BONN-TH-95-12
ITPSB 95-18
HEP-TH 9507072

Polynomial Identities, Indices, and Duality for the $N = 1$ Superconformal Model $SM(2, 4\nu)$

Alexander Berkovich [1]

Physikalisches Institut der
Rheinischen Friedrich-Wilhelms Universität Bonn
Nussallee 12
D-53115 Bonn, Germany

and

Barry M. McCoy [2] and William P. Orrick[3]

Institute for Theoretical Physics
State University of New York
Stony Brook, NY 11794-3840

Dedicated to the memory of Claude Itzykson

7/95

[1] berkovic@axpib.physik.uni-bonn.de

[2] mccoy@max.physics.sunysb.edu

[3] worrick@insti.physics.sunysb.edu

Abstract

We prove polynomial identities for the $N=1$ superconformal model $SM(2,4\nu)$ which generalize and extend the known Fermi/Bose character identities. Our proof uses the q-trinomial coefficients of Andrews and Baxter on the bosonic side and a recently introduced very general method of producing recursion relations for q-series on the fermionic side. We use these polynomials to demonstrate a dual relation under $q \to q^{-1}$ between $SM(2,4\nu)$ and $M(2\nu-1,4\nu)$. We also introduce a generalization of the Witten index which is expressible in terms of the Rogers false theta functions.

1. Introduction

All chiral partition functions of conformal field theory have two distinct representations; 1) a bosonic form which may be expressed in terms of theta functions from which modular transformation properties are readily apparent [1] and 2) a fermionic form in terms of q-series in which the quasiparticle spectrum of the theory is clearly seen. The bosonic form is most useful in computing the conformal dimensions. The fermionic form is best adapted to study massive perturbations. The equality of the two forms can be thought of as generalized Rogers-Ramanujan identities.

The study of the bosonic representations has been well developed for over a decade. However, with the exceptions of the pioneering work on characters of $A_1^{(1)}$ [2]–[3] and the Z_N parafermionic theories [4]–[5] the study of the fermionic representations started only several years ago and in the last few years there have been many conjectures and proofs of fermionic representations of the various characters [6]–[37].

In this paper we consider the $N=1$ superconformal model $SM(2,4\nu)$. The bosonic form of this model's characters is a special case of the general formula [38]

$$\hat{\chi}_{r,s}^{(p,p')}(q) = \hat{\chi}_{p-r,p'-s}^{(p,p')}(q) = \frac{(-q^{\epsilon_{r-s}})_\infty}{(q)_\infty} \sum_{j=-\infty}^{\infty} \left(q^{\frac{j(jpp'+rp'-sp)}{2}} - q^{\frac{(jp+r)(jp'+s)}{2}}\right) \tag{1.1}$$

where

$$(A)_k = \begin{cases} \prod_{j=0}^{k-1}(1-Aq^j), & k=1,2,\cdots \\ 1, & \text{k=0} \end{cases} \tag{1.2}$$

and

$$\epsilon_a = \begin{cases} \frac{1}{2} & \text{if } a \text{ is even (Neveu-Schwarz (NS) sector)} \\ 1 & \text{if } a \text{ is odd (Ramond (R) sector)} \end{cases} \tag{1.3}$$

Here $r=1,2,\cdots,p-1$ and $s=1,2,\cdots,p'-1$ and p and $\frac{(p'-p)}{2}$ are coprime.

Setting $p=2$, $p'=4\nu$ in (1.1) we have for $n=0,\pm1$

$$\hat{\chi}^{(2,4\nu)}_{1,2\nu-2s'+|n|-1}(q)\equiv B^{(\nu,n)}_{s'}(q)=\frac{(-q^{\frac{1+|n|}{2}})_\infty}{(q)_\infty}\sum_{j=-\infty}^{\infty}(-1)^j q^{\nu j^2+j(s'+\frac{1-|n|}{2})} \tag{1.4}$$

where here and throughout the rest of the paper

$$s'=0,1,2,...,\nu-1 \tag{1.5}$$

and $n=0(\pm1)$ corresponds to the $NS(R)$ sector.

The Fermionic representations of $SM(2,4\nu)$ characters are given in terms of the function $F^{(\nu,n)}_{s'}(q)$ defined for $n=0,\pm1$ as follows

$$F^{(\nu,n)}_{s'}(q)=\sum_{m_1,n_2,\cdots,n_\nu\geq0}\frac{q^{Qf+Lf_{n,s'}}}{(q)_{n_2}(q)_{n_3}\cdots(q)_{n_\nu}}\begin{bmatrix}N_2\\m_1\end{bmatrix}_q, \tag{1.6}$$

where the q-binomial coefficient is defined in a slightly unconventional way as

$$\begin{bmatrix}l\\m\end{bmatrix}_q=\begin{cases}\frac{(q)_l}{(q)_m(q)_{l-m}} & \text{if } 0\leq m\leq l\\ 1 & \text{if } m=0,\ l\leq-1\\ 0 & \text{otherwise,}\end{cases} \tag{1.7}$$

the quadratic form Qf and linear form $Lf_{n,s'}$ are

$$Qf=\frac{m_1^2}{2}-m_1N_2+\sum_{j=2}^{\nu}N_j^2 \tag{1.8}$$

$$Lf_{n,s'}=n\frac{m_1}{2}+\sum_{l=\nu-s'+1}^{\nu}N_l, \tag{1.9}$$

with

$$N_k=\sum_{j=k}^{\nu}n_j. \tag{1.10}$$

Once again $n=0$ corresponds to the NS sector and $n=+1(-1)$ corresponds to the first (second) representation for the Ramond sector which we will call $R^+(R^-)$. We note in passing that the reason for existence of these two representations can be traced back to the fact that zero-modes of fermionic fields act nontrivially on the highest weight vectors.

