Non-perturbative Particle Dynamics in (2+1)-Gravity *

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Abstract

We construct a non-perturbative, single-valued solution for the metric and the motion of two interacting particles in $(2+1)$-Gravity, by using a Coulomb gauge of conformal type. The method provides the mapping from multivalued (minkowskian) coordinates to single-valued ones, which solves the non-abelian monodromies due to particles' momenta and can be applied also to the general N-body case.

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The classical [1]-[3] and quantum [4]-[6] structure of (2+1)-Gravity coupled to matter has been thoroughly investigated in the past by using locally Minkowskian coordinates and/or its topological relation to Chern-Simons Poincare' gauge theories [7]-[8].

The choice of Minkowskian coordinates is possible because in $(2+1)$-dimensions the space is flat outside the (pointlike) matter sources. However, the localized curvature due to particles' momenta implies that the Minkowskian coordinates are not single-valued, but are changed by a Poincare' transformation by parallel transport around the sources (DJH matching conditions [2]). This implies that the metric description requires singularity tails carried by each particle [9]-[10].

On the other hand, in order to define the scattering problem, and in general local particle properties, it is convenient to look for regular gauges, in which the metric is not Minkowskian, but is single-valued and is singular only at the particle sites. A method for constructing the coordinate transformation from singular to regular gauges was given in Ref. [9], but an explicit solution was exhibited only in the massless limit and in an Aichelburg-Sexl [9]-[11] gauge, of covariant type. In the general massive case only partial perturbative results are available [12]-[13].

The purpose of this note is to propose an alternative non-perturbative method to construct the above coordinate transformation, and thus the regular metric for any number of particles, and to determine the main features of the two-body problem in an "exact" way. A key ingredient of the present solution is our choice of gauge [13]-[14], which is of conformal type and is also of Coulomb type [4]-[15] , i.e., it yields an instantaneous propagation.

To set up the problem, let $X^a \equiv (T/Z/\bar{Z})$ denote the Minkowskian coordinates and $x^\mu \equiv (t/z/\bar{z})$ the single-valued ones. They are related by a dreibein $E^a_\mu = (A^a, B^a, \widetilde{B}^a)$ such that

$$dX^a = E^a_\mu dx^\mu = A^a(x)dt + B^a(x)dz + \widetilde{B}^a(x)d\bar{z}, \tag{1}$$

where the A' s and B' s are to be determined by conditions to be given shortly.

Since the X's are Minkowskian, the line element is given by

$$ds^2 = \eta_{ab} dX^a dX^b = g_{\mu\nu} dx^\mu dx^\nu \tag{2}$$

and therefore the single-valued metric tensor $g_{\mu\nu}$ is obtained as

$$g_{\mu\nu} = E^a_\mu E_{a\nu}, \tag{3}$$

where the a indices are lowered by the Minkowskian metric η_{ab}, with non-vanishing components $\eta_{00} = -2\eta_{z\bar{z}} = 1$.

For the (multivalued) coordinates X^a to exist , the dreibein must satisfy the integrability conditions

$$\partial_{[\mu} E^a_{\nu]} = 0 \quad (\mu, \nu = 0, z, \bar{z}). \tag{4}$$

The latter hold in the region outside the singularity tails departing from each particle source, which are needed in order to define a Riemann surface for the X's , and carry a non-trivial, localized spin connection, discussed elsewhere [9]-[10].

Following Ref. [13], we choose to work in a generalized Coulomb gauge of conformal type, which in the present first-order formalism is defined by[†]

$$\partial \cdot E^a = \partial_z E^a_{\bar{z}} + \partial_{\bar{z}} E^a_z = 0 \tag{5}$$

$$g_{zz} = g_{\bar{z}\bar{z}} = 0. \tag{6}$$

Because of Eqs. (4) and (5), the dreibein components satisfy the conditions

$$\partial_{\bar{z}} B^a = \partial_z \widetilde{B}^a = 0 \tag{7}$$

$$\begin{aligned} \partial_z A^a &= \partial_0 B^a(z,t), \\ \partial_{\bar{z}} A^a &= \partial_0 \widetilde{B}^a(\bar{z},t), \end{aligned} \tag{8}$$

Therefore, $B^a(z,t)$ ($\widetilde{B}^a(\bar{z},t)$) are analytic (anti-analytic) functions and $A^a(z,\bar{z},t)$ are harmonic functions, i.e., sums of analytic and anti-analytic ones.

