CERN-TH/95-195
UWThPh-20-1995
hep-th/9507074

Field Theory on a Supersymmetric Lattice

H. Grosse[1]

Institute for Theoretical Physics, University of Vienna ,
Boltzmanngasse 5, A-1090 Vienna, Austria,

C. Klimčík

Theory Division CERN, CH-1211 Geneva 23,
Switzerland

and

P. Prešnajder

Dept. of Theoretical Physics, Comenius University,
Mlynská dolina F1, SK-84215 Bratislava, Slovakia

Abstract

A lattice-type regularization of the supersymmetric field theories on a supersphere is constructed by approximating the ring of scalar superfields by an integer-valued sequence of finite dimensional rings of supermatrices and by using the differencial calculus of non-commutative geometry. The regulated theory involves only finite number of degrees of freedom and is manifestly supersymmetric.

CERN-TH/95-195
July 1995

[1]Part of Project No. P8916-PHY of the 'Fonds zur Förderung der wissenschaftlichen Forschung in Östereich'.

1 Introduction

The idea that a fine structure of space-time should be influenced by quantum gravity phenomena is certainly not original but so far there was a little success in giving it more quantitative expression. String theory constitutes itself probably the most promising avenue to a consistent theory of quantum gravity it is therefore of obvious interest to study the structure of spacetime from the point of view. Though string theory incorporates a minimal lenght the physical quantities computed in its framework reflect the symmetry properties of continuous space-time. The situation is somewhat analogous to ordinary quantum mechanics: though the phase space acquires itself a cell-like structure its symmetries remain intact, in general. In a sense the space-time possesses the cell-like structure also in string theory e.g. the quantum WZNW model for a compact group has as effective target, perceived by a string center of mass, a truncated group manifold or, in other words, a 'manifold' with a cell-like structure (see [1]). Indeed, the zero-modes' subspace of the full Hilbert space contains only the irreducible representations of a spin lower than the level k. Because this subspace describe the scalar excitations, it is clear that high frequency (or spin) modes in an effective field theory are absent. In this way string theory leads to the UV finite behaviour of physical amplitudes as was probably realized by several researchers in past (e.g.[2]).

In our contribution we would like to initiate an investigation of similar regularization in pure field theory context. That is we wish to consider fields living on truncated compact manifolds, endow them with dynamics and establish rules of their quantization. Among advantages of such a development, there would be not only the manifest preservation of all symmetries of a theory but also an expected compatibility with quantum gravity and string phenomena. In some sense we shall construct a lattice-type of regularization but the 'lattice' will not approximate the underlying spacetime (and hence the ring of functions on it) but directly the ring. As the starting point of our treatment we choose a 2d field theory on a truncated two-sphere [2].

The truncated sphere was extensively studied in past two decades for various reasons. Apparently, the structure was introduced by Berezin in 1975 [3] who quantized the (symplectic) volume two-form on the ordinary

[2]also referred to as "fuzzy", "non-commutative" or "quantum" sphere in literature [5, 4, 3].

two-sphere. He ended up with a series of possible quantizations parametrized by the size of quantum cells. In 1982, Hoppe [4] investigated properties of spherical membranes. As a technical tool he introduced the truncation of high frequency excitations which effectively lead to the quantum sphere. In 1991 the concept was reinvented by Madore [5] (see also [6]). His motivation originated in the so-called non-commutative geometry, i.e. the generalization of the ordinary differential geometry to non-commutative rings of 'functions'. The truncated algebra of ordinary functions is just the example of such a non-commutative ring.

For our purposes, we shall use the results of all those previous works, however, we shall often put emphasis on different aspects of formalism as comparing to the previous investigations. Our main concern will consist in developing basic differential and integral structures for non-commutative sphere which are needed to define a classical (and quantum) field dynamics. We shall require that the symmetries of the undeformed theory are preserved in the non-commutative deformation such as space-time supersymmetry, global isospin, local (non)abelian gauge or chiral symmetry[3] and, obviously, that the commutative limit should recover the standard formulation of the dynamics of the field theory.

In many respects a canonical procedure for endowing non-commutative rings with differential and integral calculus is known for several years from basic studies of A. Connes [7]. From his work it follows that geometrical properties of a non-commutative manifold are encoded in a fundamental triplet (A, H, D) where A is the representation of a non-commutative algebra $\mathcal{A}$ of 'functions' on the manifold in some Hilbert space. Elements of A are linear operators acting on H in such a way that the multiplication of elements of the 'abstract' algebra $\mathcal{A}$ is represented by the composition of the operators from A which represent them. D is a self-adjoint operator (called the Dirac operator) odd with respect to an appropriate grading[4] H is interpreted as a spinor bundle over the non-commutative manifold and the action of the algebra A on it makes possible to define the action of a (truncated) gauge group on spinors.

Noncommutative geometry has been already applied in theoretical physics

[3]An attempt to formulate a field theory on the fuzzy sphere was published in [5, 9, 10]. However, the crucial concept of chirality was not studied there.

[4]We ignore in this paper aspects concerning the norms of the operators from A and commutators of the form $[D, A]$ because all algebras we consider are finite-dimensional.

by providing the nice geometrical description of the standard model action including the Higgs fields [7, 8]. The latter were interpreted as the components of a noncommutative gauge connection. Starting in this paper, we hope to provide another relevant application of non-commutative geometry with the aim to understand the short distance behaviour of field theory. We believe that non-commutative geometry can provide powerful technical tools for performing new and nontrivial relevant calculations.

In the present contribution, we construct the fundamental triplet (A, H, D) and use the construction for developing the supersymmetric regularization of field theories. Though the uniqueness of (A, H, D) for a given fundamental algebra $\mathcal{A}$ is by no means guaranteed we give a highly natural choice stemming from the following construction. First we give a suitable description of spinors on the ordinary sphere as components of a scalar superfield on a supersphere. Then we represent the standard Dirac operator on the sphere in terms of the superdifferential generators of $OSp(2,1)$ algebra which is the supersymmetry superalgebra of the supersphere. The standard Dirac operator on the sphere turns out to be nothing but the fermionic part of the Casimir of $OSp(2,1)$ written in the superdifferential representation (the bosonic part is the standard Laplace operator on the sphere). Then we shall mimick the same construction for the non-commutative sphere. We describe spinors on the non-commutative sphere as the suitable components of a scalar superfield on a non-commutative supersphere. In other words, we perform the supergeometric Berezin-like quantization of the supersphere[5] but in the language of Madore. The resulting quantized ring of scalar superfields will reveal a cell-like structure of the non-commutative supersphere. The algebra $\mathcal{A}$ will be the enveloping algebra of $OSp(2,1)$ in its irreducible representation with a spin $j/2$. As $j \to \infty$ one recovers the standard ring of superscalar functions on the supersphere. The quantized ring constitutes itself the representation space of the adjoint action of $OSp(2,2)$ in the irreducible representation with the $OSp(2,1)$ superspin $j/2$. We postulate that the fermionic part of the $OSp(2,1)$ Casimir in this adjoint representation is the Dirac operator on the non-commutative sphere. We shall find that it is selfadjoint and odd. We shall compute its complete spectrum of eigenvalues and eigenfunctions and find a striking similarity with the commutative case. Namely, the non-

[5]Recently several papers have appeared dealing with supergeometric quantization of the Poincaré disc [11, 12, 13].

commutative Dirac operator turns out simply to be a truncated commutative one![6] We then construct both Weyl (chiral) and Majorana fermions.

The building of the supersymmetric theories requires even more structure. We shall demonstrate that enlarging the superalgebra $OSp(2,1)$ to $OSp(2,2)$ the additional odd generators can be identified with the supersymmetric covariant derivatives and the additional even generator with the grading of the Dirac operator. All encountered representations of $OSp(2,1)$ will turn out to be also the representations of $OSp(2,2)$.

In the following section (which does not contain original results) we repeat the known construction of the standard non-commutative sphere in a language suitable for SUSY generalization. In section 3 we give the full account of the spectrum of the standard Dirac operator on the commutative sphere. Though not the results themselves, but the (algebraic) method of their derivation is probably new and very suitable for the later non-commutative analysis. From the fourth section we present original results. We start with the description of the (untruncated) Dirac operator in terms of the fermionic part of the $OSp(2,1)$ Casimir acting on the ring of superfields on the supersphere and we quantize that ring. Then we identify the Dirac operator on the non-commutative sphere, give full account of its spectrum and describe the grading of the non-commutative spinor bundle, completing thus the construction of the fundamental triplet (A, H, D). In section 5 we apply the developed constructions in (supersymmetric) field theories. We shall construct (super)symmetric action functionals of the deformed theories containing only finite number of degrees of freedom. We finish with conclusions and outlook concerning the construction of a noncommutative de Rham complex, a non-commutative gauge connection, chiral symmetry, dynamics of gauge fields and construction of twisted bundles over the non-commutative sphere needed for the description of 'truncated' monopoles.

[6]This suggests, in turn, that in the regulated field theory one should avoid the problem of fermion doubling [14].

