hep-th/9507075
KOBE-TH-95-02
July 1995

Dual Polyhedra, Mirror Symmetry and Landau-Ginzburg Orbifolds

Hitoshi Sato

Graduate School of Science and Technology, Kobe University
Rokkodai, Nada, Kobe 657, Japan
email address : UTOSA@JPNYITP.BITNET

ABSTRACT

New geometrical features of the Landau-Ginzburg orbifolds are presented, for models with a typical type of superpotential. We show the one-to-one correspondence between some of the (a, c) states with $U(1)$ charges $(-1, 1)$ and the integral points on the dual polyhedra, which are useful tools for the construction of mirror manifolds. Relying on toric geometry, these states are shown to correspond to the $(1, 1)$ forms coming from blowing-up processes. In terms of the above identification, it can be checked that the monomial-divisor mirror map for Landau-Ginzburg orbifolds, proposed by the author, is equivalent to that mirror map for Calabi-Yau manifolds obtained by the mathematicians.

Mirror symmetry was first discovered in the context of string compactification [1, 2, 3]. Due to the relative sign of the two $U(1)$ charges, one $(2,2)$ superconformal field theory allows two geometrical interpretations, i.e. topologically distinct Calabi-Yau manifolds $\mathcal{M}$ and $\mathcal{W}$. Assuming that mirror symmetry is true, some Yukawa couplings can be determined exactly [4, 5, 6]. However, recent analysis of mirror symmetry is purely geometrical.

Batyrev [7] proposed a powerful method for constructing the mirror manifolds of a certain class of Calabi-Yau manifolds. He showed that a pair of (Δ, Δ^*) gives a Calabi-Yau manifold, where Δ is a (Newton) polyhedron corresponding to monomials and Δ^* is a dual (or polar) polyhedron describing the resolution of singularities, i.e. a point on a one- or two-dimensional face of Δ^* corresponds to a $(1,1)$ form coming from resolution. Batyrev observed that the exchange of the roles of (Δ, Δ^*) produces a mirror manifold.

Landau-Ginzburg models of $N=2$ superconformal field theories are closely related to Calabi-Yau manifolds because of their (anti-)chiral ring structures. If we consider the theory with $c=9$, the (p,q) forms on a Calabi-Yau manifold can be identified with $(3-p,q)$ states of the (c,c) ring or $(-p,q)$ states of the (a,c) ring, where c (a) stands for (anti-)chiral and the states are labeled by the $U(1)$ charges. These (c,c) and (a,c) rings can be described in terms of the Landau-Ginzburg models.

In this paper, we will find a very simple relation between a $(-1,1)$ state and a point on a face of Δ^*, when a typical type of Landau-Ginzburg models are considered. Hence we can identify a $(-1,1)$ state and a $(1,1)$ form coming from blowing-up processes in a simple and exact way. Furthermore, we will show that the monomial-divisor mirror map for Landau-Ginzburg orbifolds proposed in ref.[8] is equivalent to that mirror map of Calabi-Yau manifolds [9, 5]. These are useful extensions of the results in the previous paper [8]. Our method gives us the possibility to study the new geometric content of a class of $N=2$ superconformal field theories.

In this paper, we will restrict our attention to the superpotential of a form $W(X_i) = X_1^{a_1}+X_2^{a_2}+X_3^{a_3}+X_4^{a_4}+X_5^{a_5}$, which corresponds to the Fermat type hypersurface in WCP^4. The Landau-Ginzburg orbifolds are obtained by quotienting with an Abelian symmetry group G of $W(X_i)$, whose element g acts as an $N \times N$ diagonal matrix, $g: X_i \to e^{2\pi i \tilde{\theta}_i^g} X_i$, where $0 \le \tilde{\theta}_i^g < 1$. Of course the $U(1)$ twist $j: X_i \to e^{2\pi i q_i} X_i$ generates the symmetry group of $W(X_i)$, where $q_i = \frac{w_i}{d}$, $W(\lambda^{w_i} X_i) = \lambda^d W(X_i)$ and $\lambda \in \mathbb{C}^*$. In this paper, we further require that $w_5 = 1$ since the toric description of the corresponding Calabi-Yau mirror manifolds are well-known [7, 5].