The relation between the bosonic and fermionic forms depends on the characters studied. We consider three separate cases

1) For the Neveu-Schwarz sector we have:

$$B_{s'}^{(\nu,0)}(q) = F_{s'}^{(\nu,0)}(q); \tag{1.11}$$

2) For R^+

$$B_{s'}^{(\nu,1)}(q) = \begin{cases} \frac{1}{2}(F_{s'}^{(\nu,1)}(q) + F_{s'-1}^{(\nu,1)}(q)) & \text{for } s' \neq 0 \\ F_0^{(\nu,1)}(q) & \text{for } s' = 0; \end{cases} \tag{1.12}$$

3) For R^-

$$F_{s'}^{(\nu,-1)}(q) = \begin{cases} B_{\nu-1}^{(\nu,-1)}(q) & \text{for } s' = \nu - 1 \\ B_{s'}^{(\nu,-1)}(q) + B_{s'+1}^{(\nu,-1)}(q) & \text{for } s' \neq \nu - 1 \end{cases} \tag{1.13}$$

or, equivalently

$$B_{s'}^{(\nu,-1)}(q) = \sum_{l=s'}^{\nu-1} (-1)^{l+s'} F_l^{(\nu,-1)}(q). \tag{1.14}$$

In the Neveu-Schwarz sector the identities (1.11) are the generalizations to arbitrary ν by Andrews [39] (for $s' = 0$) and Bressoud [40] of the $\nu = 2$ results due to Slater (eqns. (34), (36) of [41]) also known as Göllnitz-Gordon identities [42] and [43]. For the Ramond cases R^- with $s' = \nu - 1$ and R^+ with $s' = 0$ the identities (1.12) and (1.13) have been conjectured by Melzer [37]. For all other values of s' the results of (1.12) and (1.13) are new. In general, the Ramond sector Fermi forms should also be compared with the result of Burge [44] (stated at the bottom of page 204 with the misprint $(q^2, q^2)_{n_{k-1}}$ corrected to $(q^2, q^2)_{n_{k-2}}$) where a free Fermi term is factored out and the number of variables in the sum is reduced to $\nu - 1$. A direct proof of the equivalence of (1.12) with [44] does not seem to be known.

The first purpose of this paper is to generalize both the bosonic and the fermionic expressions from infinite series to polynomials. Indeed, we will see that there are not one, but many distinct polynomials which generalize (1.1) and (1.6). We will then prove Fermi/Bose identities for these polynomials by obtaining recursion relations between several different polynomials which are related to a given character. These polynomial identities will reduce to (1.11)–(1.13) when the degree of polynomials goes to infinity. Our tools in this proof will be the use of the q-trinomial coefficients of Andrews and Baxter [45]–[47] on the bosonic side and the methods of ref. [22] on the fermionic side.

By the very name the $N = 1$ superconformal models have an interpretation in terms of a fermion and a boson, and one aspect of this interpretation is seen in the factorization of the bosonic form (1.1) into a free fermionic factor $(-q^{\epsilon_{r-s}})_\infty$ and another factor which looks as if it is obtained from a free boson by projecting out null states. Correspondingly, there should be an interpretation of the Fermi form (1.6) which separates the quasiparticles into one which represents the fermion and the rest which represent what in the bosonic form was called the projected boson. One such interpretation is instantly suggested by the form (1.6) itself where m_1 and n_i appear in quite different ways. We will thus adopt the tentative interpretation that m_1 is related to the fermion number operator F or perhaps more accurately that $(-1)^{m_1}$ is related to the chirality operator $(-1)^F$. With this identification we can consider the object

$$\tilde{F}^{(\nu,n)}_{s'}(q) = \sum_{m_1,n_2,\cdots,n_\nu \geq 0} \frac{(-1)^{m_1} q^{Qf+Lf_{n,s'}}}{(q)_{n_2}(q)_{n_3}\cdots(q)_{n_\nu}} \begin{bmatrix} N_2 \\ m_1 \end{bmatrix}_q \tag{1.15}$$

and ask what relation it has with

$$\mathrm{Tr}(-1)^F \exp(-H). \tag{1.16}$$

In the NS sector this relation is straightforward. Replacing $\sqrt{q}$ by $-\sqrt{q}$ in (1.11) we immediately note

$$\tilde{F}^{(\nu,0)}_{s'}(q) = F^{(\nu,0)}_{s'}(e^{2\pi i}q) = B^{(\nu,0)}_{s'}(e^{2\pi i}q) \tag{1.17}$$

Clearly, $\tilde{F}^{(\nu,0)}_{s'}(q)$ is the T-modular transform of $F^{(\nu,0)}_{s'}(q)$ and therefore must be equal to (1.16) according to [49]. In the Ramond sector we again find that there are two distinct cases. For R^+ we define in analogy with (1.12)

$$\tilde{B}^{(\nu,1)}_{s'}(q) = \begin{cases} \frac{1}{2}(\tilde{F}^{(\nu,1)}_{s'}(q) - \tilde{F}^{(\nu,1)}_{s'-1}(q)) & \text{for } s' \neq 0 \\ \tilde{F}^{(\nu,1)}_{0}(q) & \text{for } s' = 0. \end{cases} \tag{1.18}$$

