

CONSTRAINTS ON QCD SUM-RULES FROM THE HÖLDER INEQUALITIES

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(July 14, 1995)

Abstract

A new technique based on Hölder's integral inequality is applied to QCD sum-rules to provide fundamental constraints on the sum-rule parameters. These constraints must be satisfied if the sum-rules are to consistently describe integrated physical cross-sections, but these constraints do not require any experimental data and therefore can be applied to any hadronic spectral function. As an illustration of this technique the Laplace sum-rules of the light-quark correlation function for the vector and the axial-vector currents are examined in detail. We find examples of inconsistency between the inequalities and sum-rule parameters used in some previous analyses of the vector and axial-vector channels.

QCD sum-rules [1–4] have demonstrated their utility in numerous theoretical determinations of hadronic properties. In this approach the QCD condensates parametrize non-perturbative aspects of the vacuum, and through the operator-product expansion [5], the

condensates generate power-law corrections to correlation functions of hadronic currents. These power-law contributions are absent in a purely perturbative calculation and are an essential feature of the sum-rules used to determine hadronic properties.

Despite the success of QCD sum-rules, there remain several fundamental issues concerning their application. In particular, the values of the QCD condensates, applicability and implementation of the continuum (duality) hypothesis, and the energy range in which the sum-rules are reliable are significant issues in the use of QCD sum-rules in hadronic physics. One of these issues is well illustrated by the dimension-six quark condensate in (light-quark) vector current correlation functions where estimates differ by factors of 2 or more [1,6–11].

In this paper we will present a method based on Hölder’s integral inequality which provides fundamental constraints on the QCD sum-rules. These constraints must be satisfied if the sum-rules are to consistently describe integrated physical cross-sections. Using this technique, non-trivial information relating the continuum threshold, sum-rule energy scale, and QCD condensate parameter space will be obtained. These constraints then provide insight into the issues concerning the continuum hypothesis and the energy range in which the sum-rules are reliable. Although Schwarz and Hölder inequalities have been studied in connection with lattice gauge theories to demonstrate some general properties of the hadronic spectrum [12–14] they have not previously been applied to the QCD sum-rules.

The Hölder inequality technique will be illustrated by the Laplace sum-rules involving the light quark vector current (related to the ρ) and light quark axial vector current (related to the A_1). These channels have been chosen because they have been extensively studied, particularly for the vector channel where the analysis of the ρ meson has become a paradigm for sum-rule techniques.

Hölder’s inequality [15,16] for integrals defined over a measure $d\mu$ is

$$\left| \int_{t_1}^{t_2} f(t)g(t)d\mu \right| \leq \left(\int_{t_1}^{t_2} |f(t)|^p d\mu \right)^{1/p} \left(\int_{t_1}^{t_2} |g(t)|^q d\mu \right)^{1/q}, \tag{1}$$

$$\frac{1}{p} + \frac{1}{q} = 1; \quad p, q \geq 1 \quad .$$

When $p = q = 2$ the Hölder inequality reduces to the well known Schwarz inequality. The key idea in applying Hölder's inequality to sum-rules is recognizing that for a typical correlation function $\Pi(Q^2)$, $Im \Pi(Q^2)$ is positive because of its relation to physical cross-sections and can thus serve as the measure $d\mu = Im \Pi(t)dt$ in (1).

Laplace sum-rules are also related to $Im \Pi(t)$ through a Borel transform of a dispersion relation

$$\mathcal{R}_k(\tau, s_0) = \int_{t_0}^{s_0} Im \Pi(t) t^k e^{-t\tau} dt \quad , \quad k = \text{integer}, \quad (2)$$

where t_0 is a physical threshold and s_0 is the continuum representing the minimum energy needed for local duality [17]. In the sum-rule method the QCD contributions to $\mathcal{R}_k(\tau, s_0)$ on the left hand side of (2) are used to extract the phenomenological content of $Im \Pi(t)$. Among other issues, the applicability of the QCD continuum hypothesis (duality) used to model $Im \Pi(t)$ above the energy scale s_0 can be examined through the Hölder inequalities.

