CERN-TH/95-194
IFUM 511/FT

Three-body relativistic flux tube model from QCD Wilson-loop approach

N. Brambilla, G.M. Prosperi

Dipartimento di Fisica dell'Università, Milano,
INFN, Sezione di Milano, Via Celoria 16, 20133 Milano, Italy

and A. Vairo

Theory Division, CERN, CH-1211 Geneva 23, Switzerland,
INFN, Sezione di Bologna, Via Irnerio 46, 40126 Bologna, Italy

Abstract

First we review the derivation of the relativistic flux tube model for a quark-antiquark system from Wilson area law as we have given in a preceding paper. Then we extend the method to the three-quark case and obtain a Lagrangian corresponding to a star flux tube configuration.

A Hamiltonian can be explicitly constructed as an expansion in $1/m^2$ or in the string tension σ. In the first case it reproduces the Wilson loop three-quark semirelativistic potential; in the second one, very complicated in general, but it reproduces known string models for slowly rotating quarks.

CERN-TH/95-194
IFUM 511/FT
July 1995

1 Introduction

In a recent paper we have shown that, for the quark-antiquark system, neglecting spin-dependent terms, it is possible to derive rigorously the so-called relativistic flux tube model from the Wilson area law in QCD.

To achieve the result, the essential ingredients were the path-integral representation of the Pauli-type propagator we used in the derivation of the semirelativistic potential, the replacement of the $1/m^2$ expansion of the purely kinetic terms by their closed form and the explicit integration over the momenta of the functional integral.

In this letter we want to show how a similar result can be worked out for the three-quark system starting from the corresponding representation for the three-quark propagator we have considered in ref. [1]. The resulting Lagrangian turns out to correspond to a star flux tube configuration. As in the quark-antiquark case, the Hamiltonian can be expressed only in the form of an expansion in $1/m^2$ or in the string tension σ. In the first situation we obtain the Wilson loop semirelativistic potential, but for the spin-dependent part. In the second one we end up with a very complicated result which, however, reduces to the three-quark string model already considered in the literature for low angular momentum.

In Section 2 we review the quark-antiquark case to clarify the procedure and to establish notation; in Sec. 3 we derive the new Lagrangian and in Sec. 4 we obtain the Hamiltonian of the resulting model.

2 Quark-antiquark flux tube model

In QCD the quark-antiquark gauge invariant propagator can be written

$$\begin{aligned} & G(x_1, x_2, y_1, y_2) = \\ & \quad = \frac{1}{3}\langle 0|\mathrm{T}\psi_2^c(x_2)U(x_2,x_1)\psi_1(x_1)\overline{\psi}_1(y_1)U(y_1,y_2)\overline{\psi}_2^c(y_2)|0\rangle = \\ & \quad = \frac{1}{3}\mathrm{Tr}\langle U(x_2,x_1)S_1^{\mathrm{F}}(x_1,y_1|A)U(y_1,y_2)C^{-1}S_2^{\mathrm{F}}(y_2,x_2|A)C\rangle \, . \end{aligned} \tag{1}$$

Here c denotes the charge-conjugate fields, C is the charge-conjugation matrix, U the path-ordered gauge string

$$U(b,a) = \mathrm{P}\exp\left(ig\int_a^b dx^\mu \, A_\mu(x)\right) \, , \tag{2}$$

(the integration path being the straight line joining a to b), $S_1^{\rm F}$ and $S_2^{\rm F}$ the quark propagators in an external gauge field A^μ. Furthermore, in principle the angular brackets should be defined as

$$\langle f[A] \rangle = \frac{\int \mathcal{D}A\, M_f(A) f[A] e^{iS[A]}}{\int \mathcal{D}A\, M_f(A) e^{iS[A]}} \,, \tag{3}$$

$S[A]$ being the pure gauge field action and $M_f(A)$ the determinant resulting from the explicit integration on the fermionic fields. In practice in this paper we take $M_f(A) = 1$ (quenched approximation).

From (1), by a Foldy-type transformation and with the other elaborations described in ref. [1], after setting $y_1^0 = y_2^0 = t_{\rm i}$, $x_1^0 = x_2^0 = t_{\rm f}$, we obtain the following Pauli-type two-particle propagator

$$\begin{aligned} & K(\mathbf{x}_1, \mathbf{x}_2; \mathbf{y}_1, \mathbf{y}_2; t_{\rm f} - t_{\rm i}) = \\ & \int_{\mathbf{y}_1}^{\mathbf{x}_1} \mathcal{D}\mathbf{z}_1 \mathcal{D}\mathbf{p}_2 \int_{\mathbf{y}_2}^{\mathbf{x}_2} \mathcal{D}\mathbf{z}_2 \mathcal{D}\mathbf{p}_2 \exp \Bigg\{ i \Bigg[\int_{t_{\rm i}}^{t_{\rm f}} dt \sum_{j=1}^{2} \left(\mathbf{p}_j \cdot \dot{\mathbf{z}}_j - \frac{\mathbf{p}_j^2}{2m_j} + \frac{\mathbf{p}_j^4}{8m_j^3} + \dots \right) - \\ & \qquad -i \ln W_{q\bar{q}} + \text{spin dependent terms} \Bigg] \Bigg\} \,, \end{aligned} \tag{4}$$