Furthermore, because of Eq. (6) , B^a and $\widetilde{B}^a$ are null-vectors so that, by using straightforward conjugation properties we can parametrize

$$\begin{aligned} B^a &= N(z,t) W^a(z,t) \;, \quad \widetilde{B}^a = \bar{N}(\bar{z},t) \widetilde{W}^a(\bar{z},t), \\ W^a &\equiv (f')^{-1}(f/1/f^2) \;, \quad \widetilde{W}^a \equiv (\bar{f}')^{-1}(\bar{f}/\bar{f}^2/1) \;, \end{aligned} \tag{9}$$

with $W^2 = \widetilde{W}^2 = 0$, and

$$A^a = (a/A/\bar{A}) \;, \quad a = \bar{a}, \tag{10}$$

where $N(z,t)$, $f(z,t)$ and $f' = df/dz$ are analytic functions, and $a(z,\bar{z},t), A(z,\bar{z},t)$ are harmonic ones.

It is now straightforward to obtain the components of the metric tensor (3) in the form

$$-2g_{z\bar{z}} \equiv e^{2\phi} = \left|\frac{N}{f'}\right|^2 (1 - |f|^2)^2 = |N|^2(-2W \cdot \widetilde{W}), \tag{11a}$$

$$g_{0z} \equiv \frac{1}{2}\bar{\beta} e^{2\phi} = N W_a A^a \;, \quad g_{0\bar{z}} \equiv \frac{1}{2}\beta e^{2\phi} = \bar{N} \widetilde{W}_a A^a \;, \tag{11b}$$

$$g_{00} \equiv \alpha^2 - |\beta|^2 e^{2\phi} \;, \quad \alpha = V_a A^a, \tag{11c}$$

[†]The condition (5) insures also the vanishing of the extrinsic curvature $\Gamma_{0,z\bar{z}} = 0$, and is therefore equivalent to the instantaneous gauge obtained in Ref. [13] .

where we have defined the vector

$$V^a \equiv (1-|f|^2)^{-1}(1+|f|^2 \ / \ 2\bar{f} \ / \ 2f) = \epsilon^a_{bc} W^b \widetilde{W}^c (W \cdot \widetilde{W})^{-1}. \tag{12}$$

Eqs. (11) and (12) express the fields α, β, ϕ corresponding to four real variables, in terms of the functions f, N, A, a corresponding to seven real variables. This is because the metric determines the dreibein only up to local Lorentz transformations, in this case the 3-parameter $O(2,1)$ group.

If N, f (a , A) are analytic (harmonic) everywhere in the z-plane, then Eq. (11) describes a pure gauge degree of freedom, satisfying the Einstein equations with vanishing energy-momentum tensor, and we end up with a truly Minkowskian geometry.

Particle sources with masses m_i and Minkowskian momenta P_i^a ($i = 1, ..., N$) yield instead singularities of the dreibein at the particle sites $z = \xi_i(t)$. They will be introduced in the following by the DJH matching conditions [2], i.e., by the requirement that

$$(dX^a)_{II} = (L_i)^a_b (dX^b)_I \ , \quad i = 1, ..., N, \tag{13}$$

where

$$L_i = exp(iJ_a P_i^a) \ , \quad (iJ_a)_{bc} = \epsilon_{abc} \ , \tag{14}$$

denote the holonomies of the spin connection ‡, corresponding to loops around the singularity of particle i, and labels I (II) denote determinations of the X^a coordinates before (after) the loop operation.