Returning to (1) with $d\mu = Im \Pi(t)dt$, $f(t) = t^\alpha e^{-at\tau}$, $g(t) = t^\beta e^{-bt\tau}$ and appropriate integration limits we find

$$\mathcal{R}_{\alpha+\beta}(\tau, s_0) \leq \mathcal{R}_{\alpha p}^{1/p}(ap\tau, s_0) \mathcal{R}_{\beta q}^{1/q}(bq\tau, s_0); \quad a + b = 1 \quad . \quad (3)$$

Imposing restrictions that we have the integer values k needed for the sum-rules (2) leads to the following set of inequalities.

$$\mathcal{R}_0[\omega\tau_{min} + (1 - \omega)\tau_{max}, s_0] \leq \mathcal{R}_0^\omega[\tau_{min}, s_0] \mathcal{R}_0^{1-\omega}[\tau_{max}, s_0], \quad (4)$$

$$\mathcal{R}_1[\omega\tau_{min} + (1 - \omega)\tau_{max}, s_0] \leq \mathcal{R}_1^\omega[\tau_{min}, s_0] \mathcal{R}_1^{1-\omega}[\tau_{max}, s_0], \quad (5)$$

$$\mathcal{R}_1\left[\frac{\tau_{min} + \tau_{max}}{2}, s_0\right] \leq \mathcal{R}_2^{1/2}[\tau_{min}, s_0] \mathcal{R}_0^{1/2}[\tau_{max}, s_0], \quad (6)$$

$$\mathcal{R}_1\left[\frac{\tau_{min} + \tau_{max}}{2}, s_0\right] \leq \mathcal{R}_2^{1/2}[\tau_{max}, s_0] \mathcal{R}_0^{1/2}[\tau_{min}, s_0], \quad (7)$$

$$0 \leq \omega \leq 1 \quad ; \quad \tau_{min} \leq \tau_{max} \quad (8)$$

Similar inequalities can be obtained for higher sum-rules with $k \geq 2$. However, as k increases, the leading QCD condensate contributions to the sum-rules begins to depend upon poorly-understood high dimension condensates, so our analysis will concentrate upon (4) and (5)

where $k < 2$. Furthermore, for small $\tau_{max} - \tau_{min}$ (6) and (7) are in principle contained in the first two inequalities. Thus the following ratios reflecting the inequalities (4) and (5) will be used to study the parameter space of the QCD sum-rules.

$$\rho_0 \equiv \frac{\mathcal{R}_0[\omega\tau_{min} + (1-\omega)\tau_{max}, s_0]}{\mathcal{R}_0^\omega[\tau_{min}, s_0]\mathcal{R}_0^{1-\omega}[\tau_{max}, s_0]} \leq 1 \quad (9)$$

$$\rho_1 \equiv \frac{\mathcal{R}_1[\omega\tau_{min} + (1-\omega)\tau_{max}, s_0]}{\mathcal{R}_1^\omega[\tau_{min}, s_0]\mathcal{R}_1^{1-\omega}[\tau_{max}, s_0]} \leq 1 \quad (10)$$

In summary, if the QCD sum-rules are a valid and consistent representation of the integration of $Im\Pi(t)$ in (2) then the sum-rules $\mathcal{R}_k(\tau, s_0)$ must satisfy the above Hölder inequalities.

To analyze the implications of these inequalities, the QCD predictions for the sum-rules are needed. For the light-quark vector and axial¹ currents the results (to two-loops in perturbative corrections, leading order in QCD condensates) are [1,3]

$$8\pi^2\mathcal{R}_0[\tau, s_0] = \frac{1}{\tau}\left(1 + \frac{\alpha(1/\tau)}{\pi}\right)[1 - e^{-s_0\tau}] + C_2 + C_4\langle O_4 \rangle\tau + \frac{1}{2}C_6\langle O_6 \rangle\tau^2 + \frac{1}{3!}C_8\langle O_8 \rangle\tau^3 + \text{higher dimension condensates} \quad (11)$$

$$8\pi^2\mathcal{R}_1[\tau, s_0] = \frac{1}{\tau^2}\left(1 + \frac{\alpha(1/\tau)}{\pi}\right)[1 - (1 + s_0\tau)e^{-s_0\tau}] - C_4\langle O_4 \rangle - C_6\langle O_6 \rangle\tau - \frac{1}{2}C_8\langle O_8 \rangle\tau^2 + \text{higher dimension condensates} \quad (12)$$