Using the results of Intriligator and Vafa [10], we can construct the (c,c) and (a,c)

rings. Also we could have the left and right $U(1)$ charges of the ground state $|h\rangle_{(\mathrm{a,c})}$ in the h-twisted sector of the (a,c) ring. In terms of spectral flow, $|h\rangle_{(\mathrm{a,c})}$ is mapped to the (c,c) state $|h'\rangle_{(\mathrm{c,c})}$ with $h' = hj^{-1}$. Then the charges of the (a,c) ground state of h-twisted sector $|h\rangle_{(\mathrm{a,c})}$ are obtained to be

$$\begin{pmatrix} J_0 \\ \bar{J}_0 \end{pmatrix} \quad |h\rangle_{(\mathrm{a,c})} = \begin{pmatrix} -\sum_{\tilde{\theta}_i{}^{h'}>0}(1-q_i-\tilde{\theta}_i{}^{h'}) + \sum_{\tilde{\theta}_i{}^{h'}=0}(2q_i-1) \\ \sum_{\tilde{\theta}_i{}^{h'}>0}(1-q_i-\tilde{\theta}_i{}^{h'}) \end{pmatrix} \quad |h\rangle_{(\mathrm{a,c})}. \qquad (1)$$

Using this result, we see that the $(-1,1)$ states written in the form $\left|j^l\right\rangle_{(\mathrm{a,c})}$ can always arise from the twisted sector with $I' = 0$, where I' is the number of the invariant fields X_i under the h' action. Using the results of ref.[11], it was shown [8] that as long as we consider the Landau-Ginzburg models with no or one trivial field, the $(-1,1)$ states which can be represented by $\left|j^l\right\rangle_{(\mathrm{a,c})}$ may exist only in the twisted sector with $I' = 0$.

Let us turn our attention to geometry. Calabi-Yau manifolds are represented by hypersurfaces in WCP. In general, due to the WCP identification $z_i \sim \lambda^{w_i} z_i, \ \ \lambda \in \mathbb{C}^*$, we have some fixed sets on a hypersurface. When we consider Calabi-Yau 3-folds, possible fixed sets are fixed points and fixed curves. To obtain a smooth Calabi-Yau manifold we have to blow up these singularities.

Those Calabi-Yau resolutions can be described in terms of toric geometry [7, 5]. Toric geometry describes the structure of a certain class of geometrical spaces in terms of simple combinatorial data. To investigate the mirror symmetry, Batyrev's construction is useful. We will briefly summarize this method. Details are presented in [7, 5].

A (Newton) polyhedron $\Delta(w)$ is associated to monomials, where w means the set of weights w_i. A dual polyhedron $\Delta^*(w)$ allows us to describe the resolution of singularities. Integral points on faces of dimension one or two of $\Delta^*(w)$ correspond to exceptional divisors. More precisely, points lying on a one-dimensional edge correspond to exceptional divisors over singular curves, whereas the points lying in the interior of two-dimensional faces correspond to the exceptional divisors over singular points. So, integral points on faces of $\Delta^*(w)$ correspond to the $(1,1)$ forms coming from blowing-up processes.

Since we consider the Fermat type quasihomogeneous polynomial, the corresponding Calabi-Yau hypersurface consists of monomials z_i^{d/w_i} $(i = 1, \cdots, 5)$. The associated 4-dimensional integral convex polyhedron $\Delta(w)$ is the convex hull of the integral vectors m of the exponents of all quasi-homogeneous monomials of degree d shifted by $(-1, \ldots, -1)$, i.e. $\prod_{i=1}^{5} z_i^{m_i+1}$:

$$\Delta(w) := \{(m_1,\ldots,m_5) \in \mathbb{R}^5 | \sum_{i=1}^{5} w_i m_i = 0, m_i \geq -1\}. \tag{2}$$