The conditions (13) imply that the dreibein components are multivalued and transform as $O(2,1)$ vectors under application of the L_i's and their products, so as to yield an invariant, i.e. , single-valued metric tensor given by the explicitly scalar expressions in Eq. (11).

Suppose now we are able to find an analytic function $f(z,t)$, with branch points at the particle sites $z = \xi_i(t)$ such that, when z turns around ξ_i, the f transforms as a projective representation of the monodromies (13), i.e.,

$$f(z,t) \rightarrow \frac{a_i f(z,t) + b_i}{\bar{b}_i f(z,t) + \bar{a}_i} \ , \quad i = 1, ..., N, \tag{15}$$

where the a's and b's parametrize the spin $\frac{1}{2}$ representations of the loop transformations in (14). Then the W's , constructed out of f in Eq. (9), transform as the adjoint (vector) representation of $O(2,1)$, because they are obtained by applying the generators

$$L^a = (f\frac{\partial}{\partial f} / \frac{\partial}{\partial f} / f^2 \frac{\partial}{\partial f}) \tag{16}$$

‡The spin connection is localized on the tails [9] , but its form will not be discussed here, since we use the global property (13).

to the single-valued variable z. The $\widetilde{W}$'s do the same for the equivalent conjugate representation. It follows that $N(z,t)$ must be single-valued, and is at most meromorphic, with poles at $z = \xi_i$.

As for A^a, its vector transformation property is insured by the consistency conditions in Eq. (8), up to quadratures. Similarly, the vector character of V^a in (12) under the transformation (15) can be checked explicitly. As a consequence, a solution to the conditions (15) will automatically provide the correct transformation properties of the dreibein and a single-valued metric, and together with Eq. (8) has a good chance of determining the whole problem.

The simplest example of condition (15) is for one particle of mass m at rest. In such case the loop transformation (14) is a rotation of the deficit angle

$$2\pi(1-\alpha) = m \qquad (8\pi G = 1), \tag{17}$$

and Eq. (13) is just multiplication by the corresponding phase factor $exp(im)$. Therefore, for a particle at the origin,

$$f(z,t) = K\ z^{\lambda}\ , \quad \lambda = \frac{m}{2\pi} \quad (\text{mod } n), \tag{18}$$

where, however, the constant $K = O(V)$ should be considered as infinitesimal with the velocity [§] of the particle, so as to yield vanishing mixed components of the metric in Eq. (11). In this limit, also $N \sim K/z$ vanishes, and the only finite quantities are, up to a scale transformation and with $n = 0$,

$$\frac{f'}{N} = \frac{1}{\alpha} z^{m/2\pi}\ , \quad e^{2\phi} = \alpha^2 |z|^{-m/\pi} = -2g_{z\bar{z}}, \tag{19}$$

which yield the well-known [1]-[2] conical geometry in the conformal gauge:

$$\begin{aligned} ds^2 &= dt^2 - \alpha^2 |z|^{-m/\pi} |dz|^2 = \\ &= dT^2 - |dZ|^2\ , \quad (argZ < \pi\alpha). \end{aligned} \tag{20}$$

Next comes the two-body problem, with masses m_1 and m_2 , and momenta

$$P_1 = (E_1/P/\bar{P})\ , \quad P_2 = (E_2/-P/-\bar{P}) \tag{21}$$

in the Minkowskian c.m. frame. In terms of the rescaled variable

$$\zeta(z,t) = \frac{z-\xi_1}{\xi_2-\xi_1} = \frac{z-\xi_1}{\xi}\ , \tag{22}$$

[§]This feature is shared by the general case to be discussed below, and is rooted in the fact that the particles interaction becomes trivial in the static limit, due to the lack of a Newtonian force.