Then since we prove in sec. 5 that

$$\tilde{F}^{(\nu,1)}_{s'}(q) = 1 \tag{1.19}$$

we see that

$$\tilde{B}^{(\nu,1)}_{s'}(q) = \begin{cases} 0 & \text{for } s' \neq 0 \\ 1 & \text{for } s' = 0 \end{cases} \tag{1.20}$$

which is equal to the Witten indices [48] as studied in [49]. We want to emphasize that formulas (1.18) are not identities, but definitions. However, in sec. 5 we will find polynomial identities for $s' \neq 0$, which provide extra motivation for the definitions above. For the case

$s' = 0$ an appropriate polynomial identity is still lacking. Our motivation in this case is the analogy with (1.12) and the fact that we have an agreement with the Witten index calculations of [49].

For the Ramond case of R^- we define in analogy with (1.14)

$$\tilde{B}_{s'}^{(\nu,-1)}(q) = \sum_{l=s'}^{\nu-1} \tilde{F}_l^{(\nu,-1)}(q). \tag{1.21}$$

In sec. 5 we find the bosonic companion of $\tilde{F}_{s'}^{(\nu,-1)}(q)$. Remarkably, it is not a constant, but rather is

$$\tilde{F}_{s'}^{(\nu,-1)}(q) = \begin{cases} I_{\nu-1}^{(\nu)}(q) & \text{for } s' = \nu - 1 \\ I_{s'}^{(\nu)}(q) - I_{s'+1}^{(\nu)}(q) & \text{for } s' \neq \nu - 1 \end{cases} \tag{1.22}$$

where

$$I_{s'}^{(\nu)}(q) = 1 + \sum_{j=1}^{\infty} q^{\nu j^2}(q^{s'j} - q^{-s'j}) \tag{1.23}$$

is the false theta function introduced by Rogers [50] and extensively studied by Andrews [51]. Thus

$$\tilde{B}_{s'}^{(\nu,-1)}(q) = I_{s'}^{(\nu)}(q). \tag{1.24}$$

We show in sec. 5 that

$$\lim_{q\to 1} I_{s'}^{(\nu)}(q) = 1 - \frac{s'}{\nu} \tag{1.25}$$

which suggests that it is possible to define for R^- a fractional analogue of the Witten index.

In sec. 2 we will state in detail the polynomial analogs of identities (1.11)–(1.13) and the sets of recursion relations we will use to prove them. In sec. 3 we will show that the fermionic polynomials satisfy these recursion relations and in sec. 4 we will do it for the bosonic polynomials. In sec. 5 we will discuss the Fermi forms $\tilde{F}_{s'}^{(\nu,\pm 1)}(q)$ and the indices $\tilde{B}_{s'}^{(\nu,\pm 1)}(q)$. In sec. 6 we will use the polynomial identities to study the dual relation which exists between $SM(2, 4\nu)$ and $M(2\nu - 1, 4\nu)$ under the replacement $q \to q^{-1}$. Finally in sec. 7 we will discuss representation theoretical consequences of two partition identities due to Burge. We will conclude with some remarks about possible generalizations and open questions. Technical details concerning q-trinomial coefficients will be treated in the appendix for continuity of presentation.

2. Polynomials and Recursion relations.

The starting point for proving Rogers-Ramanujan type identities by the method of [22] is identifying an $(\vec{n},\vec{m})$-system and an associated counting problem. For the present case the appropriate $(\vec{n},\vec{m})$-system is as follows

$$\begin{aligned} n_1+m_1 &= \frac{1}{2}(L+m_1-m_2)-a_1 \\ n_2+m_2 &= \frac{1}{2}(L+m_1+m_3)-a_2 \\ n_i+m_i &= \frac{1}{2}(m_{i-1}+m_{i+1})-a_i, \qquad \text{for } 3\leq i\leq \nu-1 \\ n_\nu+m_\nu &= \frac{1}{2}(m_{\nu-1}+m_\nu)-a_\nu \end{aligned} \tag{2.1}$$

where n_i and m_i are integers and the components a_i of the vector $\vec{a}$ are either integers or half integers. This system is closely related to the TBA equations for the XXZ-model ((3.9) of [52]) with anisotropy

$$\gamma=\pi\frac{(2\nu-1)}{4\nu}. \tag{2.2}$$

In the language of our previous treatment [36] of the $M(p,p')$ minimal models, system (2.1) consists of two Takahashi zones with tadpoles at the end of each zone. The principal difference between the present case and the one considered in [36] is the appearance of two inhomogeneous terms $\frac{L}{2}$ in the first and second equations. The second inhomogeneous term arises because $N=1$ superconformal models are derived from the spin 1 XXZ chain [53] while $N=0$ models, investigated in [36], are derived from the spin $\frac{1}{2}$ chain. The presence of the first term in (2.1) indicates that the spin 1 XXZ model with γ given by (2.2) is in the regime of strong anisotropy. This inhomogeneous term is not expected to be present for any other $N=1$ $SM(p,p')$ model with $\frac{2p'}{p'-p}\geq 3$.