$$C_4\langle O_4 \rangle = \frac{\pi}{3}\langle\alpha G^2\rangle + 8\pi^2 m\langle\bar{q}q\rangle \quad \text{vector} \quad (13)$$

$$C_4\langle O_4 \rangle = \frac{\pi}{3}\langle\alpha G^2\rangle - 8\pi^2 m\langle\bar{q}q\rangle \quad \text{axial vector} \quad (14)$$

$$C_6\langle O_6 \rangle = -4\pi^3\frac{224}{81}\alpha(\langle\bar{q}q\rangle)^2 \quad \text{vector} \quad (15)$$

$$C_6\langle O_6 \rangle = 44\pi^3\frac{32}{81}\alpha(\langle\bar{q}q\rangle)^2 \quad \text{axial vector} \quad (16)$$

where the vacuum saturation hypothesis [1] has been used for the dimension-six condensates and the (small) perturbative contribution from quark masses has been neglected. For brevity, we have not explicitly shown the dimension-eight operators and refer instead to the literature [10,18,19]. Finally, although there are no vacuum condensates of dimension two,

¹For the axial current this represents the transverse projection of the correlation function.

the phenomenological possibility of such contributions (apart from the small quark mass corrections) has been suggested in the context of renormalons [20]. Such non-OPE corrections are represented by the constant term C_2 , and their effect will be investigated as part of the sum-rule parameter space.

The gluon condensate is now reasonably well established [10] to lie within the range $\langle \alpha G^2 \rangle = 3 (0.050 \pm 0.015) / \pi \text{ GeV}^4$. However, it has been suggested that the vacuum saturation hypothesis [1], leading to $C_6 \langle O_6 \rangle = -0.06 \text{ GeV}^6$ (vector) and $C_6 \langle O_6 \rangle = \frac{11}{7} 0.06 \text{ GeV}^6$ (axial), underestimates the magnitude of the dimension-six quark condensate by a factor of 2 or more [6–11]. The many dimension-eight operators fall into two distinct classes consisting of operators amenable to estimation through the vacuum saturation hypothesis and fermionic operators consisting of contractions of $\langle \bar{q} D_\mu D_\nu D_\lambda D_\rho D_\omega q \rangle$. These fermionic condensates have been estimated at lower dimension [18] by assuming that quarks have a virtuality of $M^2 \approx 0.3 \text{ GeV}^2$ which replaces each covariant derivative with a mass scale M . Using these ideas we estimate that the non-fermionic condensates dominate $C_8 \langle O_8 \rangle$ leading to a result of $C_8 \langle O_8 \rangle \approx 4 \times 10^{-3} \text{ GeV}^8$. However, the analysis of [9–11] suggests large deviations from this value. This does not necessarily reflect a complete failure of the vacuum saturation hypothesis since even a 10% deviation from vacuum saturation for each individual operator can accumulate through the combination of the many dimension-eight condensates.

To analyze the inequalities (9,10), we restrict the parameter space by performing a local analysis with $\tau_{max} - \tau_{min} = \delta\tau = 0.01 \text{ GeV}^{-2}$ and setting $\Lambda_{MS} = 0.15 \text{ GeV}$. Further decrease and moderate increase in the value of $\delta\tau$ does not affect the conclusions presented below, and the effects of changing Λ_{MS} are negligible. The QCD condensates are then fixed to a particular set of values and the regions of s_0, τ parameter space leading to $\rho_0 < 1$ and $\rho_1 < 1$ for all $0 < \omega < 1$ are determined. The values of the condensates are then varied and the process is repeated.

The results of this analysis for both the vector and axial-vector channels are illustrated in the Figures, corresponding to specific values of the condensates used or determined in

the literature. In each figure the shaded region represents the admissible s_0, τ parameter space where the inequalities are satisfied. The boxed regions in the figures are the values of s_0 and τ range given in the literature. As is evident from the figures there are several cases where the parameters used in a particular sum-rule analysis are inconsistent with the inequalities. To determine whether this inconsistency is significant we have analyzed the sum-rules to determine how the uncertainties inherent to the sum-rule technique (such as truncation of the OPE beyond condensates of a certain dimension) affect the inequalities². This has been done for Figures 1-3 by assuming that the power-law corrections in \mathcal{R}_0 and \mathcal{R}_1 have an intrinsic error of 10% at 1.0 GeV. This is a generous estimate of the uncertainty by comparison with the assumptions of the standard sum-rule error analysis [1–3] which leads to less than a 1% error in the power-law corrections at 1.0 GeV. Our uncertainty is then modelled by a condensate representing the first truncated term in the OPE, and then considering values of the condensate which have a 10% effect in the power law corrections at 1.0 GeV. Clearly other error models could be chosen but for the purpose of this work we wish to emphasize the method based on the inequalities. For Figure 4, an alternative approach for studying the effects of truncation in the OPE will be discussed below.