2 The non-commutative sphere

2.1 The commutative warm-up

A very convenient manifestly $SU(2)$ invariant description of the (L_2-normed) algebra of functions $\mathcal{A}_\infty$ on the ordinary sphere can be obtained by factorizing the algebra $\mathcal{B}$ of analytic functions of three real variables by its ideal $\mathcal{I}$, consisting of all functions of a form $h(x^i)(\sum {x^i}^2 - \rho^2)$. The scalar product on $\mathcal{A}_\infty$ is given by[7]

$$(f,g)_\infty \equiv \frac{1}{2\pi\rho}\int_{R^3} d^3x^i \delta({x^i}^2-\rho^2)f^*(x^i)g(x^i), \qquad f,g\in\mathcal{A}_\infty \tag{1}$$

Here $f(x^i), g(x^i) \in \mathcal{B}$ are some representatives of f and g. The algebra $\mathcal{A}_\infty$ is obviously generated by functions[8] $x^i, i=1,2,3$ which commute with each other under the usual pointwise multiplication. Their norms are given by

$$||x^i||^2_\infty = \frac{\rho^2}{3}. \tag{2}$$

Consider the vector fields in R^3 generating $SU(2)$ rotations of $\mathcal{B}$. They are given by explicit formulae

$$R_j = -i\epsilon_{jkl}x^k\frac{\partial}{\partial x^l} \tag{3}$$

and obey the $SU(2)$ Lie algebra commutation relations

$$[R_i, R_j] = i\epsilon_{ijk}R_k \tag{4}$$

The action of R_i on $\mathcal{B}$ leaves the ideal $\mathcal{I}$ invariant hence it induces an action of $SU(2)$ on $\mathcal{A}_\infty$. The generators $x^i\in\mathcal{A}_\infty$ form a spin 1 irreducible representation of $SU(2)$ algebra under the action (hence they are linear combinations of the spherical functions with $l=1$). They fulfil an obvious relation

$${x^i}^2 = \rho^2. \tag{5}$$

Higher powers of x^i can be rearranged into irreducible multiplets corresponding to higher spins. For instance, the multiplet of spin l is conveniently constructed subsequently applying the lowering operator $R_- \equiv R_1 - iR_2$ on the

[7]The normalization ensures that the norm of the unit element of $\mathcal{A}_\infty$ is 1.

[8]Speaking more precisely, x^i denote the corresponding equivalence classes in $\mathcal{B}$.

highest weight vector x^{+l}. It is well-known (cf. any textbook on quantum mechanics) that the full decomposition of $\mathcal{A}_\infty$ into the irreducible representations of $SU(2)$ is given by the infinite direct sum

$$\mathcal{A}_\infty = 0 + 1 + 2 + \dots, \tag{6}$$

where the integers denote the spins of the representations.

2.2 The truncation of $\mathcal{A}_\infty$

We define the family of non-commutative spheres $\mathcal{A}_j$ by furnishing the truncated sum of the irreducible representations

$$\mathcal{A}_j = 0 + 1 + \dots + j, \tag{7}$$

with an associative product and a scalar product which in the limit $j \to \infty$ give the standard products in $\mathcal{A}_\infty$. To do this consider the space $\mathcal{L}(j/2, j/2)$ of linear operators from the representation space of the irreducible representation with the spin $j/2$ into itself. Clearly, $SU(2)$ algebra acts on $\mathcal{L}(j/2, j/2)$ by the adjoint action. This 'adjoint' representation is reducible and the standard Clebsch-Gordan series for $SU(2)$ [15] gives its decomposition

$$\mathcal{L}(j/2, j/2) = 0 + 1 + \dots + j \equiv \mathcal{A}_j. \tag{8}$$

The scalar product on $\mathcal{A}_j$ is defined by[9]

$$(f, g)_j \equiv \frac{1}{j+1}\mathrm{Tr}(f^* g), \qquad f, g \in \mathcal{A}_j, \tag{9}$$

and the associative product is defined as the standard composition of operators from the space $\mathcal{L}(j/2, j/2)$. Now we make more precise the notion of the commutative limits of the scalar product and the associative product. There is a natural chain of the linear embeddings of the vector spaces

$$\mathcal{A}_1 \hookrightarrow \mathcal{A}_2 \hookrightarrow \dots \hookrightarrow \mathcal{A}_j \hookrightarrow \dots \hookrightarrow \mathcal{A}_\infty \tag{10}$$

Any (normalized) element from $\mathcal{A}_j$ of the form

$$c_{j,lp} R_-^p X_j^{+l} \tag{11}$$

[9]The normalization ensures that the norm of the identity matrix is 1.

is mapped in an (normalized) element from $\mathcal{A}_k$ given by

$$c_{k,lp} R^p_- X_k^{+l}. \tag{12}$$

Here X_j^α are representatives of the $SU(2)$ generators in the irreducible representation with spin $j/2$ ($X_\infty^\alpha \equiv x^\alpha$). They are normalized so that

$$[X_j^m, X_j^n] = i \frac{\rho}{\sqrt{\frac{j}{2}(\frac{j}{2}+1)}} \epsilon_{mnp} X^p, \tag{13}$$

and $c_{j(k),lp}$ are the (real) normalization coefficients given by the requirement that the embedding conserves the norm. Note that X_j^{+l} are the highest weight vectors in $\mathcal{A}_j$. Because the adjoint action of the $SU(2)$ algebra is hermitian for arbitrary $\mathcal{A}_j$ (as it can be easily seen from the definitions of the scalar products (1),(9)) the embeddings are in fact isometric. Indeed, the scalar product of the eigenvectors of the hermitian operator vanishes if the corresponding eigenvalues are different. Obviously different l's give different eigenvalues of the (hermitian) adjoint Casimir. The commutative limit of the associative product is more involved, however[10]. Clearly, the embeddings cannot be (and should not be) the homomorphisms of the associative products! For instance the product of two elements from $\mathcal{A}_j$ with the maximal spin j has again a maximal spin j because it is from $\mathcal{A}_j$ but could have a spin $2j$ component if the product is taken in a sufficiently larger algebra $\mathcal{A}_k$.

Consider more closely the behaviour of the product as the function of k. According (10), arbitrary two elements f, g of $\mathcal{A}_j$ can be canonically considered as the elements of $\mathcal{A}_k$ for whatever $k > l$ (including $k = \infty$). Their product in every $\mathcal{A}_k$ can also be embedded in $\mathcal{A}_\infty$. Denote the corresponding element of $\mathcal{A}_\infty$ as $(fg)_k$. We shall argue that

$$\lim_{k\to\infty} (fg)_k = fg \tag{14}$$

where fg is the standard commutative pointwise multiplication in $\mathcal{A}_\infty$.

Before plunging into proof of this statement we try to formulate its meaning more 'physically'. It is not true that the algebra $\mathcal{A}_j$ tends to be commutative for large j (as the matrix algebra it, in fact, cannot.) What is the case

[10] The nice establishment of the correct commutative limit of the product was given in [6] using the coherent states for $SU(2)$.

that for large j the elements with much lower spins than j almost commute. In the field theory language: long distance limit corresponds to the standard commutative theory but for short distances the structure is truly non-commutative. This non-commutativeness, however, preserves the symmetry of the space-time. The algebra $\mathcal{A}_j$ is finite-dimensional with the dimension being $(j+1)^2$. That means that the sphere is effectively divided in $(j+1)^2$ cells of an average area $\frac{4\pi\rho^2}{(j+1)^2}$. A theory based on the non-commutative ring $\mathcal{A}_j$ has, therefore, a minimal lenght $\frac{2\rho}{j+1}$ incorporated.

Now it is easy to prove (14). Actually because of relation (13), which ensures the commutativity of the limit, it is enough to show that the normalization coefficients $c_{j,lp}$ defined in (11,12) have the property

$$\lim_{k\to\infty} c_{k,lp} = c_{\infty,lp}. \tag{15}$$

Due to the rotational invariance of the inner products in all $\mathcal{A}_k (k = 1, \ldots, \infty)$ it is enough to demonstrate it just for the highest weight element X_k^{+l}. Then

$$c_{k,l0}^{-2} = (X_k^{+l}, X_k^{+l})_k = \rho^{2l} \frac{(2l)!!}{(2l+1)!!} \frac{(k+l+1)!}{(k+1)(k)^l(k+2)^l(k-l)!} \tag{16}$$

The last equality follows from a formula derived in [16] (p. 618, Eq. (36)).

The relation (15) then obviously holds since the last fraction tends to 1 and it can be simply computed from (1) that

$$c_{\infty,l0}^{-2} = \rho^{2l} \frac{(2l)!!}{(2l+1)!!}. \tag{17}$$

Note that the generators X_k^i are themselves normalized as

$$(X_k^i, X_k^i)_k = \frac{\rho^2}{3} \tag{18}$$

and the standard relation defining the surface S_2 holds in the non-commutative case

$$X_k^{i\,2} = \rho^2. \tag{19}$$

We observe from (2) and (18) that for every j $X_j^i \in \mathcal{A}_j$ are embedded in $\mathcal{A}_\infty$ as just the standard commutative generators x^i and in $\mathcal{A}_k, k > j$ as $X_k^i \in \mathcal{A}_k$. The notation is therefore justified and in what follows we shall often write just X^i in the non-commutative case and x^i in the commutative one.