The $(\vec{n},\vec{m})$-system (2.1) describes ν Fermi bands. Each band consists of n_i+m_i consecutive integers with only n_i distinct integers being occupied by the n_i quasiparticles. The remaining m_i integers can be thought of as holes. If one allows particles to move freely in each band (subject only to fermionic exclusion rules) then one is naturally led to the following counting problem

$$F(L)=\sum_{n_i,m_i\geq 0}\prod_{i=1}^{\nu}\begin{bmatrix} n_i+m_i \\ n_i \end{bmatrix} \tag{2.3}$$

where the summation variables n_i, m_i are related by (2.1) with $\vec{a}$ fixed to be zero for the time being. To calculate $F(L)$ we use three simple consequences of (2.1)

$$L = n_1 + m_\nu + \sum_{i=2}^{\nu}(2i-3)n_i \tag{2.4}$$

$$m_i = m_\nu + 2\sum_{j=i+1}^{\nu} N_j, \qquad i \geq 2 \tag{2.5}$$

$$n_1 + m_2 = N_2 \tag{2.6}$$

along with the generating function technique (sec. 2 of [22]) to obtain

$$F(L) = B(L) \tag{2.7}$$

where

$$B(L) = \sum_{j=-\infty}^{\infty}(-1)^j\left(\binom{L}{2\nu j}_2 + \binom{L}{2\nu j+1}_2\right) \tag{2.8}$$

and N_j was defined in (1.10).

The trinomial coefficients $\binom{L}{i}_2$ which appear in the above equation are given by

$$(z+1+\frac{1}{z})^L = \sum_{i=-L}^{L}\binom{L}{i}_2 z^i. \tag{2.9}$$

In what follows we will consider three different q-analogs of (2.7) associated with the NS and $R^{\pm}$ sectors. We remark that these q-deformations amount to prescribing the linear dispersion law for the quasiparticles described above. We also point out that one can use (2.1) and (2.4) to find a pictorial representation for quasiparticles in the spirit of [26]. This representation will be given elsewhere.

Motivated by (2.3) we now introduce the polynomial generalization of the fermionic form $F_{s'}^{(\nu,n)}(q)$ (1.6)-(1.9)

$$F_{r',s'}^{(\nu,n)}(L,q) = \sum_{\mathcal{D}_{r',s'}} q^{Qf+Lf_{n,s'}} \prod_{i=1}^{\nu}\begin{bmatrix} n_i + m_i \\ n_i \end{bmatrix}_q \tag{2.10}$$

where the "finitization" parameter r' is

$$r' = 0, 1, 2, \cdots, \nu - 2 \tag{2.11}$$

and the variables n_i, m_i are related by (2.1) with the vector $\vec{a}$ defined by

$$\begin{aligned} &\vec{a} = \vec{a}^{(r')} + \vec{a}^{(s')} \\ &a_i^{(k)} = \begin{cases} \frac{1}{2}(\delta_{i,\nu} - \delta_{i,\nu-k}) & \text{for } 0 \le k \le \nu - 2 \\ \frac{1}{2}(\delta_{i,\nu} + \delta_{i,1}) & \text{for } k = \nu - 1. \end{cases} \end{aligned} \tag{2.12}$$

The domain of summation, $\mathcal{D}_{r',s'}$ is best described in terms of $\vec{n}$ and m_ν which are subject to the constraint

$$L = (n_1 + a_1) + m_\nu + \sum_{i=2}^{\nu}(2i-3)(n_i + a_i). \tag{2.13}$$

All other variables are given by

$$m_1 = N_2 - n_1 \tag{2.14}$$

$$m_i = m_\nu + 2\sum_{j=i+1}^{\nu}(j-i)(n_j + a_j), \qquad i = 2, 3, \cdots, \nu - 1. \tag{2.15}$$

Keeping in mind that

$$\begin{bmatrix} \text{neg. int.} \\ 0 \end{bmatrix}_q = 1, \tag{2.16}$$

we define $\mathcal{D}_{r',s'}$ for $s' \ge r'$ as the union of the sets of solutions to (2.13) satisfying

$$\begin{aligned} &0: \quad n_i,\ m_\nu \ge 0, \\ &1: \quad n_\nu = 0, m_\nu = -2,\ n_1, \cdots, n_{\nu-1} \ge 0, \\ &2: \quad n_\nu = n_{\nu-1} = 0,\ m_\nu = -4,\ n_1, \cdots n_{\nu-2} \ge 0, \\ &\qquad \cdots \\ &r': \quad n_\nu = n_{\nu-1} = \cdots = n_{\nu-r'+1} = 0, \quad m_\nu = -2r', \quad n_1, \cdots, n_{\nu-r'} \ge 0; \end{aligned} \tag{2.17}$$

and for $s' < r'$ the definition is the same as above with $r' \to s'$.

Using the asymptotic formula

$$\lim_{A\to\infty} \begin{bmatrix} A \\ B \end{bmatrix}_q = \frac{1}{(q)_B} \tag{2.18}$$

and the simple consequence of (2.1)

$$n_i + m_i = L + m_1 + n_i - 2\sum_{j=2}^{i}(j-1)(n_j + a_j) - 2\sum_{j=i+1}^{\nu}(i-1)(n_j + a_j); \quad i \ge 2 \tag{2.19}$$

along with (2.14), we establish relations between $F^{(\nu,n)}_{r',s'}(L,q)$ and the fermionic forms (1.6)

$$\lim_{L\to\infty} F^{(\nu,n)}_{r',s'}(L,q) = F^{(\nu,n)}_{s'}(q) \tag{2.20}$$

which hold for all r'.