This implies that only the origin is the point in the interior of Δ. Its dual polyhedron is defined by

$$\Delta^* = \{\ (x_1,\ldots,x_4) \mid \sum_{i=1}^{4} x_i y_i \geq -1 \text{ for all } (y_1,\ldots,y_4) \in \Delta\ \}. \tag{3}$$

In our case it is known that (Δ, Δ^*) is a reflexive pair. An l-dimensional face $\Theta \subset \Delta$ can be represented by specifying its vertices $\mathrm{v}_{i_1}, \cdots, \mathrm{v}_{i_k}$. Then the dual face Θ^* is a $(4-l-1)$-dimensional face of Δ^* and defined by

$$\Theta^* = \{x \in \Delta^* \mid (x, \mathrm{v}_{i_1}) = \cdots = (x, \mathrm{v}_{i_k}) = -1\}, \tag{4}$$

where $(*, *)$ is the ordinary inner product.

For our type of models, we then always obtain as vertices of $\Delta(w)$

$$\begin{aligned} &\nu_1 = (d/w_1 - 1, -1, -1, -1),\ \nu_2 = (-1, d/w_2 - 1, -1, -1),\ \nu_3 = (-1, -1, d/w_3 - 1, -1), \\ &\nu_4 = (-1, -1, -1, d/w_4 - 1),\ \nu_5 = (-1, -1, -1, -1), \end{aligned} \tag{5}$$

and for the vertices of the dual polyhedron $\Delta^*(w)$ one finds

$$\begin{aligned} \nu_1^* &= (1,0,0,0), \quad \nu_2^* = (0,1,0,0), \quad \nu_3^* = (0,0,1,0), \quad \nu_4^* = (0,0,0,1), \\ \nu_5^* &= (-w_1, -w_2, -w_3, -w_4). \end{aligned} \tag{6}$$

For the Fermat type hypersurfaces of degree d, the explicit form of the monomial-divisor mirror map has been already studied. Through this map, integral points μ in $\Delta^*(w)$ are mapped to monomials of the homogeneous coordinates of WCP^4 by

$$\mu = (\mu_1, \mu_2, \mu_3, \mu_4) \mapsto \frac{\prod_{i=1}^{4} z_i^{\mu_i d/w_i}}{\left(\prod_{i=1}^{5} z_i\right)^{(\sum_{i=1}^{4} \mu_i) - 1}}. \tag{7}$$

In the following we will associate an integral point inside $\Delta^*(w)$, i.e. an exceptional divisor, with a $(-1,1)$ state which can be written in the form $\left|j^{-l}\right\rangle_{(\mathrm{a,c})}$.

To explain our method, we define the phase symmetries ρ_i which act on X_i as

$$\rho_i X_i = e^{2\pi i q_i} X_i, \tag{8}$$

with trivial action for other fields. The operator ρ_i can be represented by a diagonal matrix whose diagonal matrix elements are 1 except for $(\rho_i)_{i,i} = e^{2\pi i q_i}$. It is obvious that

$$j = \rho_1 \cdots \rho_5. \tag{9}$$

In ref.[8] the mirror map for the (a,c) ground states in the j^{-l}-twisted sector $\left|j^{-l}\right\rangle_{(a,c)}$ are considered. In the j^{-l}-twisted sector, if a field X_i is invariant then

$$\rho_i^{-l} = \rho_i^{-l_i} = \text{identity}, \tag{10}$$

where $-l_i \equiv -l \bmod a_i$ and one gets

$$j^{-l} = \prod_{-lq_i \notin \mathbf{Z}} \rho_i^{-l_i}. \tag{11}$$

So, we may represent $\left|j^{-l}\right\rangle_{(a,c)} = \left|\prod_{-lq_i \notin \mathbf{Z}} \rho_i^{-l_i}\right\rangle_{(a,c)}$. Furthermore, we can calculate the $U(1)$ charges of this state using eq.(1) and the result is

$$\Big(-\sum_{-lq_i \notin \mathbf{Z}} l_i q_i, \sum_{-lq_i \notin \mathbf{Z}} l_i q_i\Big). \tag{12}$$

The eq.(11) is the key equation for our purpose. This implies

$$-lq_i = u_i - l_i q_i \qquad \text{for } i = 1 \sim 5, \tag{13}$$

where $-l_i$ are defined to be $-a_i + 1 \leq -l_i \leq 0$. Thus u_i are uniquely determined. Clearly, u_i are integers and $u_i = 0$ if X_i is invariant under j^{-l} action (see eq.(11)). u_5 always vanish since $w_5 = 1$.