3 The Dirac operator on S_2 and its spectrum

The construction of the spinor bundle[11] over S_2 is standard part of any textbook of quantum field theory (e.g. see [17]) though, perhaps, it is not stressed explicitly. Also the spectrum of the Dirac operator acting on this bundle is known in that context, the eigenfunctions are nothing but the so-called spinorial harmonics [17]. We present the manifestly rotation invariant description of the spectrum in the spirit of the previous section.

Consider the trivial spinor bundle S_B over $\mathbf{R}^3$. Its sections are ordinary quantum mechanical two-component spinorial wave-functions of the form

$$\begin{pmatrix} \Psi_+ \\ \Psi_- \end{pmatrix}, \qquad \Psi_+, \Psi_- \in \mathcal{B}. \tag{20}$$

The action of the $SU(2)$ algebra is described by the generators

$$J_i \equiv R_i + \frac{1}{2}\sigma_i, \tag{21}$$

where σ_i are the standard Pauli matrices. Hence, S_B is the representation space of some (reducible) representation of $SU(2)$. Now R_3 can be viewed as the fibration of S_2 by the half-lines in R_3 starting in its centre. The position of a point on the fiber we measure by the radial coordinate r. The subbundle S_{A_∞} of the sections of S_B independent on the fiber coordinate r can be interpreted as the spinor bundle over the base manifold S_2 of the fibration. Clearly, S_{A_∞} is the $SU(2)$ subrepresentation of S_B. The decomposition of S_{A_∞} into irreducible representation follows from the standard Clebsch-Gordan series [15] for the tensor product of the representations $\mathcal{A}_\infty$ and $1/2$

$$S_{A_\infty} = 2(1/2 + 3/2 + 5/2 + \ldots). \tag{22}$$

Here the factor 2 in front of the bracket means that each representation in the bracket occurs in the direct sum twice. This doubling may be interpreted as the sum of the left and right chiral spinor bundles. We shall argue that the standard Dirac operator corresponding to the round metric on S_2 can be

[11] We have in mind the trivial bundle, twists by $U(1)$ bundles needed for the inclusion of monopoles will be considered in a forthcoming paper.

written solely in terms of the $SU(2)$ generators as follows[12]

$$D = \frac{1}{\rho}(\sigma_i R_i + 1). \tag{23}$$

Here ρ is the radius of the sphere. This operator is self-adjoint with respect to the scalar product on S_{A_∞} given by

$$(\Psi, \Xi) \equiv \frac{1}{2\pi\rho}\int d^3x^i \delta(x^{i^2} - \rho^2)(\Psi_+^* \Xi_+ + \Psi_-^* \Xi_-), \qquad \Psi, \Xi \in S_{A_\infty}. \tag{24}$$

The easy way of deriving (23) consists in comparing a three dimensional flat Dirac operator D_3 on S_B written in the spherical coordinates with the two dimensional round Dirac operator D_2 on the sphere in the same coordinates. Due to the rotational invariance the choice of a coordinate chart is irrelevant and we may proceed by choosing (and fixing) the poles of the sphere. The Dirac operator D in arbitrary coordinates in a general (curved) Riemannian manifold is given by

$$D_2 = -i\gamma^a e_a^\mu(\partial_\mu + \frac{1}{4}\omega_{\mu ab}[\gamma^a, \gamma^b]), \tag{25}$$

where γ^a are generators of the flat Clifford algebra

$$\{\gamma^a, \gamma^b\} = 2\delta^{ab}, \qquad \gamma^{a\,2} = 1, \qquad \gamma^{a\,\dagger} = \gamma^a, \tag{26}$$

e_a^μ is the vielbein and $\omega_{\mu ab}$ the spin connection defined by

$$\partial_\mu e_\nu^a - \Gamma_{\mu\nu}^\lambda e_\lambda^a + \omega_\mu{}^a{}_b e_\nu^b = 0. \tag{27}$$

For S_2 in the spherical coordinates

$$e_1^\theta = \frac{1}{\rho}, \qquad e_2^\phi = \frac{1}{\rho\sin\theta}, \qquad \omega_{\phi 12} = -\omega_{\phi 21} = -\cos\theta. \tag{28}$$

All remaining components of the vielbein and the connection vanish. For R^3 in the spherical coordinates

$$e_1^\theta = \frac{1}{r}, \qquad e_2^\phi = \frac{1}{r\sin\theta}, \qquad e_3^r = 1 \tag{29}$$

[12]The same formula was already given in [18, 19]. We give the different evidence of its validity, however.

and

$$\omega_{\phi 21} = -\omega_{\phi 12} = \cos\theta, \qquad \omega_{\phi 23} = -\omega_{\phi 32} = \sin\theta, \qquad \omega_{\theta 13} = -\omega_{\theta 31} = 1. \tag{30}$$

Thus

$$D_2 = -i\gamma^1 \frac{1}{\rho}(\partial_\theta + \frac{1}{2}\mathrm{ctg}\theta) - i\gamma^2 \frac{1}{\rho\sin\theta}\partial_\phi. \tag{31}$$

and

$$D_3 = -i\gamma^1 \frac{1}{r}(\partial_\theta + \frac{1}{2}\mathrm{ctg}\theta) - i\gamma^2 \frac{1}{r\sin\theta}\partial_\phi + -i\gamma^3(\partial_r + \frac{1}{r}). \tag{32}$$

We observe a simple relation between D_3 restricted on S_{A_∞} and D_2 namely

$$-i\gamma^3 D_3|_{restr.} + 1/\rho = D_2. \tag{33}$$

(note that $-i\gamma^3\gamma^a, a = 1,2$ fulfil the defining relations of the Clifford algebra (26)).

D_3 can be expressed also in the flat coordinates in R^3

$$D_3 = -i\sigma_i\partial_i, \tag{34}$$

where σ_i are the Pauli matrices which also generate the Clifford algebra (26). A simple algebra gives

$$D_3 = (\frac{\sigma_k x_k}{r})^2 D_3 = -i(\frac{\sigma_k x_k}{r})(\frac{x_i}{r}\partial_i - \frac{1}{r}\sigma_i R_i). \tag{35}$$

Because $\frac{x_i}{r}\partial_i = \partial_r$ and the vector fields R_i have no radial component it follows from (32) and (35) that

$$\gamma_3 = (\frac{\sigma_k x_k}{r}). \tag{36}$$

Inserting γ_3 from (36) and D_3 from (35) into Eq.(33) we get the $SU(2)$ covariant form (23) of the round Dirac operator on S_2.

The spectrum of D_2 readily follows from the group representation considerations. Consider a (normalized) spinor

$$\frac{\Theta^+}{\rho} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{37}$$

It is obviously the eigenvector of D_2 with an eigenvalue 1. Moreover it is the highest weight state of one of the spin 1/2 representations in the decomposition (22) as it can be directly checked using the generators J_i from (21). Indeed

$$J_+\Theta^+ = 0, \qquad J_iJ_i\ \Theta^+ = 3/4. \tag{38}$$

The construction of the other (normalized) highest weight states in the irreducible representations with the higher spins is obvious. They are given by

$$\Psi_{l,h.w.} = \rho^{-l-1}\sqrt{\frac{(2l+1)!!}{(2l)!!}}x^{+l}\Theta^+. \tag{39}$$

Here l is the spin of the irreducible representation. A direct computation shows

$$D_2\Psi_{l,h.w.} = (l+1)\Psi_{l,h.w.}. \tag{40}$$

Due to the rotational invariance of D_2 the other eigenvectors within the irreducible representation are obtained by the action of the lowering generator J_-, i.e.

$$\Psi_{l,m} = \rho^{-l-1}\sqrt{\frac{(2l+1-m)!}{(2l+1)!m!}\frac{(2l+1)!!}{(2l)!!}}J_-^m x^{+l}\Theta^+. \tag{41}$$

The eigenvalue corresponding to the eigenvector $\Psi_{l,m}, \quad m = 0,\ldots,2l$ is obviously $l+1$. So far we have constructed only one branch of the spectrum. However, due to an obvious relation

$$D_2\gamma^3 + \gamma^3 D_2 = 0 \tag{42}$$

also spinors $\gamma^3\Psi_{l,m}$ are the eigenvectors of D_2 with the eigenvalues $-(l+1)$. In this way we found the complete spectrum because all eigenvectors $\Psi_{l,m}$ and $\gamma^3\Psi_{l,m}$ form the basis of the spinor bundle S_{A_∞}.

4 Non-commutative supersphere

Having in mind the goal of constructing a non-commutative spinor bundle, we have to look for a language to describe the commutative case which would be best suited for performing the non-commutative deformation. We shall argue that the very structure to be exploited is $OSp(2,2)$ superalgebra which is somewhat hidden in the presentation given in the previous section. We shall proceed conceptually as follows: The non-commutative sphere, described in section 2, emerged naturally from the quantization of the algebra of the scalar fields on the ordinary sphere. Hence, it is natural to expect that the quantization of the supersphere would give a deformed ring of the scalar superfields on the supersphere. Those superfields contain as their components the ordinary fermion fields on the sphere, therefore the deformation of the algebra of the superfield should give (and it does give) the non-commutative spinor bundle on the non-commutative sphere, i.e. the structure we are looking for.