The details of the individual figures vary, but two common features persist: the existence of a lower bound on the continuum threshold s_0 and an upper bound on the sum-rule energy parameter τ . An interesting feature of the bound on s_0 is that it is generally smaller in the vector channel than in the axial channel, in agreement with the trend observed in phenomenology.

In Fig. 1 the allowed s_0 - τ parameter space for the vector channel is shown for three sets of the condensates. In all three cases the dimension-eight and higher condensates are ignored. The bottom graph corresponds to the standard Shifman-Vainshtein-Zakharov values of the condensates [1,2], the middle graph incorporates a larger value of the gluon condensate,

²We are grateful to the referee for this suggestion.

and the top graph represents twice vacuum saturation for the dimension six-condensate [4]. The square boxes represent the range of values for s_0 and τ used in the literature for the sum-rule analysis corresponding to the condensates ³. The dashed line represents the border of the parameter space when a dimension-two phenomenological contribution of $C_2 = -0.09 \text{ GeV}^2$ is included as suggested by the upper bounds in [20]. The dotted lines represent the border of the parameter space consistent with the inequalities after including the effect of uncertainty in the power-law corrections. For the lower two graphs (standard values) the shaded area (and its extension to the dotted line after including uncertainties) overlaps with a large portion of the boxed region, so the sum-rule analysis [2] is consistent with the inequalities. By contrast, the upper graph has a very small overlap between the shaded region (with its extension to the dotted line) and boxed region so there is a minimal consistency between the inequalities and sum-rule parameters. In general, Fig. 1 shows that the vector channel parameter space consistent with the inequalities lies within the bounds $s_0 > 1.0 \text{ GeV}^2$, $\tau < 1.8 \text{ GeV}^{-2}$ for the standard values and $s_0 > 1.5 \text{ GeV}^2$, $\tau < 1.4 \text{ GeV}^{-2}$ for twice vacuum saturation.

Fig. 2 represents a similar analysis for three sets of condensates in the axial-vector channel. In all cases the dimension-eight and higher condensates are ignored. The bottom two graphs correspond to standard values of the condensates as used in [3] with the middle graph using a slightly larger gluon condensate. The top graph again corresponds to twice vacuum saturation for the dimension-six condensate [4]. All other features are the same as in Fig. 1. We see from Fig. 2 that in all cases the sum-rule analyses of the axial vector channel are inconsistent with the inequalities even when the effect of uncertainties are considered. Fig. 2 shows that the axial-vector channel parameter space consistent with the inequalities lies within the bounds $s_0 > 2.5 \text{ GeV}^2$, $\tau < 1.1 \text{ GeV}^{-2}$ for the standard values and $s_0 > 3.0 \text{ GeV}^2$, $\tau < 0.8 \text{ GeV}^{-2}$ for twice vacuum saturation.

³In analyses where no range was reported for s_0 a range of 0.5 GeV^2 has been assumed.

It is evident from both Fig. 1 and Fig. 2 that the inequalities are insensitive to a reasonable variation in the gluon condensate, but rather sensitive to the value of the dimension-six condensate. The dimension-two phenomenological condensate also has a relatively minor effect on the parameter space consistent with the inequalities.

In Fig. 3 and 4 we repeat the procedure including the effects of higher dimension condensates. The value of dimension eight condensates were determined for both the vector and axial-vector channels by Dominguez and Sòla [9] using finite energy (FESR) and Laplace sum-rules. The dimension-eight condensates were found to improve duality between the experimental data and QCD and their values hint to a possible larger fermionic contribution to $C_8\langle O_8 \rangle$. In Fig. 3, we display our results based on the average values for the higher dimension condensates as given in Table 1 of [9]. As in the other figures, we also show (square boxes) the ranges of values for s_0 and τ employed in this same work [9]. For the vector channel, the shaded area (and its extension to the dotted line after including uncertainties) does not overlap with the box and therefore the values of the condensates are *inconsistent* with the $s_0 - \tau$ region as analysed in Dominguez-Sòla work. The admissible parameter space lies within the bounds $s_0 \geq 2.3 \text{ GeV}^2$ and $\tau \leq 0.9 \text{ GeV}^{-2}$ for the vector channel while $s_0 \geq 0.5 \text{ GeV}^2$ and $\tau \leq 1.3 \text{ GeV}^{-2}$ for the axial channel.