the function $f(z,t)$ has now branch points at $\zeta = 0$ and $\zeta = 1$ (and $\zeta = \infty$), around which it has to transform as in Eq. (15), with

$$\begin{aligned} a_i &= cos\,\frac{m_i}{2} + i\,\gamma_i\,sin\frac{m_i}{2}, \quad b_i = -i\,\gamma_i\,\bar{V}_i\,sin\frac{m_i}{2}\ , \\ V_{1,2} &\equiv \pm\frac{P}{E_{1,2}}\ , \quad \gamma_i \equiv (1-|V_i|^2)^{-\frac{1}{2}}\ , \quad i=1,2. \end{aligned} \tag{23}$$

The difficulty now lies in the fact that L_1, L_2 do not commute, because of the relative speed, and thus cannot be brought together to the form of a phase transformation. Nevertheless, we can use the analyticity properties of the solution of second order differential equations around Fuchsian singularities [16] in order to obtain $f(z,t)$ as the ratio of properly chosen independent solutions with three singularities, i.e., essentially hypergeometric functions.

Indeed, after some algebra, we find the expression

$$f(z,t) = e^{-i\theta_V}\,\frac{f_{(1)}(z,t) - th\frac{1}{2}\eta_1}{1 - th\frac{1}{2}\eta_1 f_{(1)}(z,t)}\ , \tag{24}$$

where

$$f_{(1)}(z,t) \;=\; cth\frac{1}{2}(\eta_1-\eta_2)\;\zeta^{\lambda}\;\frac{\widetilde{F}(\frac{1}{2}(1+\mu+\lambda-\nu),\frac{1}{2}(1-\mu+\lambda-\nu);1+\lambda;\zeta)}{\widetilde{F}(\frac{1}{2}(1+\mu-\lambda-\nu),\frac{1}{2}(1-\mu-\lambda-\nu);1-\lambda;\zeta)} \tag{25}$$

has the meaning of f-function in the particle 1 rest frame, $\eta_i = th^{-1}V_i$ denote the velocity boosts, θ_V the relative velocity phase, $\widetilde{F}(a,b,c;z) \equiv \Gamma(a)\Gamma(b)\Gamma^{-1}(c)F(a,b,c;z)$ is a modified hypergeometric function, and the indices λ, ν and μ are related to the masses m_1 , m_2 and invariant mass $\mathcal{M}$ as follows

$$\begin{aligned} \lambda &= \pm\frac{m_1}{2\pi}\ (\text{mod. } n_1)\ , \quad \nu = \pm\frac{m_2}{2\pi}\ (\text{mod.} n_2), \\ \mu &= \pm\left(\frac{\mathcal{M}}{2\pi} - 1\right)\ (\text{mod.} - n_1 - n_2 + 2n), \end{aligned} \tag{26}$$

where $\mathcal{M}$, corresponding to the topological invariant $Tr(L_1L_2)$, is given by [8]

$$cos\frac{\mathcal{M}}{2} = cos\frac{m_1}{2}cos\frac{m_2}{2} - sin\frac{m_1}{2}sin\frac{m_2}{2}\ \frac{P_1\cdot P_2}{m_1m_2}. \tag{27}$$

The solutions (24) and (25) are obtained by observing [17] that, if y_1 and y_2 are independent solutions of the equation

$$y'' + \frac{1}{4}\left(\frac{1-\lambda^2}{\zeta^2} + \frac{1-\nu^2}{(1-\zeta)^2} + \frac{1-\lambda^2-\nu^2+\mu^2}{\zeta(1-\zeta)}\right)y = 0, \tag{28}$$

then $f = y_1/y_2$ transforms according to a subgroup of $SL(2,C)$ around the branch points $\zeta = 0$ and $\zeta = 1$. By adjusting the y's and their indices to our $O(2,1)$ case in Eq. (23), Eqs. (24) and (25) follow.
Note that $f(z,t)$ is time-dependent only through the rescaled variable $\zeta(z,t)$ in Eq.(22), because the momenta P_1, P_2 are the constants of motion of our problem [9]-[13]. Note also that different determinations of f due to different choices of the integers n_1, n_2, n, correspond in general to different behaviours close to the singularities and for $z \to \infty$. In the following, we shall choose the $+$ determination of signs and we shall also set $n_1 = n_2 = n = 0$, in order to match with perturbative results [13].