4.1 (Super)commutative supersphere

Consider a three-dimensional superspace $\mathbf{SR^3}$ with coordinates x^i, θ^α; the super-coordinates are the $SU(2)$ Majorana spinors. Consider an algebra $\mathcal{SB}$ of analytic functions on the superspace with the Grassmann coefficients in front of the odd monomials in θ. $\mathcal{SB}$ can be factorized by its ideal $\mathcal{SI}$, consisting of all functions of a form $h(x^i, \theta^\alpha)(\sum {x^i}^2 + C_{\alpha\beta}\theta^\alpha\theta^\beta - \rho^2)$. Here

$$C = i\sigma^2. \tag{43}$$

We refer to the quotient $\mathcal{SA}_\infty$ as to the algebra of superfields on the supersphere. An $OSp(2,2)$ invariant inner product of two elements Φ_1, Φ_2 of $\mathcal{SA}_\infty$ is given by[13]

$$(\Phi_1, \Phi_2)_\infty \equiv \frac{\rho}{2\pi}\int_{R^3} d^3x^i d\theta^+ d\theta^- \delta({x^i}^2 + C_{\alpha\beta}\theta^\alpha\theta^\beta - \rho^2)\Phi_1^\ddagger(x^i, \theta^\alpha)\Phi_2(x^i, \theta^\alpha), \tag{44}$$

Here $\Phi_1(x^i, \theta^\alpha), \Phi_2(x^i, \theta^\alpha) \in \mathcal{SB}$ are some representatives of Φ_1 and Φ_2 and the (graded) involution [20, 21] is defined by

$${\theta^+}^\ddagger = \theta^-,\ {\theta^-}^\ddagger = -\theta^+,\ \ (AB)^\ddagger = (-1)^{degA\ degB} B^\ddagger A^\ddagger. \tag{45}$$

The algebra $\mathcal{SA}_\infty$ is obviously generated by (the equivalence classes) x^i ($i = 1,2,3$) and θ^α ($\alpha = +,-$) which (anti)commute with each other under the usual pointwise multiplication, i.e.

$$x^i x^j - x^j x^i = x^i\theta^\alpha - \theta^\alpha x^i = \theta^\alpha\theta^\beta + \theta^\beta\theta^\alpha = 0. \tag{46}$$

[13]The normalization ensures that the norm of the unit element of $\mathcal{SA}_\infty$ is 1. The inner product is supersymmetric but it is not positive definite. However, such a property of the product is not needed for our purposes.

Their norms are given by

$$||x^i||^2_\infty = ||\theta^\alpha||^2_\infty = \rho^2. \tag{47}$$

Consider the vector fields in $\mathbf{SR^3}$ generating $OSp(2,2)$ superrotations of $\mathcal{SB}$. They are given by explicit formulae

$$v_+ = -\frac{1}{2}(x^3\partial_{\theta^-} - (x^1+ix^2)\partial_{\theta^+}) + \frac{1}{2}(-\theta^+\partial_{x^3} - \theta^-(\partial_{x^1}+i\partial_{x^2})), \tag{48}$$

$$v_- = -\frac{1}{2}(x^3\partial_{\theta^+} + (x^1-ix^2)\partial_{\theta^-}) + \frac{1}{2}(\theta^-\partial_{x^3} - \theta^+(\partial_{x^1}-i\partial_{x^2})), \tag{49}$$

$$d_+ = -\frac{1}{2}r(1+\frac{2}{r^2}\theta^+\theta^-)\partial_- + \frac{\theta^-}{2r}R_+ - \frac{\theta^+}{2r}(x^i\partial_i - R_3), \tag{50}$$

$$d_- = \frac{1}{2}r(1+\frac{2}{r^2}\theta^+\theta^-)\partial_+ + \frac{\theta^+}{2r}R_- - \frac{\theta^-}{2r}(x^i\partial_i + R_3) \tag{51}$$

$$\Gamma_\infty = (\frac{\theta^+x^3}{r} + \frac{\theta^-x^+}{r})\partial_+ + (\frac{\theta^+x^-}{r} - \frac{\theta^-x^3}{r})\partial_- \equiv 2(\theta^-v_+ - \theta^+v_-). \tag{52}$$

$$r_+ = x^3(\partial_{x^1}+i\partial_{x^2}) - (x^1+ix^2)\partial_{x^3} + \theta^+\partial_{\theta^-}, \tag{53}$$

$$r_- = -x^3(\partial_{x^1}-i\partial_{x^2}) + (x^1-ix^2)\partial_{x^3} + \theta^-\partial_{\theta^+}, \tag{54}$$

$$r_3 = -ix^1\partial_{x^2} + ix^2\partial_{x^1} + \frac{1}{2}(\theta^+\partial_{\theta^+} - \theta^-\partial_{\theta^-}) \tag{55}$$

and they obey the $OSp(2,2)$ Lie superalgebra graded commutation relations [13, 21]

$$[r_3, r_\pm] = \pm r_\pm, \qquad [r_+, r_-] = 2r_3, \tag{56}$$

$$[r_3, v_\pm] = \pm\frac{1}{2}v_\pm, \qquad [r_\pm, v_\pm] = 0, \qquad [r_\pm, v_\mp] = v_\pm, \tag{57}$$

$$\{v_\pm, v_\pm\} = \pm\frac{1}{2}r_\pm, \qquad \{v_\pm, v_\mp\} = -\frac{1}{2}r_3. \tag{58}$$

$$[\Gamma_\infty, v_\pm] = d_\pm, \qquad [\Gamma_\infty, d_\pm] = v_\pm, \qquad [\Gamma_\infty, r_i] = 0, \tag{59}$$

$$[r_3, d_\pm] = \pm\frac{1}{2}d_\pm, \quad [r_\pm, d_\pm] = 0, \quad [r_\pm, d_\mp] = d_\pm, \tag{60}$$

$$\{d_\pm, v_\pm\} = 0, \qquad \{d_\pm, v_\mp\} = \pm\frac{1}{4}\Gamma_\infty, \tag{61}$$

$$\{d_\pm, d_\pm\} = \mp\frac{1}{2}r_\pm, \qquad \{d_\pm, d_\mp\} = \frac{1}{2}r_3. \tag{62}$$

Note, that all introduced generators do annihilate the quadratic form ${x^i}^2 + C_{\alpha\beta}\theta^\alpha\theta^\beta$ hence they induce the action of $OSp(2,2)$ on $\mathcal{SA}_\infty$[14] .

In order to demonstrate the $OSp(2,2)$ invariance of the inner product (44) we have to settle the properties of the $OSp(2,2)$ generators with respect to the graded involution. It holds

$$(\Phi_1, r_i\Phi_2)_\infty = (r_i\Phi_1, \Phi_2)_\infty \tag{63}$$

$$(\Phi_1, v_\mp\Phi_2)_\infty = \pm(v_\pm\Phi_1, \Phi_2)_\infty. \tag{64}$$

$$(\Phi_1, d_\mp\Phi_2)_\infty = \mp(d_\pm\Phi_1, \Phi_2)_\infty. \tag{65}$$

$$(\Phi_1, \Gamma_\infty\Phi_2)_\infty = (\Gamma_\infty\Phi_1, \Phi_2)_\infty. \tag{66}$$

Consider now the variation of a superfield Φ

$$\delta\Phi = i(\varepsilon_+ v_+ + \varepsilon_- v_-)\Phi, \tag{67}$$

where ε_α is a constant Grassmann Majorana spinor, i.e.

$$\varepsilon_+^\ddagger = \varepsilon_-, \quad \varepsilon_-^\ddagger = -\varepsilon_+ \tag{68}$$

and, much in the same manner, a variation

$$\delta\Phi = i(\varepsilon_- d_+ + \varepsilon_+ d_-)\Phi. \tag{69}$$

Using the relations (63-66) it is straightforward to observe the invariance of the inner product with respect to the defined variations.

As it is well known [21] the typical irreducible representations of $OSp(2,2)$ consist of quadruplets of the $SU(2)$ irreducible representations $j \oplus j - \frac{1}{2} \oplus j - \frac{1}{2} \oplus j - 1$. The number j is an integer or a half-integer and it is referred to as the $OSp(2,2)$ superspin. The generators $x^i, \theta^\alpha \in \mathcal{SA}_\infty$ together with

$$\frac{1}{\rho^2}(\theta^+ x^3 + \theta^- x^+), \qquad j = \frac{1}{2},\ j_3 = \frac{1}{2}, \tag{70}$$

[14]The appearance of r in Eqs.(50-52) may seem awful because we have considered the ring of superanalytic functions on SR_3. However, this is only a formal drawback, which can be cured by a completion of the space of superanalytic functions with respect to an appropriate inner product. In fact, we need not even do that for our purposes because the terms involving r become anyway harmless after the factorization by the ideal SI.