$W_{q\bar{q}}$ being the so-called Wilson loop integral

$$W_{q\bar{q}} = \frac{1}{3} \left\langle \mathrm{Tr}\, \mathrm{P} \exp \left(ig \oint_\Gamma dx^\mu\, A_\mu(x) \right) \right\rangle \,. \tag{5}$$

Here the integration loop Γ is assumed to be made by the quark world line Γ_1, the antiquark world line Γ_2 (described in the reverse direction) and the two straight lines which connect $\mathbf{x}_1$ to $\mathbf{x}_2$, $\mathbf{y}_2$ to $\mathbf{y}_1$ and close the contour (see Fig. 1). As usual $A_\mu(x) = \frac{1}{2}\lambda_a A_\mu^a(x)$, P prescribes the ordering of the colour matrices (from right to left) according to the direction fixed on the loop, and the "functional measures" are given by

$$\mathcal{D}\mathbf{z} = \left(\frac{m}{2\pi i \varepsilon} \right)^{\frac{3N}{2}} d\mathbf{z}_1 \dots d\mathbf{z}_{N-1} \,, \quad \mathcal{D}\mathbf{p} = \left(\frac{i\varepsilon}{2\pi m} \right)^{\frac{3N}{2}} d\mathbf{p}_1 \dots d\mathbf{p}_{N-1} d\mathbf{p}_N \,, \tag{6}$$

where $\varepsilon = t/N$, the limit $N \to \infty$ is understood and the end points x and y stand for the conditions $z_0 = y$, $z_N = x$.

After neglecting the spin-dependent terms and reintegrating the kinematical terms in their closed form, (4) becomes

$$K(\mathbf{x}_1, \mathbf{x}_2; \mathbf{y}_1, \mathbf{y}_2; t_\mathrm{f} - t_\mathrm{i}) = \int_{\mathbf{y}_1}^{\mathbf{x}_1} \mathcal{D}\mathbf{z}_1 \mathcal{D}\mathbf{p}_2 \int_{\mathbf{y}_2}^{\mathbf{x}_2} \mathcal{D}\mathbf{z}_2 \mathcal{D}\mathbf{p}_2 \times$$

$$\exp\left\{ i \left[\int_{t_\mathrm{i}}^{t_\mathrm{f}} dt \sum_{j=1}^{2} \left(\mathbf{p}_j \cdot \dot{\mathbf{z}}_j - \sqrt{m_j^2 + \mathbf{p}_j^2} \right) - i \ln W_{q\bar{q}} \right] \right\} . \tag{7}$$

As usual we assume that $i \ln W_{q\bar{q}}$ can be written as the sum of a perturbative term (at the lowest order) and an area and a perimeter term

$$i \ln W_{q\bar{q}}^{\mathrm{SR}} = \frac{4}{3} g^2 \int_{\Gamma_1} dx_1^\mu \int_{\Gamma_2} dx_2^\nu \, i D_{\mu\nu}(x_1 - x_2) + \sigma S_{\min} + CP \, , \tag{8}$$

$S_{\min}$ being the minimal area enclosed by Γ, P its perimeter and $D_{\mu\nu}$ the usual gluon propagator. Furthermore, to evaluate $S_{\min}$, we adopt the *straight line approximation*, consisting of replacing the minimal area with the area spanned by equal time straight lines joining the quark and the antiquark

$$S_{\min} = \int_{t_\mathrm{i}}^{t_\mathrm{f}} dt \int_0^1 ds |\mathbf{z}_1 - \mathbf{z}_2| \left[1 - (s\dot{\mathbf{z}}_{1\mathrm{T}} + (1-s)\dot{\mathbf{z}}_{2\mathrm{T}})^2 \right]^{\frac{1}{2}} \, , \tag{9}$$

having set $\dot{z}_{j\mathrm{T}}^h = (\delta^{hk} - \hat{r}^h \hat{r}^k) \dot{z}_j^k$ (transversal velocity), $\hat{r}^h = r^h / r$ and $\mathbf{r} = \mathbf{z}_1 - \mathbf{z}_2$. We also use the Coulomb gauge and adopt the *instantaneous approximation* on the perturbative term. So we can write

$$\begin{aligned} i \ln W_{q\bar{q}} = & -\frac{4}{3} \frac{\alpha_s}{r} \left[1 - \frac{1}{2} (\delta^{hk} + \hat{r}^h \hat{r}^k) \dot{z}_1^h \dot{z}_2^k \right] + \\ & + \sigma r \int_0^1 ds \left[1 - (s\dot{\mathbf{z}}_{1\mathrm{T}} + (1-s)\dot{\mathbf{z}}_{2\mathrm{T}})^2 \right]^{1/2} + C \left(\sqrt{1 - \dot{\mathbf{z}}_1^2} + \sqrt{1 - \dot{\mathbf{z}}_2^2} \right) . \end{aligned} \tag{10}$$

Notice that in the first term in (8) and in the last in (10), we have neglected the contributions coming from the end straight lines ($t = t_\mathrm{f}$ and $t = t_\mathrm{i}$) having in mind the limit $t_\mathrm{f} - t_\mathrm{i} \to \infty$. Furthermore, (9) not being Lorentz invariant, our assumptions are understood to be made in the centre-of-mass frame.