Let $u \equiv (u_1, u_2, u_3, u_4)$ be an integral vector. In the following we will show that u is on a face of $\Delta^*(w)$. So we should assert that u is just an integral point inside $\Delta^*(w)$, which can be identified with the exceptional divisors. Through this identification, we obtain the one-to-one correspondence between the $(-1,1)$ state $\left|j^{-l}\right\rangle_{(a,c)}$ and the exceptional divisor. Moreover, once this identification is made, we can see that the monomial-divisor mirror map for Calabi-Yau manifolds, i.e. eq.(7), is equivalent to that mirror map for Landau-Ginzburg orbifolds conjectured in ref.[8].

First, we show that u is a point on a dual face Θ^* of $\Delta^*(w)$. A dual face Θ^* is specified by some of the vertices $\{\nu_1, \nu_2, \nu_3, \nu_4, \nu_5\} \subset \Theta$ through the eq.(4). So, we have to show that there are some vectors ν_j satisfying

$$(u, \nu_j) = -1 \qquad \text{for some } j. \tag{14}$$

To prove this, we consider the invariant field X_j under j^{-l} action. This implies $-lq_j = u_j$. It can be shown that the corresponding vector ν_j satisfies eq.(14). Denote by $(\nu_j)_i$ the i-th component of the vector ν_j. Multiplying u_i by $(\nu_j)_i$ and using eq.(13), one finds

$$(u, \nu_j) = -\sum_{i=1}^{5} l_i q_i, \tag{15}$$

where we have used $\sum_{i=1}^{5} q_i = 1$, $u_5 = 0$ and $-lq_j = u_j$.

It should be noticed that the right hand side of eq.(15) is nothing but the left $U(1)$ charge of the state $\left|j^{-l}\right\rangle_{(\mathrm{a,c})}$. Since we consider the $(-1,1)$ state, it has been proved that eq.(14) holds for the vector ν_j with $-lq_j = u_j$. Note that Θ^* is specified by the vectors $\nu_j \subset \Theta$ through eq.(4), whose corresponding fields X_j with the same index are invariant under j^{-l} action.

Now we turn our attention to the monomial-divisor mirror map. As mentioned above, the monomial-divisor mirror map for Calabi-Yau manifolds is summarized in eq.(7). In ref.[8], it was conjectured that the monomial-divisor mirror map for Landau-Ginzburg orbifolds is obtained to be

$$\left|\prod_{-lq_i \notin \mathbf{Z}} \rho_i^{-l_i}\right\rangle_{(\mathrm{a,c})} \stackrel{\text{mirror pair}}{\longleftrightarrow} \prod_{-lq_i \notin \mathbf{Z}} \overline{X_i}^{l_i} |0\rangle, \tag{16}$$

where $\overline{X_i}$ are the fields of the so-called transposed potential $\overline{W}$ [12]. Evidently, $X_i = \overline{X_i}$ for our Fermat type potentials.

By identifying the vector μ in eq.(7) with our vector u, we easily obtain

$$\begin{aligned} u = (u_1, u_2, u_3, u_4) \quad &\mapsto \quad \frac{\prod_{i=1}^{4} z_i^{u_i d/w_i}}{\left(\prod_{i=1}^{5} z_i\right)^{(\sum_{i=1}^{4} u_i)-1}} \\ &= \quad \prod_{-lq_i \notin \mathbf{Z}} z_i^{l_i}. \end{aligned} \tag{17}$$

Since the vector u corresponds to the $(-1,1)$ state $\left|j^{-l}\right\rangle_{(\mathrm{a,c})}$ and z_i to $\overline{X_i}$, one finds that the two monomial-divisor mirror map, i.e. eq.(7) and (16), are equivalent.