The complete determination of the meromorphic function $N(z,t)$ appears to be harder. We shall exclude a holomorphic (constant) behaviour because at least pole singularities are needed to build non-trivial sources. Assuming simple poles (corresponding to δ-function energy-momentum density), we think that the residues should be related in order to avoid zeros of the determinant¶

$$\sqrt{|g|} = |E| = \alpha e^{2\phi} = \left|\frac{N}{f'}\right|^2 (1-|f|^2)^2 (V_a A^a). \tag{29}$$

Therefore, we shall take the ansatz

$$N(z,t) = \frac{R(\xi(t)))}{(z-\xi_1)(\xi_2 - z)} = \frac{R(\xi(t))}{\xi(t)^2}\frac{1}{\zeta(1-\zeta)}, \tag{30}$$

where $\xi(t) \equiv \xi_2(t) - \xi_1(t)$. A form of type (30) checks also with perturbative results [12]-[13].

We are now in a position to discuss the functional relation of the coordinates X^a and x^μ implied by Eqs. (1), (8), (9) and by Eqs. (25) and (30). By integrating Eq. (1) out of particle 1, say, we obtain

$$X^a = X_1^a(t) + \int_{\xi_1}^{z} dz\; N W^a(z,t) + \int_{\bar{\xi}_1}^{\bar{z}} d\bar{z}\; \bar{N}\widetilde{W}^a(\bar{z},t), \tag{31}$$

where we denote the Minkowskian 1-trajectory by

$$X_1^a(t) = B_1^a + V_1^a T_1(t) \quad (V_1^a \equiv P_1^a/E_1). \tag{32}$$

By then inserting the ansatz (30) into Eq. (31) we obtain

$$X^a = B_1^a + V_1^a T_1 + R(\xi(t)) \int_0^{\zeta(z,t)} \frac{d\zeta}{\zeta(1-\zeta)} W^a(\zeta) + \bar{R}(\bar{\xi}(t)) \int_0^{\bar{\zeta}(z,t)} \frac{d\bar{\zeta}}{\bar{\zeta}(1-\bar{\zeta})} \widetilde{W}^a(\bar{\zeta}), \tag{33}$$

¶We are indebted to Camillo Imbimbo for a discussion on this point. The presence of zeros could also be cancelled by additional spurious singularities in $f(z,t)$. We are assuming here a minimal set related to the particle sites, which appears appropriate in the c.m. system.

and, by a time-derivative,

$$A^a = V_1^a \dot{T}_1 + \partial_t \left(R(\xi(t))\, I^a(0, \zeta(z,t)) + \bar{R}(\bar{\xi}(t)) \tilde{I}^a(0, \bar{\zeta}) \right), \tag{34}$$

where we have introduced the notation

$$I^a(0,\zeta) = \int_0^\zeta \frac{d\zeta}{\zeta(1-\zeta)} W^a(\zeta) \tag{35}$$

and a similar one for $\widetilde{W}$.

The expression for A in Eq. (34) satisfies the consistency condition (8) by construction, and the monodromy vector transformation by inspection. Furthermore, the z-dependence in Eqs. (33) and (34) is embodied in the integrals $I^a(0, \zeta(z,t))$, which in turn are determined by the functional form of $f(\zeta)$ in Eqs. (24) and (25).