To write the bosonic polynomials one needs the q-analogs of the trinomial coefficients $\binom{L}{A}_2$ introduced in (2.9). Following Andrews and Baxter [45] we define

$$\binom{L, A-n; q}{A}_2 = \sum_{j\geq 0} t_n(L,A;j), \qquad n \in Z \tag{2.21}$$

and

$$T_n(L,A;q^{\frac{1}{2}}) = q^{\frac{L(L-n)-A(A-n)}{2}} \binom{L, A-n; q^{-1}}{A}_2 \tag{2.22}$$

where

$$t_n(L,A;j) = \frac{q^{j(j+A-n)}(q)_L}{(q)_j (q)_{j+A} (q)_{L-2j-A}}. \tag{2.23}$$

We note the elementary property

$$T_n(L,A;q^{\frac{1}{2}}) = T_n(L,-A;q^{\frac{1}{2}}) \tag{2.24}$$

and remark that

$$T_n(L,A;-q^{\frac{1}{2}}) = \begin{cases} (-1)^{L+A} T_n(L,A;q^{\frac{1}{2}}) & \text{for } n \text{ even} \\ T_n(L,A;q^{\frac{1}{2}}) & \text{for } n \text{ odd} \end{cases} \tag{2.25}$$

Consequently, $T_n(L,A;q^{\frac{1}{2}})$ is actually a polynomial in q for n odd or for n even and $L+A$ even, while for n even and $L+A$ odd $T_n(L,A,q^{\frac{1}{2}})$ contains only odd powers of $q^{\frac{1}{2}}$.

We then have the following definition of bosonic polynomials:

1) For the Neveu-Schwarz sector

$$\begin{aligned} B^{(\nu,0)}_{r',s'}(L,q) = \sum_{j=-\infty}^{\infty} (-1)^j q^{\nu j^2 + (s'+\frac{1}{2})j} \Big(& T_0(L, 2\nu j + s' - r'; q^{\frac{1}{2}}) \\ & + T_0(L, 2\nu j + s' + 1 + r'; q^{\frac{1}{2}}) \Big); \end{aligned} \tag{2.26}$$

2) For the Ramond sector R^+

$$\begin{aligned} & B^{(\nu,1)}_{r',s'}(L,q) = \\ & \frac{1}{2} \sum_{j=-\infty}^{\infty} (-1)^j q^{\nu j^2 + s' j} \Big(T_{-1}(L, 2\nu j + s' - r'; q^{\frac{1}{2}}) + T_{-1}(L, 2\nu j + s' + 1 + r'; q^{\frac{1}{2}}) \\ & \qquad + T_{-1}(L, 2\nu j + s' - r' - 1; q^{\frac{1}{2}}) + \ T_{-1}(L, 2\nu j + s' + r'; q^{\frac{1}{2}}) \Big); \end{aligned} \tag{2.27}$$

3) For the Ramond sector R^-

$$B^{(\nu,-1)}_{r',s'}(L,q)=\sum_{j=-\infty}^{\infty}(-1)^j q^{\nu j^2+s'j}\sum_{i=-r'}^{r'}(-1)^{r'+i}T_1(L,2\nu j+s'+i;q^{\frac{1}{2}}) \tag{2.28}$$

where $s'=0,1,2,\cdots,\nu-1$ and $r'=0,1,2,\cdots,\nu-2$.

Using the limiting formula of the appendix

$$\lim_{L\to\infty}T_n(L,A;q^{\frac{1}{2}})=\begin{cases}\frac{(-q^{\frac{(1-n)}{2}})_\infty+(q^{\frac{(1-n)}{2}})_\infty}{2(q)_\infty} & \text{if } L-A \text{ is even}\\ \frac{(-q^{\frac{(1-n)}{2}})_\infty-(q^{\frac{(1-n)}{2}})_\infty}{2(q)_\infty} & \text{if } L-A \text{ is odd.}\end{cases} \tag{2.29}$$

and noting the special case

$$\lim_{L\to\infty}T_1(L,A;q^{\frac{1}{2}})=\frac{(-q)_\infty}{(q)_\infty} \tag{2.30}$$

we find the relation between the polynomials $B^{(\nu,n)}_{r',s'}(L,q)$ and the characters (1.4)

$$\begin{aligned}&\lim_{L\to\infty}B^{(\nu,0)}_{r's'}(L,q)=B^{(\nu,0)}_{s'}(q)\\ &\lim_{L\to\infty}B^{(\nu,1)}_{r',s'}(L,q)=\lim_{L\to\infty}B^{(\nu,-1)}_{r',s'}(L,q)=B^{(\nu,\pm1)}_{s'}(q)\end{aligned} \tag{2.31}$$

which holds for all r'.