Note that through the eqs.(13) and (17), the origin $\nu_0^* = (0,0,0,0)$ in $\Delta^*(w)$ corresponds to the $(-1,1)$ state $|j^{-1}\rangle_{(\mathrm{a,c})}$ and its mirror partner corresponds to the monomial $z_1 z_2 z_3 z_4 z_5$. They always exist for our type of models.

Let us demonstrate our method by taking some examples. Since we have shown $u \in \Theta^*$, we change our notation of u into ν_6^* (ν_7^*, $\cdots$ if several u exist).

First we consider the Landau-Ginzburg model with the superpotential

$$W_1 = X_1^4 + X_2^4 + X_3^4 + X_4^8 + X_5^8, \tag{18}$$

with $U(1)$ charges of X_i being

$$\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8} \right), \tag{19}$$

which were studied in ref.[5, 8]. The orbifold model W_1/j has a corresponding $\mathbb{Z}_2$ fixed curve which can be written

$$z_1^4 + z_2^4 + z_3^4 = 0 \quad \text{in } WCP^4_{(1,1,2,2,2)}. \tag{20}$$

In this model, we have one twisted ground state $|j^{-4}\rangle_{(\mathrm{a,c})}$ which corresponds to the resulting one $(1,1)$ form after blowing-up. Since $j^{-4} = \rho_4^{-4}\rho_5^{-4}$, we can calculate $\nu_6^* = (-1,-1,-1,0)$. Also it can be seen that ν_6^* is on the dual face Θ^* specified by the vectors ν_1, ν_2, ν_3 (in the following we denote by $\Theta^*(1,2,3)$ this face). From eq.(16), ν_6^* is mapped to the monomial $\overline{X_4^4 X_5^4}$. These are the same results as the ones obtained in [5].

As a more complicated example, we take the following Landau-Ginzburg superpotential

$$W_2 = X_1^3 + X_2^3 + X_3^6 + X_4^9 + X_5^{18}, \tag{21}$$

with $U(1)$ charges

$$\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{6}, \frac{1}{9}, \frac{1}{18} \right). \tag{22}$$

This model is considered in [6, 8]. The orbifold model W_2/j has one corresponding $\mathbb{Z}_2$ fixed curve, one corresponding $\mathbb{Z}_3$ fixed curve and corresponding $\mathbb{Z}_6$ fixed points on the intersections of these curves. They can be written as

$$\mathbb{Z}_2 \text{ fixed curve } z_1^3 + z_2^3 + z_4^9 = 0 \tag{23}$$

$$\mathbb{Z}_3 \text{ fixed curve } z_1^3 + z_2^3 + z_3^6 = 0 \tag{24}$$

$$\mathbb{Z}_6 \text{ fixed points } z_1^3 + z_2^3 = 0 \quad \text{in } WCP^4_{(6,6,3,2,1)}. \tag{25}$$

After blowing-up, one obtains four $(1,1)$ forms whose mirror partners can be associated to monomial deformation. It is easy to find the corresponding $(-1,1)$ states written in the form $\left|j^{-l}\right\rangle_{(\mathrm{a,c})}$, the vectors ν_j^*, dual faces Θ^* on which ν_j^* are lying, and mirror partners. The results are displayed in Table 1, where we have omitted the bar over X_i.

This result agrees with the one obtained in [6] . Note that we do not need any geometrical informations such as the number of fixed sets or the relations among them.