So far, the time-dependent residue function $R(\xi(t))$ in Eq. (34) appears undetermined and so is, therefore, the relative motion trajectory $\xi(t)$. In fact, we have still to insure that we are not in a rotating frame at space infinity or , in other words, that the affine connection vanishes fast enough asymptotically. This asymptotic condition implies that $A^a(t, z, \bar{z})$ should be at most logarithmic, for large $|z|$, and therefore by Eq. (34), that

$$R(\xi)\ \zeta \partial_\zeta I^a(0,\zeta) \to I^a(0,\zeta)\ \xi \partial_\xi R(\xi) \quad , \quad \left(|\zeta| \simeq \left| \frac{z}{\xi} \right| \gg 1 \right). \tag{36}$$

On the other hand, it is easy to check that, by Eqs. (24) and (25), I^a increases as

$$I^a(0,\zeta) \simeq C^a \zeta^{1-\mathcal{M}/2\pi} \quad , |\zeta| \gg 1, \quad 0 < \mathcal{M} < 2\pi, \tag{37}$$

where $\mathcal{M}$ is the invariant mass (27). Therefore, by Eq. (36) we must require

$$R(\xi(t)) = C(\xi(t))^{1-\frac{\mathcal{M}}{2\pi}} \tag{38}$$

which determines R up to a scale factor, and thus N, A^a and the metric as functions of $\zeta(z,t)$, $\xi(t)$ and of the constants of motion.

Finally, we have still to use the equations of motion for particle 2, which in integrated form read, by Eq. (33),

$$B_2^a - B_1^a + T_2 V_2^a - T_1 V_1^a = C \xi^{1-\mathcal{M}/2\pi} I^a(0,1) + \bar{C} \bar{\xi}^{1-\mathcal{M}/2\pi} \tilde{I}^a(0,1). \tag{39}$$

Since I^a and $\tilde{I}^a$ are calculable constants, functions of P_1^a and P_2^a, Eq. (39) determines the relative time variable $T_1(t) - T_2(t)$ and the trajectory $\xi(t)$ up to an overall time reparametrization and a scale freedom provided by C.

Without discussing the solution of Eq. (39) in detail, it is rather clear that $\xi^{1-\mathcal{M}/2\pi}$ should have, for large times, the same phase as $(V_1 - V_2)t + iB$, where B is the relative impact parameter. It follows that ξ should rotate by $\pi(1-\mathcal{M}/2\pi)^{-1}$ as time varies from $-\infty$ to $+\infty$, and that the corresponding scattering angle is

$$\Theta(\mathcal{M}) = \frac{\mathcal{M}}{2}(1-\frac{\mathcal{M}}{2\pi})^{-1} \tag{40}$$

consistently with an early suggestion by 't Hooft [4].

We have so far analyzed in detail the two-body case. However, the method just outlined applies in general to N particles, provided we are able to solve the monodromies (13) by the auxiliary function f, transforming as in Eq. (15).

For N particles, we expect that the corresponding second-order differential equation should have at least $N+1$ regular singularities, one of which at infinity. Since only the difference of indices, related to physical masses, matters for the branch point behaviour of f, it seems that $N+1$ singularities are not enough for $N > 2$: they provide $2N-1$ parameters, instead of the $3N-3$ which are needed (N three-momenta with $O(2,1)$ invariance). Hence, for $N > 2$, some extra singularities are expected in the Schwartzian derivative [17] of f, which are not branch points of f, but rather zeros of f' .

Several comments are in order. First of all, the basic simplification which allows to deal with the monodromy properties in a single complex plane is rooted in the 3-dimensional nature of the problem, according to which the Coulomb condition in Eq. (5) implies the analyticity (harmonicity) properties in Eq. (7) (Eq. (8)). This in turn is equivalent to the instantaneous propagation in a second-order formalism [13], and is due to the absence of physical (transverse) gravitons. For this reason the time-evolution is coupled to the z dependence only through the rescaled variable $\zeta(z,t)$.