$$\frac{1}{\rho^2}(\theta^+ x^- - \theta^- x^3), \qquad j=\frac{1}{2},\ j_3=-\frac{1}{2}, \tag{71}$$

$$1+\frac{1}{\rho^2}\theta^+\theta^-, \qquad j=0,\ j_3=0. \tag{72}$$

indeed form the (typical) superspin 1 irreducible representation of $OSp(2,2)$ algebra under the action of the vector fields (48-55). The numbers j, j_3 in (70-72) correspond to the total $SU(2)$ spin and its third component. The supermultiplet with the superspin 1 can be conveniently constructed applying subsequently the lowering operators v_- and d_- on the highest weight vector x^+. Supermultiplets with higher superspins can be obtained in the same way starting with the highest weight vectors x^{+^l}. Thus the full decomposition of $\mathcal{SA}_\infty$ into the irreducible representations of $OSp(2,2)$ can be written as the infinite direct sum

$$\mathcal{SA}_\infty = 0+1+2+\ldots, \tag{73}$$

where the integers denote the $OSp(2,2)$ superspins of the representations[15]. From the point of view of the $SU(2)$ representations, the algebra of the superfields consists of two copies of $\mathcal{A}_\infty$ and the spinor bundle $\frac{1}{2}\otimes\mathcal{A}_\infty$ (see Eq. (22)) Note that the generators of $\mathcal{SA}_\infty$ fulfil the obvious relation

$$x^{i^2} + C_{\alpha\beta}\theta^\alpha\theta^\beta = \rho^2. \tag{74}$$

The big algebra $\mathcal{SB}$ has a natural grading as the vector space, given by the parity of the total power of the Grassmann coordinates θ^α. Because we factorized over the quadratic surface in the superspace, this grading induces the grading in $\mathcal{SA}_\infty$. It is easy to see that the odd elements of $\mathcal{SA}_\infty$ with respect to this grading can be identified with the fermion fields on the sphere. Indeed, they can be written as

$$\Psi = \Psi_\alpha(x^i)\frac{\theta^\alpha}{\rho}, \tag{75}$$

where the (Grassmann) components Ψ_α belong to $\mathcal{A}_\infty$[16]. But this is the standard spinor bundle on the sphere

$$\begin{pmatrix}\Psi_+(x^i)\\ \Psi_-(x^i)\end{pmatrix}, \tag{76}$$

[15] The 'baryon' number of those representations, in the sense of Ref.[21], is zero.

[16] The factorization by the relation $\sum x^{i^2} - \rho^2 = 0$ and the relation $\sum x^{i^2} + C_{\alpha\beta}\theta^\alpha\theta^\beta - \rho^2 = 0$ is effectively the same in this case because the term quadratic in θ is killed upon the multiplication by another θ in Eq. (75).

described in section 3. The scalar product on the bundle is inherited from the inner product (44)

$$(\Psi, \Xi) \equiv \frac{\rho}{2\pi} \int d^3x^i \delta(x^{i^2} - \rho^2) d\theta^+ d\theta^- \Psi^\ddagger \Xi, \tag{77}$$

and (up to a sign) it coincides with the scalar product (24). The Pauli matrices, as the operators acting on the two-component spinors, can be expressed in the superfield formalism as follows

$$\sigma^3 = \theta^+ \partial_{\theta^+} - \theta^- \partial_{\theta^-}, \qquad \sigma^\pm = 2\theta^\pm \partial_{\theta^\mp}. \tag{78}$$

In what follows we shall refer to the odd (even) elements with respect to the described grading as to the fermionic (bosonic) superfields in order to make a difference with the even and odd superfields in the standard (Grassmann) sense.

The $OSP(2,1)$ superalgebra generated by $r_i, v_\pm$ has a quadratic Casimir

$$K_2 = (r_3^2 + \frac{1}{2}\{r_+, r_-\}) + (v_+v_- - v_-v_+) \equiv B_2 + F_2. \tag{79}$$

Using Eqs. (78), it is easy now to check that the fermionic part F_2 of the Casimir is directly related to the Dirac operator (23)

$$\rho D = \sigma^i R_i + 1 = 2F_2 - \frac{1}{2} = 2(v_+v_- - v_-v_+) - \frac{1}{2}. \tag{80}$$

The grading γ^3 of the Dirac operator is just the $OSp(2,2)$ generator Γ_∞. Its eigenfuctions are obviously the Weyl spinors. A Majorana spinors are given by the restriction

$$\psi_+^\ddagger = \psi_-, \qquad \psi_-^\ddagger = -\psi_+ \tag{81}$$

which can be easily derived from the reality condition on the superfield Φ.

4.2 The truncation of $\mathcal{SA}_\infty$

We define the family of non-commutative superspheres $\mathcal{SA}_j$ by furnishing the truncated sum of the irreducible representations of $OSp(2,2)$

$$\mathcal{SA}_j = 0 + 1 + \ldots + j, \qquad j \in \mathbf{Z} \tag{82}$$

with an associative product and an inner product which in the limit $j \to \infty$ give the standard products in $\mathcal{SA}_\infty$. In order to do this consider the space $\mathcal{L}(j/2, j/2)$ of linear operators from the representation space of the $OSp(2,1)$ irreducible representation with the $OSp(2,1)$ superspin $j/2$ into itself. (Note that the $OSp(2,1)$ irreducible representation with the $OSp(2,1)$ superspin j has the $SU(2)$ content $j \oplus j - \frac{1}{2}$ [21]). The action of the superalgebra $OSp(2,2)$ itself on $\mathcal{L}(j/2, j/2)$[17] is described by operators $R_i, V_\alpha, D_\alpha, \gamma \in \mathcal{L}(j/2, j/2)$ given by [23]

$$R_i = \begin{pmatrix} R_i^{\frac{j}{2}} & 0 \\ 0 & R_i^{\frac{j}{2}-\frac{1}{2}} \end{pmatrix}, \qquad \gamma = \begin{pmatrix} -j\, Id & 0 \\ 0 & -(j+1)Id \end{pmatrix}. \tag{83}$$

$$V_\alpha = \begin{pmatrix} 0 & V_\alpha^{\frac{j}{2},\frac{j}{2}-\frac{1}{2}} \\ V_\alpha^{\frac{j}{2}-\frac{1}{2},\frac{j}{2}} & 0 \end{pmatrix}, \qquad D_\alpha = \begin{pmatrix} 0 & V_\alpha^{\frac{j}{2},\frac{j}{2}-\frac{1}{2}} \\ -V_\alpha^{\frac{j}{2}-\frac{1}{2},\frac{j}{2}} & 0 \end{pmatrix}, \tag{84}$$

where

$$\langle l, l_3+1|R_+^l|l, l_3\rangle = \sqrt{(l-l_3)(l+l_3+1)}, \tag{85}$$

$$\langle l, l_3-1|R_-^l|l, l_3\rangle = \sqrt{(l+l_3)(l-l_3+1)}, \tag{86}$$

$$\langle l, l_3|R_3^l|l, l_3\rangle = l_3, \tag{87}$$

$$\langle l_3 + \frac{1}{2}|V_+^{\frac{j}{2},\frac{j}{2}-\frac{1}{2}}|l_3\rangle = -\frac{1}{2}\sqrt{\frac{j}{2} + l_3 + \frac{1}{2}}, \tag{88}$$

$$\langle l_3 - \frac{1}{2}|V_-^{\frac{j}{2},\frac{j}{2}-\frac{1}{2}}|l_3\rangle = -\frac{1}{2}\sqrt{\frac{j}{2} - l_3 + \frac{1}{2}}, \tag{89}$$

$$\langle l_3 + \frac{1}{2}|V_+^{\frac{j}{2}-\frac{1}{2},\frac{j}{2}}|l_3\rangle = -\frac{1}{2}\sqrt{\frac{j}{2} - l_3}, \tag{90}$$

$$\langle l_3 - \frac{1}{2}|V_-^{\frac{j}{2}-\frac{1}{2},\frac{j}{2}}|l_3\rangle = \frac{1}{2}\sqrt{\frac{j}{2} + l_3}. \tag{91}$$

Every $\Phi \in \mathcal{L}(j/2, j/2)$ can be written as a matrix

$$\Phi = \begin{pmatrix} \phi_R & \psi_R \\ \psi_L & \phi_L \end{pmatrix}, \tag{92}$$

[17] The so-called non-typical irreducible representation of $OSp(2,2)$ [21, 22] is in the same time also the $OSp(2,1)$ irreducible representation with the $OSp(2,1)$ superspin $j/2$.

where ϕ_R and ϕ_L are square $(j+1)\times(j+1)$ and $j\times j$ matrices respectively and ψ_R and ψ_L are respectively rectangular $(j+1)\times j$ and $j\times(j+1)$ matrices. The meaning of the indices R and L will become clear in the next subsection. A fermionic element is given by a supermatrix with vanishing diagonal blocks and a bosonic element by one with vanishing off-diagonal blocks. Clearly, $OSp(2,2)$ superalgebra acts on $\mathcal{L}(j/2,j/2)$ by the superadjoint action

$$\mathcal{R}_i\Phi \equiv [R_i,\Phi], \qquad \Gamma\Phi \equiv [\gamma,\Phi]. \tag{93}$$

$$\mathcal{V}_\alpha\Phi_{even} \equiv [V_\alpha,\Phi_{even}], \qquad \mathcal{V}_\alpha\Phi_{odd} \equiv \{V_\alpha,\Phi_{odd}\}. \tag{94}$$

$$\mathcal{D}_\alpha\Phi_{even} \equiv [D_\alpha,\Phi_{even}], \qquad \mathcal{D}_\alpha\Phi_{odd} \equiv \{D_\alpha,\Phi_{odd}\}. \tag{95}$$