Then we replace (10) in (7), expand the kinetic terms around $\mathbf{p}_j = m_j \dot{\mathbf{z}}_j / \sqrt{1 - \dot{\mathbf{z}}_j^2}$

$$\begin{aligned} \mathbf{p} \cdot \dot{z} - \sqrt{m^2 + \mathbf{p}^2} = & -m\sqrt{1 - \dot{\mathbf{z}}^2} - \frac{1}{2m} (1 - \dot{\mathbf{z}}^2)^{\frac{1}{2}} (\delta^{hk} - \dot{z}^h \dot{z}^k) \\ & \left(p^h - \frac{m\dot{z}^h}{\sqrt{1 - \dot{\mathbf{z}}^2}} \right) \left(p^k - \frac{m\dot{z}^k}{\sqrt{1 - \dot{\mathbf{z}}^2}} \right) + \ldots \, , \end{aligned} \tag{11}$$

and perform explicitly the integration over the momenta neglecting the successive terms in (11) (Gaussian approximation). We obtain

$$K(\mathbf{x}_1,\mathbf{x}_2;\mathbf{y}_1,\mathbf{y}_2;t_{\rm f}-t_{\rm i}) = \int_{\mathbf{y}_1}^{\mathbf{x}_1}\mathcal{D}\mathbf{z}_1\Delta[\mathbf{z}_1]\int_{\mathbf{y}_2}^{\mathbf{x}_2}\mathcal{D}\mathbf{z}_2\Delta[\mathbf{z}_2]\exp\left\{i\int_{t_{\rm i}}^{t_{\rm f}} dt L(\mathbf{z}_1,\mathbf{z}_2,\dot{\mathbf{z}}_1,\dot{\mathbf{z}}_2)\right\}\,, \tag{12}$$

with

$$\begin{aligned} L \;&= -\sum_{j=1}^{2} m_j\sqrt{1-\dot{\mathbf{z}}_j^2}+\frac{4}{3}\frac{\alpha_s}{r}\left[1-\frac{1}{2}(\delta^{hk}+\hat{r}^h\hat{r}^k)\dot{z}_1^h\dot{z}_2^k\right]+ \\ &\quad -\sigma r\int_0^1 ds\left[1-(s\dot{\mathbf{z}}_{1\rm T}+(1-s)\dot{\mathbf{z}}_{2\rm T})^2\right]^{1/2}-C\sum_{j=1}^{2}\int_{t_{\rm i}}^{t_{\rm f}} dt\sqrt{1-\dot{\mathbf{z}}_j^2}\,, \end{aligned} \tag{13}$$

and

$$\Delta[\mathbf{z}] = \left\{\prod_t \det\left[\frac{1}{m}(1-\dot{\mathbf{z}}^2)^{1/2}(\delta^{hk}-\dot{z}^h\dot{z}^k)\right]\right\}^{-\frac{1}{2}} = \prod_t (1-\dot{\mathbf{z}}^2)^{-\frac{3}{4}}\,. \tag{14}$$

Of course the factor $\Delta[z]$ has to be considered part of the relativistic "functional measure" in the configuration space.

Notice that if we introduce explicitly the factor $1/\hbar$ in front of the exponential in (7), the result expressed by (12) and (13) becomes exact in the limit $\hbar \to 0$. This implies that (13) provides already the exact classical Lagrangian of the system, while to go beyond the Gaussian approximation would amount to modifying the expression of $\Delta(\mathbf{z})$ alone. As a matter of fact, the additional terms would be highly singular in the time lattice spacing ε and would not match in reconstructing a time integral in the exponent (the fact can be checked explicitly performing the exact integrals in the momenta [2]).

The Lagrangian defined by (13) is the Lagrangian of the relativistic flux tube model [3,4] with the addition of a Coulombic term. The perimeter term can be absorbed in a redefinition of the quark masses ($m_j \to m_j + C$) and from here on we ignore it.

From (13) we can introduce the canonical momenta

$$\mathbf{p}_1 = \frac{\partial L}{\partial \dot{\mathbf{v}}_1} = \frac{m_1\mathbf{v}_1}{\sqrt{1-\mathbf{v}_1^2}}+\sigma r\int_0^1 ds\,\frac{s\,\mathbf{v}_{\rm t}}{\sqrt{1-\mathbf{v}_{\rm t}^2}}\,,$$

$$\mathbf{p}_2 = \frac{\partial L}{\partial \dot{\mathbf{v}}_2} = \frac{m_2 \mathbf{v}_2}{\sqrt{1-\mathbf{v}_2^2}} + \sigma r \int_0^1 ds \, \frac{(1-s)\mathbf{v}_{\rm t}}{\sqrt{1-\mathbf{v}_{\rm t}^2}} \,, \tag{15}$$

while the total linear momentum and the Hamiltonian turn out respectively to be

$$\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2 = \frac{m_1 \mathbf{v}_1}{\sqrt{1-\mathbf{v}_1^2}} + \frac{m_2 \mathbf{v}_2}{\sqrt{1-\mathbf{v}_2^2}} + \sigma r \int_0^1 ds \frac{\mathbf{v}_{\rm t}}{\sqrt{1-\mathbf{v}_{\rm t}^2}} \,, \tag{16}$$

$$H = \sum_{j=1}^{2} \mathbf{p}_j \cdot \mathbf{v}_j - L = \sum_{j=1}^{2} \frac{m_j}{\sqrt{1-\mathbf{v}_j^2}} + \sigma r \int_0^1 ds \frac{1}{\sqrt{1-\mathbf{v}_{\rm t}^2}} \,. \tag{17}$$

Here we have set $\mathbf{v}_j = \dot{\mathbf{z}}_j$ and by $\mathbf{v}_{\rm t}(s) = s\,\dot{\mathbf{z}}_{1\rm T} + (1-s)\dot{\mathbf{z}}_{2\rm T}$ we have denoted the velocity of the flux tube segment specified by s and $s + ds$.