Secondly, the general method above can be explicitly checked by the perturbative calculations available for (i) first non-trivial order in V_i and all-orders in G [12] and (ii) second-order in G and any speed [13]. For instance, in the first case we find, from Ref. [12],

$$\begin{aligned} f(z,t) &= \frac{1}{2}(\bar{V}_1-\bar{V}_2)\int_0^\zeta dt\; t^{m_1/2\pi-1}(1-t)^{m_2/2\pi-1}\; B^{-1}(\frac{m_1}{2\pi},\frac{m_2}{2\pi}) - \frac{1}{2}\bar{V}_1, \\ R(\xi) &= \frac{1}{2}(\bar{V}_1-\bar{V}_2)B^{-1}(\frac{m_1}{2\pi},\frac{m_2}{2\pi})\xi^{1-\frac{m_1+m_2}{2\pi}}, \end{aligned} \tag{41}$$

and explicit expressions, involving hypergeometric functions, for A and a. The calculable parameters of Eq. (39) are in this case, to first order in V_i's ,

$$\begin{aligned} CI^a(0,1) &= \left(i\,\frac{V_1-V_2}{2}\,J - i\,\frac{V_1}{2}I \;/\; I \;/\; 0\right), \quad I = B\left(1-\frac{m_1}{2\pi},1-\frac{m_2}{2\pi}\right)\;, \\ J &= B^{-1}\left(\frac{m_1}{2\pi},\frac{m_2}{2\pi}\right)\frac{\psi(1-m_2/2\pi)-\psi(m_1/2\pi)}{1-m_1/2\pi-m_2/2\pi}. \end{aligned} \tag{42}$$

The check above still leaves open the question about the classification and the physical meaning of alternative determinations for $f(z,t)$, corresponding to different choices of n's in Eq. (26). This gives rise to different behaviours around the singularity points, and in particular $z = \infty$, around which asymptotic conditions of type in Eq. (36) seem to work only in a limited mass range (e.g. , $0 < \mathcal{M} < 2\pi$).

In addition, one should mention the possible (non-perturbative) zero of the determinant (29), which could occur at $|f| = 1$. With our choice of indices one can show, by Eq.(25), that $|f_{(1)}(\infty)|, |f_{(1)}(1)| < 1$ in the naive mass range $0 < m_1, m_2, \mathcal{M} < 2\pi$, hence by the maximum modulus theorems [18] $|f_{(1)}(z)| < 1$ on the first sheet of the cut z-plane, and thus $|f(z)| < 1$ on any sheet , because Eq. (15) preserves this inequality. Therefore, for proper values of the masses, there are no problems with our choice.

On the other hand, if $P_1 \cdot P_2$ exceeds some critical value, the invariant mass takes the form $\mathcal{M} = 2\pi + i\sigma$ ($cos(\mathcal{M}/2) < -1$) and closed timelike curves appear [19]. In the same situation, since

$$f_{(1)}(\infty)^2 = \frac{sin\ \frac{1}{4}\ (\mathcal{M} - m_1 - m_2)\ sin\ \frac{1}{4}\ (\mathcal{M} - m_1 + m_2)}{sin\ \frac{1}{4}\ (\mathcal{M} + m_1 + m_2)\ sin\ \frac{1}{4}\ (\mathcal{M} + m_1 - m_2)}\ , \tag{43}$$

it is easy to realize that $|f(\infty)|^2 = 1$, and the gauge choice may become pathological. Thus the restriction to $cos(\mathcal{M}/2) > -1$ is natural, and also avoids CTC's.

Finally, let us remark that in our gauge, because of the instantaneous propagation, the particles interact at all times, making decoupling properties rather difficult to handle. For instance, comparing the expression (25) with Eq. (18), we see that only in the regions $|z - \xi_1| \ll |\xi|$ $(|z - \xi_2| \ll |\xi|)$ does the interacting metric look like the single particle ones. In all other regions they considerably differ, at all times.

This feature is to be contrasted with what happens in covariant-type gauges [9] in which the metric decouples in two single-particle ones at large times. In particular, in the present case, the massless limit, which exists with some care, does not correspond to shock-wave scattering of Aichelburg-Sexl type.

As a consequence, the local space-time properties, and thus the scattering angle in Eq. (40), appear to be different than the ones in covariant gauges [9]. Whether this fact is to be related with the lack of true asymptotic decoupling [20] in this gauge is an interesting question still to be investigated.

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