This 'superadjoint' representation is reducible and, in the spirit of Ref.[21, 22], it is easy to work out its decomposition into $OSp(2,2)$ irreducible representations

$$\mathcal{L}(j/2,j/2) = 0 + 1 + \ldots + j. \tag{96}$$

The associative product in $\mathcal{L}(j/2,j/2)$ is defined as the composition of operators and the $OSp(2,2)$ invariant inner product on $\mathcal{L}(j/2,j/2)$ is defined by[18]

$$(\Phi_1,\Phi_2)_j \equiv \mathrm{STr}(\Phi_1^\ddagger,\Phi_2), \qquad \Phi_1,\Phi_2 \in \mathcal{L}(j/2,j/2). \tag{97}$$

Here STr is the supertrace and $\ddagger$ is the graded involution. Although these concepts are quite standard in the literature it is instructive to work out their content in our concrete example. The supertrace is defined as usual

$$STr\Phi \equiv Tr\phi_R - Tr\phi_L \tag{98}$$

and the graded involution as [20]

$$\Phi^\ddagger \equiv \begin{pmatrix} \phi_R^\dagger & \mp\psi_L^\dagger \\ \pm\psi_R^\dagger & \phi_L^\dagger \end{pmatrix}. \tag{99}$$

$\dagger$ means the standard hermitian conjugation of a matrix and the upper (lower) sign refers to the case when the entries consists of odd (even) elements of a Grassmann algebra. Note that

$$R_i^\ddagger = R_i, \qquad V_+^\ddagger = V_-, \quad V_-^\ddagger = -V_+. \tag{100}$$

[18]The normalization ensures that the norm of the identity matrix is 1.

Now we identify $\mathcal{SA}_j$ with even elements of $\mathcal{L}(j/2, j/2)$ which means that the entries of the (off)-diagonal matrices are (anti)-commuting variables. This correspond to the similar requirement in the untruncated case because in the truncated case the spinors form the off-diagonal part of the superfield.

We can demonstrate the $OSp(2,2)$ invariance of the inner product (97) again by settling the properties of the $OSp(2,2)$ generators with respect to the graded involution (99). They read

$$(\Phi_1, \mathcal{R}_i\Phi_2)_j = (\mathcal{R}_i\Phi_1, \Phi_2)_j. \tag{101}$$

$$(\Phi_1, \mathcal{V}_\mp\Phi_2)_j = \pm(\mathcal{V}_\pm\Phi_1, \Phi_2)_j. \tag{102}$$

$$(\Phi_1, \mathcal{D}_\mp\Phi_2)_j = \mp(\mathcal{D}_\pm\Phi_1, \Phi_2)_j. \tag{103}$$

$$(\Phi_1, \Gamma\Phi_2)_j = (\Gamma\Phi_1, \Phi_2)_j. \tag{104}$$

Consider now the variation of a superfield Φ

$$\delta\Phi = i(\epsilon_+\mathcal{V}_+ + \epsilon_-\mathcal{V}_-)\Phi, \tag{105}$$

where ϵ_α is given by

$$\epsilon_\alpha = \begin{pmatrix} \varepsilon_\alpha & 0 \\ 0 & -\varepsilon_\alpha \end{pmatrix} \tag{106}$$

and ε_α are the usual Grassmann variables with the involution properties

$$\varepsilon_+^\ddagger = \varepsilon_-, \quad \varepsilon_-^\ddagger = -\varepsilon_+. \tag{107}$$

Much in the same manner, consider also a variation

$$\delta\Phi = i(\epsilon_-\mathcal{D}_+ + \epsilon_+\mathcal{D}_-)\Phi. \tag{108}$$

Using the relations (101-104) it is straightforward to observe the invariance of the inner product with respect to the defined variations. Note that ϵ_α do anticommute with D_α and V_α as they should.

We can choose a basis in $\mathcal{SA}_j$ formed by eigenvectors of the Hermitian operators

$$\mathcal{Q}^2 \equiv \mathcal{R}_i^2 + C_{\alpha\beta}\mathcal{V}_\alpha\mathcal{V}_\beta, \tag{109}$$

$\mathcal{R}_i^2$ and $\mathcal{R}_3$. The spectrum of (the $OSp(2,1)$ Casimir) $\mathcal{Q}^2$ consists of numbers $q(q+1/2)$ where the $OSp(2,1)$ superspin q runs over all integers and half-integers from 0 to j [23]; the remaining two operators have the standard spectra known in the $SU(2)$ context.

Now we make more precise the notion of the commutative limits of the inner product and the associative product. There is a natural chain of the linear embeddings of the vector spaces

$$\mathcal{SA}_1 \hookrightarrow \mathcal{SA}_2 \hookrightarrow \ldots \hookrightarrow \mathcal{SA}_j \hookrightarrow \ldots \hookrightarrow \mathcal{SA}_\infty \tag{110}$$

Any (normalized) element from $\mathcal{SA}_j$ of a form

$$s_{j,lpq}\mathcal{V}_-{}^p\mathcal{D}_-{}^q X_j^{+l} \tag{111}$$

is mapped into an element from $\mathcal{SA}_k$ of the form

$$s_{k,lpq}\mathcal{V}_-{}^p\mathcal{D}_-{}^q X_k^{+l}. \tag{112}$$

Here X_j^i (and $\Theta_j^\pm \equiv -\mathcal{V}_\mp X_j^\pm$) are the representatives of the $OSp(2,1)$ generators in the $OSp(2,1)$ irreducible representation with the $OSp(2,1)$ superspin $j/2$ ($X_\infty^i \equiv x^i$ and $\Theta_\infty^\alpha \equiv \theta^\alpha$). They are normalized so that

$$[X^m, X^n] = i\frac{\rho}{\sqrt{\frac{j}{2}(\frac{j}{2}+\frac{1}{2})}}\epsilon_{mnp}X^p, \tag{113}$$

$$[X^i, \Theta^\alpha] = \frac{\rho}{2\sqrt{\frac{j}{2}(\frac{j}{2}+\frac{1}{2})}}\sigma^{i\beta\alpha}\Theta^\beta, \tag{114}$$

$$\{\Theta^\alpha, \Theta^\beta\} = \frac{\rho}{2\sqrt{\frac{j}{2}(\frac{j}{2}+\frac{1}{2})}}(C\sigma^i)^{\alpha\beta}X^i, \tag{115}$$

Hence

$$(X_j^i, X_j^i)_j = (\Theta_j^\alpha, \Theta_j^\alpha)_j = \rho^2. \tag{116}$$

$s_{j,lpq}$ are (real) normalization coefficients given by the requirement that the embedding is norm-conserving. Because the operators $\mathcal{Q}^2, \mathcal{R}_i^2$ and $\mathcal{R}_3$ are hermitian for arbitrary $\mathcal{SA}_j$ (as it can be easily seen from the definitions of the inner products (44),(97)) the embeddings are in fact isometric. Indeed, the inner product of the eigenvectors of hermitian operators vanishes if the corresponding eigenvalues are different. The commutative limit of the associative product is more involved, however. We proceed in an analogous way as in the purely bosonic case $SU(2)$.

Consider more closely the behaviour of the product as the function of k. According the relation (110), arbitrary two elements Φ_1, Φ_2 of $\mathcal{SA}_j$ can be canonically considered as the elements of $\mathcal{SA}_k$ for whatever $k > l$ (including $k = \infty$). Their product in every $\mathcal{SA}_k$ can also be embedded in $\mathcal{SA}_\infty$. Denote the corresponding element of $\mathcal{SA}_\infty$ as $(\Phi_1\Phi_2)_k$. We shall argue that

$$\lim_{k\to\infty} (\Phi_1\Phi_2)_k = \Phi_1\Phi_2 \tag{117}$$

where $\Phi_1\Phi_2$ is the standard supercommutative pointwise multiplication in $\mathcal{SA}_\infty$.

For proving the relation (117), it is convenient to realize that $\mathcal{SA}_j$ can be generated by taking products of generators X^i_j and Θ^α_j of $OSp(2,1)$ in the irreducible representation with the $OSp(2,1)$ superspin $j/2$. This statement follows from the Burnside lemma [24], but its validity can be seen directly. Indeed, from the $OSp(2,2)$ commutation relations it follows easily that every element of the form (111) can be expressed in terms of X^i_j and Θ^α_j. Hence the relations (113-115) ensure the (graded) commutativity in the limit $j \to \infty$ and it is therefore sufficient just to show that the normalization coefficients $s_{j,lpq}$ defined in (112) have the property

$$\lim_{k\to\infty} s_{k,lpq} = s_{\infty,lpq}. \tag{118}$$

Because of the $OSp(2,2)$ invariance of the inner products in all $\mathcal{SA}_k$ ($k = 1,\ldots,\infty$), it is in fact enough to demonstrate it just for the highest weight elements X^{+l}_k. Then it is a straighforward computation to check that

$$\lim_{k\to\infty} s^{-2}_{k,l00} \equiv \lim_{k\to\infty} (X^{+l}_k, X^{+l}_k)_k = (2l+1)c^{-2}_{\infty,l0}, \tag{119}$$

where $c^{-2}_{\infty,l0}$ have been given in Eq.(17). But $s^{-2}_{\infty,l00}$ can be directly computed from (44) giving

$$s^{-2}_{\infty,l00} = (2l+1)c^{-2}_{\infty,l0}. \tag{120}$$

We have thus proven the commutative limit relation (117).