Eqs. (15) cannot be inverted in closed form, but it can be by an expansion in $1/m^2$ or in σ. In the first case we reobtain the semirelativistic Wilson-loop potential [1] (apart from the spin-dependent terms), in the second one we find [4] at the first order in σ

$$\begin{aligned} H &= \sqrt{m_1^2+q^2} + \sqrt{m_2^2+q^2} + \frac{\sigma r}{2} \frac{1}{\sqrt{m_1^2+q^2}+\sqrt{m_2^2+q^2}} \\ &\times \Bigg\{ \sqrt{\frac{m_2^2+q^2}{m_1^2+q^2}} \sqrt{m_1^2+q_r^2} + \sqrt{\frac{m_1^2+q^2}{m_2^2+q^2}} \sqrt{m_2^2+q_r^2} + \\ &+ \left(\frac{\sqrt{m_1^2+q^2}\sqrt{m_2^2+q^2}}{q_{\rm T}} \right) \left(\arcsin \frac{q_{\rm T}}{\sqrt{m_1^2+q^2}} + \arcsin \frac{q_{\rm T}}{\sqrt{m_2^2+q^2}} \right) \Bigg\} \,, \end{aligned} \tag{18}$$

where relevant simplifications have been obtained setting explicitly $\mathbf{p}_1 = -\mathbf{p}_2 = \mathbf{q}$ (centre-of-mass frame) and where we have defined $\mathbf{q}_r = (\mathbf{q} \cdot \hat{r})\hat{r}$.

Notice that if we use (18) in the phase space path integral and try to go back to Eq. (12) by integrating again over the momenta, we would find a different $\Delta(\mathbf{z})$ due to the occurrence in (18) of a momentum-dependent interaction part. This means that to be consistent we have actually to introduce an appropriate normalization factor in front of the expression of $W_{q\bar{q}}$ as would be given by (8) (see [7]).

3 Three-quark flux tube Lagrangian

The three-quark gauge invariant propagator is

$$
\begin{aligned}
&G(x_1, x_2, x_3, y_1, y_2, y_3) = \\
&\qquad\qquad = \frac{1}{3!}\varepsilon_{a_1a_2a_3}\varepsilon_{b_1b_2b_3} \\
&\langle 0|\mathrm{T}\, U^{a_3c_3}(x_M, x_3)U^{a_2c_2}(x_M, x_2)U^{a_1c_1}(x_M, x_1)\psi_{3c_3}(x_3)\psi_{2c_2}(x_2)\psi_{1c_1}(x_1) \\
&\quad \overline{\psi}_{1d_1}(y_1)\overline{\psi}_{2d_2}(y_2)\overline{\psi}_{3d_3}(y_3)U^{d_1b_1}(y_1, y_M)U^{d_2b_2}(y_2, y_M)U^{d_3b_3}(y_3, y_M)|0\rangle \\
&\qquad\qquad = \frac{1}{3!}\varepsilon_{a_1a_2a_3}\varepsilon_{b_1b_2b_3} \\
&\Big\langle \Big(U(x_M, x_1)S_1^{\mathrm{F}}(x_1, y_1|A)U(y_1, y_M)\Big)^{a_1b_1} \\
&\quad \Big(U(x_M, x_2)S_2^{\mathrm{F}}(x_2, y_2|A)U(y_2, y_M)\Big)^{a_2b_2} \\
&\quad \Big(U(x_M, x_3)S_3^{\mathrm{F}}(x_3, y_3|A)U(y_3, y_M)\Big)^{a_3b_3}\Big\rangle\,,
\end{aligned} \tag{19}
$$

where we assume $x_1^0 = x_2^0 = x_3^0 = x_M^0 = t_{\mathrm{f}}$, $y_1^0 = y_2^0 = y_3^0 = y_M^0 = t_{\mathrm{i}}$, $\mathbf{x}_M$ and $\mathbf{y}_M$ are points chosen inside the triangles $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ and $(\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3)$ in such a way that $\sum_{j=1}^3 |\mathbf{x}_j - \mathbf{x}_M|$ and $\sum_{j=1}^3 |\mathbf{y}_j - \mathbf{y}_M|$ are minima and a_i, b_i are colour indices.