In Fig. 4, we display the results based on the value of the condensates as given in Table 4 of [11] which contains condensates up to dimension sixteen. The shaded region is the $s_0 - \tau$ parameter space consistent with the inequalities using the condensates up to dimension sixteen. Since these values of the condensates are reasonably consistent with [9] for dimension 8 and less, the effect of truncating the OPE can be explicitly studied in this case by omitting the condensates above dimension 8 resulting in a shift of the border of the parameter space to the dotted line. The effect in this case is more significant than the error model considered in the other figures because the contribution of condensates of dimension 10 to 16 is significantly more than 10% at 1.0 GeV. It is interesting that including higher dimension condensates does not necessarily increase the region consistent with the inequalities as evidenced by the axial vector channel. As in the other figures, the boxed

regions show the $s_0 - \tau$ interval used in this same work [11]. Clearly there are regions of $s_0 - \tau$ parameter space used in the analysis [11] which are inconsistent with the inequalities. The admissible parameter space lies within the bounds $s_0 \geq 1.6 \text{ GeV}^2$ and $\tau \leq 0.95 \text{ GeV}^{-2}$ for the vector channel while $s_0 \geq 2.2 \text{ GeV}^2$ and $\tau \leq 1.1 \text{ GeV}^{-2}$ for the axial channel.

The inequalities should be viewed as a test of both the validity of the continuum hypothesis and of the upper bound on τ (lowest energy) beyond which the neglected or unknown effects in the sum-rule become substantial. In general, features of the allowed parameter space that are independent of s_0 (such as the rising vertical sections) represent the upper range on τ for which the sum-rule becomes unreliable, and the lower horizontal portions suggest a failure of the continuum hypothesis regardless of the energy scale τ . It is significant that the bounds on the s_0 and τ parameter space are obtained only by demanding that the sum-rule be consistent with its phenomenological description in terms of an integrated cross-section through $Im \Pi(t)$, leading to the Hölder inequality constraint. This should be contrasted with the conventional approach of determining an upper bound on τ where an assumption on the uncertainties in the power-law corrections is made, and the limit on τ represents an energy at which the uncertainties reach an unacceptable level.

Although the effects of the higher dimension condensates are readily observed in the figures, the inequalities are relatively insensitive to the dimension-four gluon condensate and are virtually independent of the possible (non-OPE) dimension-two contributions represented by C_2 this implies that dimension-two phenomenological terms can be accommodated in the sum-rules without violating the fundamental constraints imposed by the Hölder inequalities.

Using Hölder's integral inequality we have constructed fundamental constraints on the QCD sum-rules that must be satisfied if the sum-rule is consistent with its phenomenological relation to the integral of $Im \Pi(t)$. As an example of the application of this idea, the s_0, τ parameter space satisfying the inequalities was determined for various choices of the condensates appearing in the literature. Except for the original analysis of the vector channel [1,2], the parameters employed in the sum-rule analyses are inconsistent with the inequalities. Including a model for the uncertainties associated with truncation of the OPE does not seem

sufficient to account for this inconsistency.

We view the application of the inequalities as a practical method for determining the energy (τ) range over which the sum-rules are valid. In contrast to conventional approaches which rely upon estimates of the uncertainties inherent in the sum-rules, the inequalities provide a simple and fundamental constraint for studying the reliable energy range of the QCD sum-rules. Although one could devise models where the intrinsic errors associated with the sum-rules are sufficiently large to accommodate violations of the inequalities, we feel that the most conservative approach is to restrict a sum-rule analysis to regions of parameter space where the inequalities are satisfied, rather than relying upon error estimates (perhaps based on prejudice) to enforce consistency of the sum-rules with the Hölder inequalities. Furthermore, rough lower bounds on the continuum threshold can be obtained, a result which is valuable in cases where phenomenological estimates of the continuum are not available.

