \frac{1}{2\xi + \sigma\rho^2} + 2\gamma(2\xi + \sigma\rho^2 - \frac{1}{4})], \tag{11}$$

where the function γ is defined by

$$\gamma(\beta) = \lim_{N\to\infty}[\sum_{n=1}^{N}\frac{n + \frac{1}{2}}{(n + \frac{1}{2})^2 + \beta} - \frac{1}{2}\ln(N^2 + \beta)]. \tag{12}$$

The γ function looks like a generalization of Euler-Mascheroni constant

$$\gamma_E = \lim_{N\to\infty}[\sum_{n=1}^{N} n^{-1} - \ln N] = 0.5772\cdots.$$

In fact, making use of the formulae e.g. in [7], one can show that $\gamma(0) = \gamma_E - 2 - 2\ln 2$. Furthermore, it is easy to show that $\gamma(\beta)$ $(\beta > -\frac{1}{4})$ is a monotonically decreasing function of β, and it approaches to $-\infty$ as β goes to ∞.

Now one can find that, for a fixed ξ, $V'(\sigma)$ monotonically decreases to $-\infty$ as σ increases. That is, a global stationary point of the effective potential V which denotes the dynamical mass generation appears only when $V'(0) > 0$. Therefore, the critical radius of S^2 is given by

$$\sigma_0\rho_c^2 = \exp(-\frac{1}{2\xi} - 2\gamma(2\xi - \frac{1}{4})). \tag{13}$$

Or, the critical curvature is given by

$$R_c = 2\sigma_0 \exp(\frac{1}{2\xi} + 2\gamma(2\xi - \frac{1}{4})). \tag{14}$$

The critical curvature which is ∞ in the limit $\xi \to 0$ decreases monotonically to 0 as ξ increases to ∞, and the dynamical mass generation takes place only when $R < R_c$ (or $\rho > \rho_c$).

3 Discussion

By evaluating the effective potential (9) in the leading order of $1/N$ expansion, we have shown that, the $O(N)$ nonlinear σ model with the gravitational coupling term

described by the Lagrangian (1), has a critical curvature (given in (13,14)) which decreases as ξ increases.

For the conformal symmetry, ξ must be $\xi_c = \frac{1}{4}[(d-2)/(d-1)]$ in a d-dimensional spacetime. That is, the conformal coupling constant ξ_c is 0 in two dimension and $\frac{1}{8}$ in three dimension. One of the common features of the $O(N)$ nonlinear σ model on S^2 and $S^2 \times R^1$ [2] is that the dynamical mass generation takes place for any finite R in the limit $\xi \to \xi_c$ while there exists a critical curvature for $\xi > \xi_c$.

Appendix

Making use of the identity

$$\int \frac{t[-(t+l)^2 + \beta]}{[(t+l)^2 + \beta]^2} dt = -\frac{l(t+l) + \beta}{(t+l)^2 + \beta} - \frac{1}{2}\ln[(t+l)^2 + \beta]$$
$$\text{for} \quad (t+l)^2 + \beta > 0, \tag{15}$$

one can find the following equality which could be used to find (11) and (12) from (10);

$$\sum_{l=1}^{\infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{t[-(t+l)^2 + x - \frac{1}{4}]}{[(t+l)^2 + x - \frac{1}{4}]^2} dt$$
$$= \frac{1}{4x} + \frac{1}{2}\ln x + \lim_{N \to \infty} \{\sum_{n=1}^{N-1} \frac{n + \frac{1}{2}}{(n + \frac{1}{2})^2 + x - \frac{1}{4}} - \frac{1}{2}\ln[(N + \frac{1}{2})^2 + x - \frac{1}{4}]\}. \tag{16}$$

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