Again, neglecting spin-independent terms, we can write

$$
\begin{aligned}
&K(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3; \mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3; t_{\mathrm{f}} - t_{\mathrm{i}}|\, \mathbf{x}_M, \mathbf{y}_M) = \\
&\qquad = \int_{\mathbf{y}_1}^{\mathbf{x}_1} \mathcal{D}\mathbf{z}_1\mathcal{D}\mathbf{p}_1 \int_{\mathbf{y}_2}^{\mathbf{x}_2} \mathcal{D}\mathbf{z}_2\mathcal{D}\mathbf{p}_2 \int_{\mathbf{y}_3}^{\mathbf{x}_3} \mathcal{D}\mathbf{z}_3\mathcal{D}\mathbf{p}_3 \\
&\qquad\quad \exp\left\{ i\left[\int_{t_{\mathrm{i}}}^{t_{\mathrm{f}}} dt\, \sum_{j=1}^{3}\left(\mathbf{p}_j\cdot\dot{\mathbf{z}}_j - \sqrt{m_j^2 + \mathbf{p}_j^2}\right) - i\ln W_{3q}\right]\right\}\,,
\end{aligned} \tag{20}
$$

which corresponds to (7). W_{3q} is the "Wilson loop integral" for three quarks defined by

$$
\begin{aligned}
W_{3q} = \frac{1}{3!}\Big\langle \varepsilon_{a_1a_2a_3}\varepsilon_{b_1b_2b_3} &\left[\mathrm{P}\exp\left(ig\int_{\overline{\Gamma}_1} dx^{\mu_1}A_{\mu_1}(x)\right)\right]^{a_1b_1} \\
&\left[\mathrm{P}\exp\left(ig\int_{\overline{\Gamma}_2} dx^{\mu_2}A_{\mu_2}(x)\right)\right]^{a_2b_2}\left[\mathrm{P}\exp\left(ig\int_{\overline{\Gamma}_3} dx^{\mu_3}A_{\mu_3}(x)\right)\right]^{a_3b_3}\Big\rangle\,,
\end{aligned} \tag{21}
$$

$\bar{\Gamma}_j$ being made by the world line Γ_j of the quark j plus the straight lines connecting y_M to y_j and x_j to x_M (see Fig. 2). As in (8) we shall set

$$i \ln W_{3q} = \frac{2}{3} g^2 \sum_{i<j} \int_{\Gamma_i} dx_i^\mu \int_{\Gamma_j} dx_j^\nu \, i D_{\mu\nu}(x_i - x_j) + \sigma S_{\min} + CP \,, \tag{22}$$

where the first term is again the lowest-order perturbative contribution; P is the total length of $\bar{\Gamma}_1$, $\bar{\Gamma}_2$ and $\bar{\Gamma}_3$, $S_{\min}$ is the area of the minimal three-sheet surface enclosed by $\bar{\Gamma}_1$, $\bar{\Gamma}_2$, $\bar{\Gamma}_3$ and a mid-point world-line Γ_M ($z_M = z_M(t)$) joining y_M and x_M ; the minimum is performed for fixed quark world line, but varying Γ_M.

Again adopting instantaneous and straight line approximations we obtain

$$\begin{aligned} i \ln W_{3q} = & \int_{t_\mathrm{i}}^{t_\mathrm{f}} dt \left\{ \sum_{j<l} \frac{-2}{3} \frac{\alpha_s}{r_{jl}} \left[1 - \frac{1}{2} (\delta^{hk} + \hat{r}_{jl}^h \hat{r}_{jl}^k) \dot{z}_j^h \dot{z}_l^k \right] + \right. \\ & + \sum_{j=1}^{3} \sigma \, r_j \int_0^1 ds_j \left[1 - \left(s_j \dot{\mathbf{z}}_{j\mathrm{T}_j} + (1 - s_j) \dot{\mathbf{z}}_{M\mathrm{T}_j} \right)^2 \right]^{1/2} + \\ & \left. + \, C \sum_{j=1}^{3} \sqrt{1 - \dot{\mathbf{z}}_j^2} \right\} \,. \end{aligned} \tag{23}$$

Here $\mathbf{r}_{ij} = \mathbf{z}_i - \mathbf{z}_j$, $\mathbf{r}_j = \mathbf{z}_j - \mathbf{z}_M$ and $\mathbf{z}_M$ has to be chosen in such a way that $\sum_{j=1}^3 r_j$ is minimum at any given time, $\dot{z}_{j\mathrm{T}_j}^h = (\delta^{hk} - \hat{r}_j^h \hat{r}_j^k) \dot{z}_j^k$. Of course, $\mathbf{z}_M(t)$ has to be coplanar with $\mathbf{z}_1(t)$, $\mathbf{z}_2(t)$ and $\mathbf{z}_3(t)$ and then two different types of configurations are possible:

I) if in the triangle ($\mathbf{z}_1(t)$, $\mathbf{z}_2(t)$, $\mathbf{z}_3(t)$) no angle exceeds 120^0, then $\mathbf{z}_M(t)$ has to be such that $\mathbf{r}_1(t)$, $\mathbf{r}_2(t)$, $\mathbf{r}_3(t)$ make angles of 120^0 with each other;

II) if the angle in $\mathbf{z}_l(t)$ reaches 120^0, then $\mathbf{z}_M = \mathbf{z}_l$.

Notice that correspondingly for $\dot{\mathbf{z}}_M$ we have

$$\dot{\mathbf{z}}_M = \begin{cases} \mathcal{R}^{-1} \sum_{j=1}^3 \left(\dot{\mathbf{z}}_{j\mathrm{T}_j} / r_j \right) & \text{I type configuration} \\ \dot{\mathbf{z}}_l & \text{II type configuration} \end{cases} \tag{24}$$

$\mathcal{R}$ being the matrix with elements $\mathcal{R}^{hk} = \sum_{j=1}^3 (\delta^{hk} - \hat{r}_j^h \hat{r}_j^k)/r_j$.