We will prove the following polynomial identities which generalize the character identities (1.11)–(1.13)

1) For NS

$$F^{(\nu,0)}_{r',s'}(L,q)=B^{(\nu,0)}_{r',s'}(L,q); \tag{2.32}$$

2) For R^+

$$B^{(\nu,1)}_{r',s'}(L,q)=\begin{cases}\frac{1}{2}(F^{(\nu,1)}_{r',s'}(L,q)+F^{(\nu,1)}_{r',s'-1}(L,q)) & \text{for } s'\neq 0\\ F^{(\nu,1)}_{r',0}(L,q) & \text{for } s'=0;\end{cases} \tag{2.33}$$

3) For R^-

$$F^{(\nu,-1)}_{r',s'}(L,q)=\begin{cases}B^{(\nu,-1)}_{r',\nu-1}(L,q) & \text{for } s'=\nu-1\\ B^{(\nu,-1)}_{r',s'}(L,q)+B^{(\nu,-1)}_{r',s'+1}(L,q) & \text{for } s'\neq\nu-1.\end{cases} \tag{2.34}$$

by showing that both $F^{(\nu,n)}_{r',s'}(L,q)$ and $B^{(\nu,n)}_{r',s'}(L,q)$ satisfy the following set of recursion relations for $\nu \geq 3$ in the variables L and r'

$$\begin{aligned}
h_0(L) &= h_1(L-1) + (q^{L-\frac{1-n}{2}}+1)h_0(L-1) + (q^{L-1}-1)h_0(L-2),\\
h_r(L) &= h_{r-1}(L-1) + h_{r+1}(L-1) + q^{L-\frac{1-n}{2}}h_r(L-1) + (q^{L-1}-1)h_r(L-2)\\
&\qquad\qquad\qquad\qquad \text{for } 1 \leq r \leq \nu-3,\\
h_{\nu-2}(L) &= h_{\nu-3}(L-1) + q^{L-\frac{1-n}{2}}h_{\nu-2}(L-1) + q^{L-1}h_{\nu-2}(L-2);
\end{aligned} \tag{2.35}$$

where

$$n = \begin{cases} 0 & \text{for } NS \\ 1 & \text{for } R^+ \\ -1 & \text{for } R^-. \end{cases} \tag{2.36}$$

Note that the first and the last equations follow from the middle equation if one introduces $h_{-1}(L)$ and $h_{\nu-1}(L)$ satisfying

$$h_{-1}(L) = h_0(L) \tag{2.37}$$

and

$$h_{\nu-1}(L) = h_{\nu-2}(L-1). \tag{2.38}$$

For $\nu = 2$ there is only the single equation

$$h_0(L) = (1 + q^{L-\frac{1-n}{2}})h_0(L-1) + q^{L-1}h_0(L-2). \tag{2.39}$$

Observe that the recursion relations in the sectors NS and $R^{\pm}$ are independent of s'. The proof of the polynomial identities will be completed by showing that (2.32)–(2.34) hold for $L = 0, 1$. We record here the values of the fermionic and bosonic forms at $L = 0, 1$, computed directly from (2.10) and (2.26)–(2.28). Notice that there is no dependence on ν. The fermionic forms are

$$\begin{aligned}
F^{(\nu,n)}_{r',s'}(0,q) &= \delta_{r',s'},\\
F^{(\nu,0)}_{r',s'}(1,q) &= \begin{cases} 1+q^{\frac{1}{2}} & \text{if } r'=s'=0 \\ q^{\frac{1}{2}} & \text{if } r'=s'\geq 1 \\ 1 & \text{if } r'=s'+1 \text{ or } s'=r'+1 \\ 0 & \text{otherwise,} \end{cases}\\
F^{(\nu,1)}_{r',s'}(1,q) &= \begin{cases} 1+q & \text{if } r'=s'=0 \\ q & \text{if } r'=s'\geq 1 \\ 1 & \text{if } r'=s'+1 \text{ or } s'=r'+1 \\ 0 & \text{otherwise,} \end{cases}\\
F^{(\nu,-1)}_{r',s'}(1,q) &= \begin{cases} 2 & \text{if } r'=s'=0 \\ 1 & \text{if } r'=s'\geq 1 \text{ or } r'=s'+1 \text{ or } s'=r'+1 \\ 0 & \text{otherwise.} \end{cases}
\end{aligned} \tag{2.40}$$

The bosonic forms are

$$
\begin{aligned}
B^{(\nu,0)}_{r',s'}(0,q) =&\delta_{r',s'},\\
B^{(\nu,0)}_{r',s'}(1,q) =&\begin{cases} 1+q^{\frac{1}{2}} & \text{if } r'=s'=0\\ q^{\frac{1}{2}} & \text{if } r'=s'\geq 1\\ 1 & \text{if } r'=s'+1 \text{ or } s'=r'+1\\ 0 & \text{otherwise,}\end{cases}\\
B^{(\nu,1)}_{r',s'}(0,q) =&\begin{cases} 1 & \text{if } r'=s'=0\\ \frac{1}{2} & \text{if } r'=s'\geq 1 \text{ or } s'=r'+1\\ 0 & \text{otherwise,}\end{cases}\\
B^{(\nu,1)}_{r',s'}(1,q) =&\begin{cases} 1+q & \text{if } r'=s'=0\\ 1+\frac{1}{2}q & \text{if } s'=1,\ r'=0\\ \frac{1}{2}+\frac{1}{2}q & \text{if } r'=s'\geq 1 \text{ or } s'=r'+1\geq 2\\ 1 & \text{if } s'=0,\ r'=1\\ \frac{1}{2} & \text{if } r'=s'+1\geq 2 \text{ or } s'=r'+2\\ 0 & \text{otherwise,}\end{cases}\\
B^{(\nu,-1)}_{r',s'}(0,q) =&\begin{cases} 0 & \text{if } r'<s'\\ (-1)^{r'+s'} & \text{if } r'\geq s',\end{cases}\\
B^{(\nu,-1)}_{r',s'}(1,q) =&\begin{cases} 1 & \text{if } r'=s'=0 \text{ or } s'=r'+1\\ (-1)^{r'+s'+1} & \text{if } r'>s'\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}
\tag{2.41}
$$

Equations (2.32)–(2.34) may be readily verified using these expressions. The character identities (1.11)–(1.13) will follow from the $L\to\infty$ limit of the polynomial identities (2.32)–(2.34) thanks to (2.20) and (2.31).