(a,c) state	ν_j^* vector	dual face	mirror partner
$\|j^{-3}\rangle_{(a,c)}$	$\nu_6^* = (-1,-1,0,0)$	$\Theta^*(1,2)$	$X_3{}^3 X_4{}^3 X_5{}^3 \|0\rangle$
$\|j^{-6}\rangle_{(a,c)}$	$\nu_7^* = (-2,-2,-1,0)$	$\Theta^*(1,2,3)$	$X_4{}^6 X_5{}^6 \|0\rangle$
$\|j^{-9}\rangle_{(a,c)}$	$\nu_8^* = (-3,-3,-1,-1)$	$\Theta^*(1,2,4)$	$X_3{}^3 X_5{}^9 \|0\rangle$
$\|j^{-12}\rangle_{(a,c)}$	$\nu_9^* = (-4,-4,-2,-1)$	$\Theta^*(1,2,3)$	$X_4{}^3 X_5{}^{12} \|0\rangle$

Table 1: The monomial-divisor mirror map for Landau-Ginzburg orbifolds of W_2

In this model, there are two $(-1,1)$ states represented by $X_1|j^{-2}\rangle_{(a,c)}$ and $X_2|j^{-2}\rangle_{(a,c)}$. The mirror partners for the corresponding $(1,1)$ forms cannot be described by monomials. Unfortunately, we have not succeeded in associating these states with toric data yet.

In conclusion, we have discovered a simple and direct connection between the Landau-Ginzburg and the toric descriptions of our class of Calabi-Yau manifolds. In other words, we find that a Landau-Ginzburg orbifold and a corresponding Calabi-Yau manifold have just the same toric data, as far as models with a typical type of superpotentials are concerned. Our method enables us to calculate toric data easily without referring to the geometrical nature of $\Delta^*(w)$. We can uniquely identify a $(-1,1)$ state $\left|j^{-l}\right\rangle_{(a,c)}$ with a $(1,1)$ form resulting from resolution. This would be a useful technique for analyzing the Yukawa couplings, especially when Calabi-Yau manifolds with several $(1,1)$ forms are considered.

The Fermat type superpotential considered in this paper corresponds to the Gepner model of A-type [13]. However, there are non-Fermat-type potentials with only the singularities which can be treated through toric geometry. For example, there is the hypersurface embedded in $WCP^4_{(1,2,2,2,3)}$, which gets two toric divisors after blowing up [14]. For this model, our method for the calculation of the vector u could not be applied as it is. Some extensions are needed. We will report it elsewhere [15].

Acknowledgements : The author would like to thank C.S. Lim for careful reading of this manuscript.

References

[1] B.R. Greene and M.R. Plesser, Nucl. Phys. **B338** (1990) 15.

[2] W. Lerche, C. Vafa and N.P. Warner, Nucl. Phys. **B324** (1989) 427.

[3] L. Dixon, in *Superstrings, Unified Theories and Cosmology 1987*, G. Furlan et.al.,eds., World Scientific, Singapore (1988).

[4] P. Candelas, X. de la Ossa, P.S. Green and L. Parkes, Phys. Lett. **B258** (1991) 118 and Nucl. Phys. **B359** (1991) 21.

[5] H. Hosono, A. Klemm, S. Theisen and S.-T. Yau, Commun. Math. Phys. **167** (1995) 301.

[6] P.S. Aspinwall, B.R. Greene and D. Morrison, Phys. Lett. **B303** (1993) 249 , Nucl. Phys. **B416** (1993) 414.

[7] V. Batyrev, J. Alg. Geom. **3** (1994) 493.

[8] H. Sato, Mod. Phys. Lett. **A9** (1994) 3721.

[9] P.S. Aspinwall, B.R. Greene and D. Morrison, Int. Math. Res. Notices (1993) 319.

[10] K. Intriligator and C. Vafa, Nucl. Phys. **B339** (1990) 95; C. Vafa, Mod. Phys. Lett. **A4** (1989) 1169.

[11] H. Sato, Mod. Phys. Lett. **A9** (1994) 885.

[12] P. Berglund and T. Hübsch, Nucl. Phys. **B 393** (1993) 377.

[13] D. Gepner, Phys. Lett. **B199** (1987) 380 , Nucl. Phys. **B296** (1988) 757.

[14] P. Berglund and S. Katz, to appear in *Essays on Mirror Manifolds II* (Ed. S.-T. Yau), hep-th/9406008

[15] H. Sato, in preparation.