Replacing as before (23) in (20) and integrating on the momenta in the Gaussian approximation we can write

$$K(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3;\mathbf{y}_1,\mathbf{y}_2,\mathbf{y}_3;t_\mathrm{f}-t_\mathrm{i}) = \int_{\mathbf{y}_1}^{\mathbf{x}_1} \mathcal{D}[\mathbf{z}_1]\Delta[\mathbf{z}_1] \int_{\mathbf{y}_2}^{\mathbf{x}_2} \mathcal{D}[\mathbf{z}_2]\Delta[\mathbf{z}_2] \int_{\mathbf{y}_3}^{\mathbf{x}_3} \mathcal{D}[\mathbf{z}_3]\Delta[\mathbf{z}_3] \exp\left\{ i \int_{t_\mathrm{i}}^{t_\mathrm{f}} dt\, L(\mathbf{z}_1,\mathbf{z}_2,\mathbf{z}_3,\dot{\mathbf{z}}_1,\dot{\mathbf{z}}_2,\dot{\mathbf{z}}_3) \right\} , \quad (25)$$

and obtain the three-quark flux tube Lagrangian

$$\begin{aligned} L = &- \sum_{j=1}^{3} m_j \sqrt{1-\dot{\mathbf{z}}_j^2} + \sum_{j<l} \frac{2}{3}\frac{\alpha_s}{r_{jl}} \left[1 - \frac{1}{2}\left(\delta^{hk} + \hat{r}_{jl}^h \hat{r}_{jl}^k\right) \dot{z}_j^h \dot{z}_l^k \right] - \\ &- \sum_{j=1}^{3} \sigma\, r_j \int_0^1 ds_j \left[1 - \left(s_j \dot{\mathbf{z}}_{j\mathrm{T}_j} + (1-s_j)\dot{\mathbf{z}}_{M\mathrm{T}_j} \right)^2 \right]^{1/2} - \\ &- C \sum_{j=1}^{3} \sqrt{1-\dot{\mathbf{z}}_j^2} \,. \end{aligned} \quad (26)$$

Notice that again the perimeter term can be absorbed in a redefinition of the quark masses.

4 Three-quark relativistic flux tube Hamiltonian

For simplicity, in this section we shall omit the perturbative part of the Lagrangian, keeping in mind that this can be simply added to the pure flux tube result. To simplify the notations let us set $\mathbf{v}_i = \dot{\mathbf{z}}_i$; $\mathbf{v}_M = \dot{\mathbf{z}}_M$, $\mathbf{v}_i^t(s_i) = s_i \mathbf{v}_{iT_i} + (1-s_i)\mathbf{v}_{MT_i}$. Let us also introduce the matrices $\mathcal{T}_i$ with elements $\mathcal{T}_i^{hk} = \delta^{hk} - \hat{r}_i^h \hat{r}_i^k$; then we have obviously

$$\mathcal{R} = \sum_{j=1}^{3} \frac{1}{r_j} \mathcal{T}_j \,, \quad (27)$$

and

$$\mathbf{v}_{iT_i} = \mathcal{T}_i \mathbf{v}_i \,, \quad \mathbf{v}_M = \mathcal{R}^{-1} \sum_{j=1}^{3} \frac{1}{r_j} \mathcal{T}_j \mathbf{v}_j \,,$$

$$\mathbf{v}_i^t(s_i) = \mathcal{T}_i(s_i \mathbf{v}_i + (1-s_i)\mathbf{v}_M) = s_i \mathcal{T}_i \mathbf{v}_i + (1-s_i) \sum_{j=1}^{3} \frac{1}{r_j} \mathcal{T}_i \mathcal{R}^{-1} \mathcal{T}_j \mathbf{v}_j \,. \quad (28)$$

So we can write the flux tube Lagrangian as

$$L = -\sum_{i=1}^{3} m_i \sqrt{1-\mathbf{v}_i^2} - \sigma \sum_{i=1}^{3} r_i \int_0^1 ds_i \sqrt{1-\mathbf{v}_i^{t\,2}} \tag{29}$$

and, we can obtain the canonical conjugate momenta

$$\mathbf{p}_i = \frac{m_i \mathbf{v}_i}{\sqrt{1-\mathbf{v}_i^2}} + \sigma \sum_{j=1}^{3} r_j \int_0^1 ds_j \frac{(s_j \delta_{ij} + (1-s_j)\mathcal{R}^{-1}\mathcal{T}_i/r_i)\,\mathbf{v}_j^t}{\sqrt{1-\mathbf{v}_j^{t\,2}}}\,, \tag{30}$$

while the total linear momentum and the Hamiltonian turn out, respectively, to be:

$$\mathbf{P} = \sum_{i=1}^{3} \mathbf{p}_i = \sum_{i=1}^{3} \frac{m_i \mathbf{v}_i}{\sqrt{1-\mathbf{v}_i^2}} + \sigma \sum_{i=1}^{3} r_i \int_0^1 ds_i \frac{\mathbf{v}_i^t}{\sqrt{1-\mathbf{v}_i^{t\,2}}}\,, \tag{31}$$

$$H = \sum_{i=1}^{3} \mathbf{p}_i \cdot \mathbf{v}_i - L = \sum_{i=1}^{3} \frac{m_i}{\sqrt{1-\mathbf{v}_i^2}} + \sigma \sum_{i=1}^{3} r_i \int_0^1 ds_i \frac{1}{\sqrt{1-\mathbf{v}_i^{t\,2}}}\,. \tag{32}$$

Again Eq. (30) can be solved with respect to $\mathbf{v}_j$ ($j = 1, 2, 3$) only as an expansion in $1/m_j^2$ or in the string tension σ. Consequently even H turns out to be expressed in such a form.