We close this presentation of results and methods with several remarks. First, attention should be drawn to the presence in the fermionic forms of solutions (2.17) to the system (2.1) with negative values for m_ν. This is the first time such solutions have been explicitly encountered, but it is expected that they will also be found in other nonunitary models such as $M(p,p')$ for $p+1\neq p'$. Secondly, we direct attention to the occurrence of linear combinations in the $R^{\pm}$ sectors (2.33)–(2.34). Such linear combinations have been seen in several other situations and are presumably a generic feature of Fermi/Bose correspondences although for the unitary model $M(p,p+1)$ the Bose and Fermi polynomials appear only singly. We also remark on the crucial role played by the fact that there are many different polynomials which “finitize” the same fermionic character. This is a general feature which, for example, occurs in the proof of the identities of the nonunitary $M(p,p')$ minimal model [36].

Finally, we comment that the Fermi and Bose polynomials and the corresponding recursion relations given here are not particularly unique. As an example, for R^+ with $s'=0$ we have two alternative representations

$$B^{(\nu,1)}_{r',0}(L,q)=B1^{(\nu)}_{r'}(L,q)=B2^{(\nu)}_{r'}(L,q) \tag{2.42}$$

where

$$\begin{aligned}B1^{(\nu)}_{r'}(L,q)=\sum_{j=-\infty}^{\infty}(-1)^j\big(&q^{\nu j^2}T_1(L,2\nu j+r'+1;q^{\frac{1}{2}})\\&+q^{\nu j(j-1)+\frac{(L-r')}{2}}T_0(L,2\nu j+r';q^{\frac{1}{2}})\big)\end{aligned} \tag{2.43}$$

and

$$\begin{aligned}B2^{(\nu)}_{r'}(L,q)=\sum_{j=-\infty}^{\infty}(-1)^jq^{\nu j^2}\Big[&\sum_{i=r'+1}^{\nu-1}(-1)^{i+1+r'}T_1(L+1,2\nu j+i;q^{\frac{1}{2}})\\&+\sum_{i=r'+1}^{\nu-2}(-1)^{i+1+r'}T_1(L,2\nu j+i;q^{\frac{1}{2}})\Big].\end{aligned} \tag{2.44}$$

More generally, there are systems of polynomials which reduce to the characters in the $L\to\infty$ limit and satisfy slightly different systems of equations from the one given here. However, in all these cases the new polynomials may be expressed as linear combinations of the polynomials given above.

3. Proof of Fermionic Recursion Relations

We now turn to the proof that the fermionic sums of sec. 2 defined by (2.10) satisfy the recursion relations (2.35). The proof is based upon the use of telescopic expansions of products of q-binomial coefficients developed in [22]. In contrast with the many identities on q-trinomial coefficients we shall use in the proof of the Bosonic identities the only identities we require for the proof of the Fermionic recursion relations are the elementary recursion relations for q-binomial coefficients

$$\begin{bmatrix}n+m\\n\end{bmatrix}_q=\begin{bmatrix}n+m-1\\n\end{bmatrix}_q+q^m\begin{bmatrix}n+m-1\\n-1\end{bmatrix}_q \tag{3.1}$$

and

$$\begin{bmatrix}n+m\\n\end{bmatrix}_q=\begin{bmatrix}n+m-1\\n-1\end{bmatrix}_q+q^n\begin{bmatrix}n+m-1\\n\end{bmatrix}_q. \tag{3.2}$$

We note that in order for these two identities to be used in our proofs without exception we need to use the definition (1.7).

In order to give a compact proof we introduce the following symbolic notation for fermionic sums

$$q^{\phi(\vec{n},\vec{A})}\left\{\begin{matrix}\vec{P}\\ \vec{Q}\end{matrix}\right\}_{\mathcal{D}} = \sum_{\mathcal{D}} q^{\phi(\vec{n},\vec{A})} \prod_{j=1}^{\nu} \begin{bmatrix} n_j + m_j + P_j \\ n_j + Q_j \end{bmatrix}_q \tag{3.3}$$

where

$$\phi(\vec{n},\vec{A}) = \frac{1}{2}(n_1^2 + N_2^2) + \sum_{i=3}^{\nu} N_i^2 + \vec{A}\cdot\vec{n}, \tag{3.4}$$

and $\mathcal{D}$ specifies the domain of summation variables m_i and n_i which are related by (2.1). In what follows we will use three domains $\mathcal{D}_{r',s'}, \tilde{\mathcal{D}}_{r',s'}, \mathcal{D}'_{r',s'}$, where

1) $\mathcal{D}_{r',s'}$ was defined in sec. 2 by (2.17);

2) $\tilde{\mathcal{D}}_{r',s'}$ is defined by (2.13)–(2.15) and

$$m_\nu = -2r', \quad n_{\nu-r'} = n_{\nu-r'+1} = \cdots = n_\nu = 0, \quad n_1, n_2, \cdots, n_{\nu-r'-1} \geq 0; \tag{3.5}$$

3) $\mathcal{D}'_{r',s'}$ is defined the same way as $\mathcal{D}_{r',s'}$ except that $n_{\nu-r'-1} \geq -1$ (whereas it was $n_{\nu-r'-1} \geq 0$ for $\mathcal{D}_{r',s'}$).