Note that the normalization of X^i_j and Θ^α_j is such that the value of the Casimir in $\frac{j}{2}$ $OSp(2,1)$ irreducible representation is equal to ρ^2, i.e.

$$X^{i\,2}_j + C_{\alpha\beta}\Theta^\alpha_j\Theta^\beta_j = \rho^2. \tag{121}$$

Thus the relation defining the supersphere is preserved also in the truncated case. We observe from Eqs. (47) and (116) that for every j $X^i_j, \Theta^\alpha_j \in \mathcal{SA}_j$ are embedded in $\mathcal{SA}_\infty$ as just the standard (super)commutative generators x^i, θ^α and in $\mathcal{SA}_k, k > j$ as $X^i_k, \Theta^\alpha_k \in \mathcal{SA}_k$. The notation is therefore justified and in what follows we shall often write just X^i and Θ^α.

4.3 Dirac operator on the truncated sphere

In an analogy with the (super)commutative case, we define the non-commutative spinor bundle on the sphere S_2 as the odd part of the truncated superfield $\Phi \in \mathcal{SA}_j$ and the Dirac operator we define as

$$\rho D \equiv 2(\mathcal{V}_+\mathcal{V}_- - \mathcal{V}_-\mathcal{V}_+) - \frac{1}{2}. \tag{122}$$

This operator is manifestly self-adjoint, $SU(2)$ invariant and it is also odd with respect to the grading Γ given by Eqs. (93) and (83) or simply, if the diagonal part of a superfield vanishes, by

$$\Gamma\Phi_{fer} = \begin{pmatrix} Id & 0 \\ 0 & -Id \end{pmatrix} \Phi_{fer}. \tag{123}$$

This explains the notation in Eq. (92): in the first (second) line there are right (left) objects with respect to the chiral grading Γ. Hence, a fermionic superfield of the upper(lower)-triangular form will be referred to as the right (left) chiral spinor on the truncated sphere.

The spectrum of D readily follows from the group representation considerations. Consider a normalized spinor Θ^+/ρ. It follows directly from $OSp(2,1)$ graded commutation relations (56-58) that this is the eigenvector of D with an eigenvalue 1. Moreover it is the highest weight state of one of the $SU(2)$ spin 1/2 representations in the decomposition (82). This can be directly checked using the generators (93-95):

$$\mathcal{R}_+\Theta^+ = 0, \qquad \mathcal{R}_i^2 = 3/4. \tag{124}$$

The construction of the other (normalized) highest weight states in the irreducible representations with the higher spins is obvious. They are given by

$$\Psi_{l,h.w.} = b_{jl}\ \rho^{-l-1}\sqrt{\frac{(2l+1)!!}{(2l)!!}}X^{+l}\Theta^+. \tag{125}$$

Here l is the spin of the $SU(2)$ irreducible representation and b_{jl} is a normalization coefficient. A direct computation shows

$$D\Psi_{l,h.w.} = (l+1)\Psi_{l,h.w.}, \qquad l \leq j-1. \tag{126}$$

Due to the rotational invariance of D the other eigenvectors within the irreducible representation are obtained by the action of the lowering generator $\mathcal{R}_-$, i.e.

$$\Psi_{l,m} = b_{jl}\, \rho^{-l-1} \sqrt{\frac{(2l+1-m)!}{(2l+1)!m!}\frac{(2l+1)!!}{(2l)!!}}\mathcal{R}_-^m X^{+l}\Theta^+. \qquad (127)$$

The eigenvalue corresponding to the eigenvector $\Psi_{l,m}, \quad m = 0, \ldots, 2l$ is obviously $l+1$. So far we have constructed only one branch of the spectrum. However, due to an obvious relation

$$D\Gamma + \Gamma D = 0 \qquad (128)$$

also spinors $\Gamma\Psi_{l,m}$ are the eigenvectors of D with the eigenvalues $-(l+1)$. In this way we found the complete spectrum because all eigenvectors $\Psi_{l,m}$ and $\Gamma\Psi_{l,m}$ form the basis of the space of the fermionic superfields from $\mathcal{SA}_j$. Thus, we have obtained precisely the truncation of the commutative Dirac operator D.

5 Supersymmetric field theories

5.1 The bosonic preliminaries

Consider the following action for a real scalar field living on the sphere S_2

$$S(\phi) = \frac{1}{2}(\phi, R_i^2\phi)_\infty \equiv \frac{1}{4\pi\rho}\int d^3x^i \delta(x^{i^2} - \rho^2)\phi(x)R_i^2\phi(x). \qquad (129)$$

It is easy to show that this is just the action of a free massless field on S_2 i.e.

$$S(\phi) = -\frac{1}{8\pi}\int d\Omega\phi \triangle_\Omega \phi, \qquad (130)$$

where $\triangle_\Omega$ is the Laplace-Beltrami operator on the sphere or, simply, the angular part of the flat Laplacian in R^3. Adding a mass and an interaction term is easy, e.g. the $P(\phi)$-models [25, 10] are described by the action

$$S_\infty = \frac{1}{2}(\phi, R_i^2\phi)_\infty + (1, P(\phi))_\infty, \qquad (131)$$

where $P(\phi)$ is a polynomial in the field variable. The non-commutative analogue of the action (129) is now obvious

$$S_j = \frac{1}{2}(\phi, \mathcal{R}_i^2 \phi)_j + (1, P(\phi))_j = \frac{1}{2j+2} Tr_j(\phi \mathcal{R}_i^2 \phi) + \frac{1}{j+1} Tr_j P(\phi). \quad (132)$$

The truncated action is manifestly $SU(2)$invariant with respect to the infinitesimal transformation of the scalar field

$$\delta\phi = \varepsilon_i \mathcal{R}_i \phi \equiv \varepsilon_i [R_i, \phi]. \quad (133)$$

Another interesting class of Lagrangians consists of the nonlinear σ-models describing the string propagation in curved backgrounds. The (truncated) action reads

$$S_j = \frac{1}{2}(\mathcal{R}_i \phi^A, g^{AB}(\phi) \mathcal{R}_i \phi^B)_j \quad (134)$$

with the obvious commutative limit. It is not difficult, in fact, to define a quantization of the truncated system via the path integral because the space of field configurations in finite-dimensional. We gave the details in a separate publication [10] with the aim to develop the efficient nonperturbative regularization of field theories which could (hopefully in many aspects) compete with the traditional lattice approach.

5.2 The supersymmetric actions

The supersymmetric case is somewhat more involved than the bosonic one not only because of the enlargement of the number of degrees of freedom. Starting from the undeformed case one could suspect that the standard free $OSp(2,1)$-supersymmetric action for a real superfield on the sphere should be written in our three dimensional formalism as

$$S_{susp} = \frac{1}{2}(\Phi, (\mathcal{R}_i^2 + C_{\alpha\beta} \mathcal{V}_\alpha \mathcal{V}_\beta)\Phi)_\infty. \quad (135)$$

Though the $OSp(2,1)$ Casimir sitting within the brackets does give the SUSY invariance it *does not* yield the correct two dimensional "world-sheet" action containing just the free massless bosonic field and free massless Majorana fermion. To get out of the trouble we may use the philosophy used about a decade ago where supersymmetric models on the homogeneous spaces have been intensively studied [26]. In particular, Fronsdal has considered the spinors on anti-de Sitter spacetime and has constructed the $OSP(4,1)$ invariant supersymmetric actions by introducing another set of odd generators [26]. They were analogues of the standard supersymmetric covariant derivatives needed to build up the super-Poincaré invariant Lagrangians.

The same approach applies in our case. The new odd generators are nothing but the additional $OSp(2,2)$ generators $\mathcal{D}_\alpha$. The standard Lagrangian of the free $OSp(2,1)$ supersymmetric theory can be written solely in terms of the 'covariant derivatives' $\mathcal{D}_\alpha$ and the grading Γ.

Let us begin with the detailed quantitative account first in the non-deformed case. It is easy to check that the operator

$$C_{\alpha\beta}d_\alpha d_\beta + \frac{1}{4}\Gamma^2_\infty \tag{136}$$

is invariant with respect to $OSp(2,1)$ supersymmetry generated by r_i and $v_\pm$. Hence we may consider the action

$$S = (\Phi, C_{\alpha\beta}d_\alpha d_\beta \Phi)_\infty + \frac{1}{4}(\Phi, \Gamma^2_\infty \Phi)_\infty \equiv$$

$$\equiv \frac{\rho}{2\pi}\int_{R^3} d^3x^i d\theta^+ d\theta^- \delta({x^i}^2 + C_{\alpha\beta}\theta^\alpha\theta^\beta - \rho^2)\Phi(x^i,\theta^\alpha)(C_{\alpha\beta}d_\alpha d_\beta + \frac{1}{4}\Gamma^2)\Phi(x^i,\theta^\alpha), \tag{136a}$$

where Φ is a real superfield, i.e. $\Phi^\ddagger = \Phi$.