Using an expansion in $1/m_j^2$ we obtain

$$\begin{aligned} H =& \sum_{i=1}^{3} m_i + \frac{1}{2}\sum_{i=1}^{3} \frac{\mathbf{p}_i^2}{m_i} - \frac{1}{8}\sum_{i=1}^{3} \frac{\mathbf{p}_i^4}{m_i^3} - \\ &- \frac{\sigma}{6}\sum_{i=1}^{3} r_i \left(\frac{\mathbf{p}_{iT_i}^2}{m_i^2} + \mathbf{v}_{MT_i}^2 + \frac{\mathbf{p}_{iT_i} \cdot \mathbf{v}_{MT_i}}{m_i} \right)\Bigg|_{\mathbf{v}_i = \mathbf{p}_i/m_i} + O\left(\frac{\sigma^2}{m^3}\right)\,, \end{aligned} \tag{33}$$

with

$$\mathbf{v}_{MT_i}\big|_{\mathbf{v}_i=\mathbf{p}_i/m_i} = \mathcal{R}^{-1} \sum_j \frac{1}{r_j} \mathcal{T}_j \frac{\mathbf{p}_{j\mathrm{T}_i}}{m_j}\,. \tag{34}$$

Eq. (33) coincides with the static plus the velocity-dependent part of the three-quark Wilson loop Hamiltonian [1].

Considering an expansion in σ in (30) we obtain the explicit relativistic Hamiltonian

$$H = \sum_{i=1}^{3} \sqrt{p_i^2 + m_i^2} + \sigma \sum_{i=1}^{3} r_i \int_0^1 ds_i \, \sqrt{1 - \mathbf{v}_i^{t\,2}} \Bigg|_{\mathbf{v}_i = \mathbf{p}_i / \sqrt{p_i^2 + m_i^2}} + O\left(\sigma^2\right) \,, \tag{35}$$

obviously now

$$\mathbf{v}_i^t|_{\mathbf{v}_i = \frac{\mathbf{p}_i}{\sqrt{m_i^2 + \mathbf{p}_i^2}}} = s_i \frac{\mathbf{p}_{i\mathrm{T}_i}}{\sqrt{m_i^2 + \mathbf{p}_i^2}} + (1 - s_i)\mathcal{R}^{-1} \sum_{j=1}^{3} \frac{1}{r_j} \mathcal{I}_j \frac{\mathbf{p}_{j\mathrm{T}_i}}{\sqrt{m_j^2 + \mathbf{p}_j^2}} \,. \tag{36}$$

Notice that like Eq. (9), even Eq. (23) and so even Eqs. (26), (29), (32) are supposed to hold in the centre-of-mass frame $\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 = 0$ and $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ are not independent. A possible choice of independent variables is given by the internal Jacobi momenta, in terms of which we can write

$$\mathbf{p}_1 = \mathbf{q}\,, \qquad \mathbf{p}_2 = -\frac{m_2}{m_2 + m_3}\mathbf{q} + \mathbf{k}\,, \qquad \mathbf{p}_3 = -\frac{m_3}{m_2 + m_3}\mathbf{q} - \mathbf{k}\,. \tag{37}$$

Eq. (37) should be used in (36), (35). The final result is very complicated. For low angular momenta, however, $v_i^{t\,2}$ should be negligible and Eq. (35) reduces to the three-quark starlike string model

$$H = \sum_{i=1}^{3} \sqrt{m_i^2 + \mathbf{p}_i^2} + \sigma \sum_{i=1}^{3} r_i \,, \tag{38}$$

already considered in the literature [5,6].

Performing explicitly the s_i integration in (35) we obtain

$$\begin{aligned} H = & \sum_{i=1}^{3} \sqrt{p_i^2 + m_i^2} + \sigma \sum_{i=1}^{3} r_i \Bigg\{ \frac{1}{2} \sqrt{1 - \mathbf{v}_{i\mathrm{T}_i}^2} + \\ & + \frac{\mathbf{v}_{M\mathrm{T}_i} \cdot (\mathbf{v}_{i\mathrm{T}_i} - \mathbf{v}_{M\mathrm{T}_i})}{2(\mathbf{v}_{i\mathrm{T}_i} - \mathbf{v}_{M\mathrm{T}_i})^2} \left[\sqrt{1 - \mathbf{v}_{i\mathrm{T}_i}^2} - \sqrt{1 - \mathbf{v}_{M\mathrm{T}_i}^2} \right] \\ & + \frac{1}{2|\mathbf{v}_{i\mathrm{T}_i} - \mathbf{v}_{M\mathrm{T}_i}|} \left[(1 - \mathbf{v}_{M\mathrm{T}_i}^2) + \frac{(\mathbf{v}_{i\mathrm{T}_i} \cdot (\mathbf{v}_{i\mathrm{T}_i} - \mathbf{v}_{M\mathrm{T}_i}))^2}{(\mathbf{v}_{i\mathrm{T}_i} - \mathbf{v}_{M\mathrm{T}_i})^2} \right] \\ & \times \left(\arcsin \frac{\mathbf{v}_{i\mathrm{T}_i} \cdot (\mathbf{v}_{i\mathrm{T}_i} - \mathbf{v}_{M\mathrm{T}_i})}{\sqrt{(1 - \mathbf{v}_{M\mathrm{T}_i}^2)(\mathbf{v}_{i\mathrm{T}_i} - \mathbf{v}_{M\mathrm{T}_i})^2 + (\mathbf{v}_{i\mathrm{T}_i} \cdot (\mathbf{v}_{i\mathrm{T}_i} - \mathbf{v}_{M\mathrm{T}_i}))^2}} \right. \\ & \left. + \arcsin \frac{\mathbf{v}_{M\mathrm{T}_i} \cdot (\mathbf{v}_{i\mathrm{T}_i} - \mathbf{v}_{M\mathrm{T}_i})}{\sqrt{(1 - \mathbf{v}_{M\mathrm{T}_i}^2)(\mathbf{v}_{i\mathrm{T}_i} - \mathbf{v}_{M\mathrm{T}_i})^2 + (\mathbf{v}_{i\mathrm{T}_i} \cdot (\mathbf{v}_{i\mathrm{T}_i} - \mathbf{v}_{M\mathrm{T}_i}))^2}} \right) \Bigg\} \end{aligned} \tag{39}$$