In conclusion, the Hölder inequalities for the QCD sum-rules provides a useful and fundamental diagnostic for any sum-rule analysis, and we encourage the use of this technique as a valuable consistency check in any sum-rule application.

Acknowledgements: MB and TGS are grateful for the financial support of the Natural Sciences and Engineering Research Council of Canada (NSERC). GO thanks V. Vento for useful discussions.

REFERENCES

- [1] M.A. Shifman, A.I. Vainshtein, V.I. Zakharov, Nucl. Phys. **B147**, 385 (1979).
- [2] M.A. Shifman, A.I. Vainshtein, V.I. Zakharov, Nucl. Phys. **B147**, 448 (1979).
- [3] L.J. Reinders, H. Rubenstein, S. Yazaki, Phys. Rep. **127**, 1 (1985).
- [4] S. Narison, *QCD Spectral Sum-Rules*, Lecture Notes in Physics v. 26 (World Scientific, Singapore, 1989).
- [5] K.G. Wilson, Phys. Rev. **179**, 1499 (1969).
- [6] S.I Eidelman, L. M. Kurdadze, A.I. Vainshtein, Phys. Lett. **B82**, 278 (1979).
- [7] G. Launer, S. Narison, R. Tarrach, Z. Phys. **C26**, 433(1984).
- [8] R.A. Bertlmann, C.A. Dominguez, M. Loewe, M. Perrottet, E. de Rafael, Z. Phys. **C39**, 231 (1988).
- [9] C.A. Dominguez, J. Solà, Z. Phys. **C40**, 63 (1988).
- [10] V. Giménez, J. Bordes, J.A. Peñarrocha, Phys. Lett. **B223**, 251 (1989).
- [11] V. Giménez, J. Bordes, J.A. Peñarrocha, Nucl. Phys. **B357**, 3 (1991).
- [12] F.T. Tomboulis, Phys. Rev. Lett. **50**, 886 (1983).
- [13] D. Weingarten, Phys. Rev. Lett. **51**, 1830 (1983).
- [14] S. Nussinov, Phys. Rev. Lett. **52**, 966 (1983).
- [15] E.F. Beckenbach, R. Bellman, *Inequalities*, (Springer, Berlin, 1961).
- [16] S.K. Berberian, *Measure and Integration*, (MacMillan, New York, 1965).
- [17] R.A. Bertlmann, G. Launer, E. de Rafael, Nucl. Phys. **B250**, 61 (1985).
- [18] S.N. Nikolaev, H.R. Radyushkin, Nucl. Phys. **B213**, 285 (1983).

[19] D.J. Broadhurst, S.C. Generalis, *Phys. Lett.* **165B**, 175 (1985).

[20] C.A. Dominguez, hep-ph/9411331, report-no: UCT-TP-221/94.

FIGURES

FIG. 1. The shaded area represents the region in the $s_0 - \tau$ parameter space for the vector channel consistent with the inequalities. The values of the condensates indicated in the figure are taken from Shifman-Vainshtein-Zakharov [1,2] analysis for the lower two graphs and twice vacuum saturation form [4] for the upper graph. The boxes represent the $s_0 - \tau$ parameter space used in the corresponding sum-rule analysis. The dashed line represents the border of the parameter space when a dimension-two phenomenological contribution [20] is included. The dotted line represents the border of the parameter space after modelling the effect of uncertainties in the power-law corrections.

FIG. 2. Same as in Fig. 1 except for the axial-vector channel. The lower two graphs correspond to the parameters of [3] and the upper graph represents the parameters from [4].

FIG. 3. The shaded area represents the region consistent with the inequalities for both vector channel and axial-vector channels using Dominguez-Solà [9] values for the condensates up to and including dimension eight. The square boxes represent the $s_0 - \tau$ parameter space used in their analysis. Dotted lines represent the effect of uncertainties as in Fig. 1.

FIG. 4. The shaded region is consistent with the inequalities for both vector channel and axial-vector channels using Giménez *et al.* [11] values for the condensates up to dimension sixteen. The square boxes represent the $s_0 - \tau$ parameter space as used in the Giménez *et al.* [11] analysis. As discussed in the text, dotted lines represent the effect of truncation of the OPE at dimension 8.