Consider now the variation of the real superfield Φ

$$\delta\Phi = i\varepsilon_\alpha v_\alpha \Phi, \tag{137}$$

which preserves the reality condition. Now Eqs.(63-66) hold also when Φ_1 is an even and Φ_2 an odd superfield in the standard Grassmann sense. Using this and the fact that $\varepsilon_\alpha v_\alpha$ commutes with the operator (136), the supersymmetry of the action S obviously follows.

It is straightforward to work out the action (136a) in the two-dimensional component language. It reads

$$S = \frac{1}{4\pi}\int d\Omega(-\frac{1}{2}\phi \triangle_\Omega \phi + \frac{1}{2}\rho^4 F^2 - \frac{1}{2}\psi^\dagger \rho^3 D_\Omega \psi), \tag{138}$$

where D_Ω is the Dirac operator on S^2 and the superfield ansatz is

$$\Phi(x^i,\theta^\alpha) = \phi(x^i) + \psi_\alpha\theta^\alpha + (F + \frac{x^i}{r^2}\partial_i\phi)\theta^+\theta^-. \tag{139}$$

Of course, ψ_α are anticommuting objects and the reality condition $\Phi^\ddagger = \Phi$ makes the fields ϕ and F real and the spinor ψ_α becomes Majorana[19], i.e.

$$\psi_+^\ddagger = \psi_-, \qquad \psi_-^\ddagger = -\psi_+. \tag{140}$$

We recognize in the expression (138) the standard free supersymmetric action in two dimensions.

Adding a (real) superpotential $W(\Phi)$ we may write down a supersymmetric action with the interaction term. It reads

$$S_\infty = (\Phi, (C_{\alpha\beta} d_\alpha d_\beta + \frac{1}{4}\Gamma_\infty^2)\Phi)_\infty + (1, W(\Phi))_\infty. \tag{141}$$

The truncated version of the action S_∞

$$S_j = (\Phi, (C_{\alpha\beta}\mathcal{D}_\alpha\mathcal{D}_\beta + \frac{1}{4}\Gamma^2)\Phi)_j + (1, W(\Phi))_j \tag{142}$$

is manifestly supersymmetric with respect to the variations

$$\delta\Phi = i\epsilon_\alpha \mathcal{V}_\alpha \Phi, \tag{143}$$

It remains to prove that S_j approaches S_∞ for $j \to \infty$. In order to do that it is convenient to rewrite both truncated and untruncated action as follows

$$S_j = (\mathcal{D}_+\Phi, \mathcal{D}_+\Phi)_j + (\mathcal{D}_-\Phi, \mathcal{D}_-\Phi)_j + \frac{1}{4}(\Gamma\Phi, \Gamma\Phi)_j + (1, W(\Phi))_j, \tag{144}$$

where the index j can be both finite and infinite and we have used the formulas (63-66) and (101-104). Now it is enough to show that

$$\lim_{k\to\infty} (\mathcal{D}_\alpha\Phi)_k = d_\alpha\Phi, \qquad \lim_{k\to\infty} (\Gamma\Phi)_k = \Gamma_\infty\Phi \tag{145}$$

(The embedding $(\Phi)_k$ was defined in Eqs.(111,112).) But this is true almost by definition because $\mathcal{D}_\alpha\Phi$ can be written as a linear superposition of the vectors of the form (111,112). As in the bosonic case we may write down the regularized action for the supersymmetric σ-models describing the superstring propagation in curved backgrounds

[19] Note, that we consider the *graded* involution defined by Eq.(45) (see also [20]).

$$S_j = (\mathcal{D}_+\Phi^A, g_{AB}(\Phi)\mathcal{D}_+\Phi^B)_j + (\mathcal{D}_-\Phi^A, g_{AB}(\Phi)\mathcal{D}_-\Phi^B)_j + \frac{1}{4}(\Gamma\Phi^A, g_{AB}(\Phi)\Gamma\Phi^B)_j. \tag{146}$$

The $OSp(2,1)$ supersymmetry and the commutative limit is obvious. The regularized action (146) can be used as the base for the path integral quantization manifestly preserving supersymmetry and still involving the finite number of degrees of freedom. Particularly this aspect of our approach seems to be very promising both in comparison with the lattice physics as well as in general. Indeed so far we are not aware of any nonperturbative regularization which would possess all those properties.

6 Conclusions and Outlook

We have regulated in the manifestly supersymmetric way the actions of the field theories on the supersphere, involving scalar and spinor fields. As a next step we plan to include in the picture the topologically non-trivial bundles and the gauge fields [27] and to study the chiral symmetry in the context. From the purely mathematical point of view we have to build up the non-commutative de Rham complex and understand the notions of one- and two-forms. It would be also interesting to establish a connection between previous works on supercoherent states [28, 12, 13] and our present treatment. In a later future we shall attempt to reach two challenging goals in our programme, namely the truncation of the four-dimensional sphere and the inclusion of gravity.

7 Acknowledgement

We are grateful to A. Alekseev, L. Álvarez-Gaumé, M. Bauer, A. Connes, V. Černý, T. Damour, J. Fröhlich, J. Ftáčnik, K. Gawędzki, J. Hoppe, B. Jurčo, E. Kiritsis, C. Kounnas, M. Rieffel, R. Stora and D. Sullivan for useful discussions. Part of the research of C.K. has been done at I.H.E.S. at Bures-sur-Yvette and of C.K. and P.P. at the Schrödinger Institute in Vienna. We thank both these institutes for hospitality.

References

[1] J. Fröhlich and K. Gawędzki, *Conformal Field Theory and Geometry of Strings*, preprint IHES/P/93/59, Proceedings of the Mathematical Quantum Theory Conference, UBC-Vancouver, August 1993, hep-th/9310187,

[2] E. Kiritsis and C. Kounnas, *Nucl. Phys.* **B442** (1995) 472,

[3] F.A. Berezin, *Comm. Math. Phys.* **40** (1975) 153,

[4] J. Hoppe, *MIT PhD Thesis*, 1982 and *Elem. Part. Res. J. (Kyoto)* **80** (1989) 145,

[5] J. Madore, *J. Math. Phys.* **32** (1991) 332 and *Class. Quant. Grav.* **9** (1992) 69,

[6] H. Grosse and P. Prešnajder, *Lett. Math. Phys.* **28** (1993) 239,

[7] A. Connes, *Noncommutative Geometry*, Academic Press London, 1994,

[8] A. Connes and J. Lott, *Nucl. Phys. Proc. Suppl.* **B18** (1991) 29,

[9] H. Grosse and J. Madore *Phys. Lett.* **B283** (1992) 218,

[10] H. Grosse, C. Klimčík and P. Prešnajder, *Towards Finite Quantum Field Theory in Non-commutative Geometry*, preprint CERN-TH/95-138, UWThPh-19-1995, hep-th/9505175,

[11] D. Borthwick, S. Klimek, A. Lesniewski and M. Rinaldi, *Comm. Math. Phys.* **153** (1993) 49,

[12] A. El Gradechi, *On the Supersymplectic Homogeneous Superspace Underlying the $OSp(2,1)$ Coherent States*, Montreal preprint CRM-1850, 1994,

[13] A. El Gradechi and L. Nieto, *Supercoherent States, Super-Kähler Geometry and Geometric Quantization*, Montreal preprint CRM-1876, 1994, hep-th/9403109,

[14] I. Montvay and G. Muenster, *Quantum fields on a lattice*, Cambridge Univ. Press, Cambridge, 1994 and references therein,

[15] L.D. Landau and E.M. Lifshitz, *Quantum Mechanics*, Nauka, Moscow, 1989 (in russian),

[16] A.P. Prudnikov, Yu.A. Brytshkov and O.I. Maritshev, *Integrals and Series*, Nauka, Moscow, 1981, (in Russian),

[17] C. Itzykson and J.B. Zuber, *Quantum Field Theory*, Mir, Moscow, 1984 (in Russian),

[18] C. Jayewardena, Helv. Phys. Acta **61** (1988) 636,

[19] H. Grosse and P. Prešnajder, *Lett. Math. Phys.* **33** (1995) 171,

[20] M. Scheunert, W. Nahm and V. Rittenberg, *J. Math. Phys.* **18** (1977) 146,

[21] M. Scheunert, W. Nahm and V. Rittenberg, *J. Math. Phys.* **18** (1977) 154,

[22] M. Marcu, *J. Math. Phys.* **21** (1980) 1284,

[23] A. Pais and V. Rittenberg, *J. Math. Phys.* **16** (1975) 2062,

[24] M.A. Naimark, *Appendix* III. in *Linear Representations of the Lorentz Group*, Pergamon Press, Oxford, 1964,

[25] J. Glimm and A. Jaffe, *Quantum Physics; A Functional Integral Point of View*, Springer-Verlag, Heidelberg, 1981,

[26] C. Fronsdal, *3+2 de Sitter Superfields* in *Essays on Supersymmetry*, ed. by C. Fronsdal, D. Reidel Publishing Company, Dordrecht, 1986,

[27] H. Grosse, C. Klimčík and P. Prešnajder, *Spinors in Non-Commutative Geometry*, in preparation

[28] M. Chaichian, D. Ellinas and P. Prešnajder, *J. Math. Phys.* **32** (1991) 3381.