which, however, does not essentially simplify after the substitution (37), contrary to what happens in the quark-antiquark case (18).

5 Conclusion

In conclusion we have shown that even for the three-quark system a flux-tube like relativistic Lagrangian can be obtained in the Wilson-loop framework, if one neglects spin. The method does not seem appropriate for a direct introduction of spin since it uses a Foldy–Wouthuysen type transformation at an early stage and so the spin dependence in the starting path-integral occurs as an expansion in $1/m^2$.

As a matter of fact, in the quark-antiquark case the relativistic flux-tube model is strictly related to the instantaneous approximation of the corresponding Bethe–Salpeter equation, and spin can be taken into account through such an equation [8]. In principle the generalization of such an equation should be the convenient formalism even for the three-quark bound state.

Notice finally that to give the quantum Hamiltonian operator corresponding to Eqs. (35) and (39), attention has to be paid to questions of ordering. If the discrete form of the quantity W_{3q} is written as [1]

$$W_{3q} = \frac{1}{3!}\Big\langle \varepsilon_{a_1a_2a_3}\varepsilon_{b_1b_2b_3} \prod_{j=1}^{3}\left[\mathrm{P}\ \exp\left(i\,g\sum_{\Gamma_j}\left(z^{\mu}_{j\,n}-z^{\mu}_{j\,n-1}\right)A_\mu\left(\frac{z_{j\,n}+z_{j\,n-1}}{2}\right)\right)\right]^{a_jb_j}\Big\rangle\,, \tag{40}$$

the correct ordering would be the Weyl's ordering.

References

[1] N. Brambilla, P. Consoli and G. M. Prosperi, *Phys. Rev.* **D50** 5878 (1994); N. Brambilla and G.M. Prosperi, *Wilson loops, $q\bar{q}$ and $3q$ potentials, Bethe–Salpeter equation*, in "Quark Confinement and the Hadron Spectrum", World Scientific Singapore 1995, (N. Brambilla and G.M. Prosperi, ed.).

[2] M.M. Mizrahi, *J. Math. Phys.* **16** 2201 (1975); **19** 298 (1978); L. Cohen, *J. Math. Phys.* 1873 **17** (1976).

[3] C. Olson, M.G. Olsson and K. Williams, *Phys. Rev.* **D45** 4307 (1992); M. G. Olsson, *Flux tube and Heavy quark Symmetry*, in Proceedings of the Conference

"Quark Confinement and the Hadron Spectrum", World Scientific Singapore 1995 and quoted references (N. Brambilla and G.M. Prosperi, ed.); see also M.G. Olsson and K. Williams, *Phys. Rev.* **D48** 417 (1993).

[4] N. Brambilla and G.M. Prosperi, *Phys. Rev.* **D47** 2107 (1993).

[5] J. Carlson, J. B. Kogut and V. R. Pandharipande, Phys. Rev. **D27** 233 (1983); S. Capstick and N. Isgur, Phys. Rev. **D34** 2809 (1986); **28** 2807 (1983); D. Gromes and I. O. Stamatescu, Nucl. Phys. **B112** 213 (1976); H. G. Dosch and V. F. Müller, Nucl. Phys. **B116** 470 (1976); R. E. Cutkosky and R. E. Hendrik, Phys. Rev. **D16** 786,793 (1977); N. Isgur and J.Paton, Phys. Lett. **124B** 247 (1983); N. Isgur and G. Karl, Phys. Rev. **D18** 4187 (1978); G. Marchesini and E. Onofri, Nuovo Cimento **65A** 298 (1981).

[6] A. Martin, *Z. Phys.* **C32** 359 (1986).

[7] A. Barchielli, N. Brambilla and G.M. Prosperi, *Nuovo Cimento* **103A** 59 (1990).

[8] N. Brambilla, E. Montaldi and G. M. Prosperi, *Bethe–Salpeter equation in QCD*, IFUM 499/FT (1995) (hep-ph/9504229); N. Brambilla and G. M. Prosperi, *Confining Bethe–Salpeter equation from scalar QCD*, IFUM 485/FT (1994) (hep-ph/9501206).

Fig. 1. Quark-antiquark Wilson loop

Fig. 2. Three-quark Wilson loop