In terms of this notation we write the fermionic polynomials for arbitrary $\vec{A}$

$$F^{\nu}_{r',s'}(L,\vec{A},q) = q^{\phi(\vec{n},\vec{A})}\left\{\begin{matrix}0\\ 0\end{matrix}\right\}_{\mathcal{D}_{r',s'}}. \tag{3.6}$$

To avoid bulky formulas we find it convenient to use shorthand notations

$$F_{r'}(L) \equiv F^{\nu}_{r',s'}(L,\vec{A},q) \tag{3.7}$$

$$\phi(\vec{n}) \equiv \phi(\vec{n},\vec{A}) \tag{3.8}$$

throughout the rest of this section.

All the equations of (2.35) are special cases of the following set of recursion relations for $\nu \geq 3$

$$F_0(L) = F_1(L-1) + (q^{L-\frac{1}{2}+\alpha}+1)F_0(L-1) + (q^{L-1+\beta}-1)F_0(L-2) \tag{3.9}$$

$$\begin{aligned} F_{r'}(L) = {} & F_{r'-1}(L-1) + F_{r'+1}(L-1) \\ & + q^{L-\frac{1}{2}+\alpha}F_{r'}(L-1) + (q^{L-1+\beta}-1)F_{r'}(L-2) \text{ for } 1 \leq r' \leq \nu-3 \end{aligned} \tag{3.10}$$

and

$$F_{\nu-2}(L) = F_{\nu-3}(L-1) + q^{L-\frac{1}{2}+\alpha} F_{\nu-1}(L-1) + q^{L-1+\beta} F_{\nu-1}(L-2) \tag{3.11}$$

where

$$\begin{aligned} \alpha &= A_2 - \tilde{a}_1^{(s')} \\ \beta &= A_1 + \alpha \\ \tilde{a}_i^{(s')} &= \sum_{j=i}^{\nu} a_j^{(s')} = \delta_{s',\nu-1}\delta_{i,1} + \frac{1}{2}\theta(\nu - s' < i) \end{aligned} \tag{3.12}$$

and $\theta(a < b) = 1$ if $a < b$ and 0 otherwise.

When $\nu = 2$ the equation (2.39) follows from the single equation

$$F_0(L) = (1 + q^{L-\frac{1}{2}+\alpha}) F_0(L-1) + q^{L-1+\beta} F_0(L-2) \tag{3.13}$$

We will find that in order for (3.9)–(3.11) to hold $\vec{A}, \vec{\tilde{a}}^{(s')}$ should satisfy

$$A_{i+1} - A_i = 2\tilde{a}_{i+1}^{(s')}, \quad \text{for } 2 \le i \le \nu - 1. \tag{3.14}$$

As a consequence of (3.14) only A_1 and A_2 may be specified independently of the inhomogeneous vector $\vec{a}^{(s')}$.

Making use of (1.8), (1.9), (3.12) and

$$m_1 = N_2 - n_1 \tag{3.15}$$

one verifies that for $\vec{A}$ defined by (3.14) with

$$A_1 = -\frac{n}{2}, \quad A_2 = \frac{n}{2} + \delta_{s',\nu-1}; \quad n = 0, \pm 1, \tag{3.16}$$

$\alpha \to \frac{n}{2}, \beta \to 0, \phi(\vec{n}) \to Qf + Lf_{n,s'}$ and therefore the fermionic forms (3.6) and recursion relations (3.9)–(3.11) reduce to (2.10) and (2.35).

Let us denote the set of solutions of (2.1) with the inhomogeneous vector (2.12) as $\{\vec{n}, \vec{m}\}_{L,r',s'}$. Then, if we define vectors $\vec{e}_l$ and $\vec{E}_{l,k}$ by

$$(\vec{e}_l)_i = \delta_{l,i} \qquad \vec{E}_{l,k} = -\sum_{i=l}^{k} \vec{e}_i \tag{3.17}$$

we may use (2.1) to verify the following relations

$$
\begin{aligned}
\{\vec{n},\vec{m}\}_{L-1,r'-1,s'} &= \{\vec{n},\vec{m}\}_{L,r',s'} + \{0,\vec{E}_{2,\nu-r'}\} \\
\{\vec{n},\vec{m}\}_{L-1,r',s'} &= \{\vec{n},\vec{m}\}_{L,r',s'} + \{-\vec{e}_2,-\vec{e}_1\} \\
\{\vec{n},\vec{m}\}_{L-2,r',s'} &= \{\vec{n},\vec{m}\}_{L,r',s'} + \{-\vec{e}_1-\vec{e}_2,0\} \\
\{\vec{n},\vec{m}\}_{L-2,r',s'} &= \{\vec{n},\vec{m}\}_{L,r',s'} + \{\vec{e}_{\nu-r'-1}-\vec{e}_{\nu-r'},2\vec{E}_{2,\nu-r'-1}\} \text{ for } \nu-r'\geq 3 \\
\{\vec{n},\vec{m}\}_{L-1,r'+1,s'} &= \{\vec{n},\vec{m}\}_{L,r',s'} + \{\vec{e}_{\nu-r'-1}-\vec{e}_{\nu-r'},\vec{E}_{2,\nu-r'-1}\} \text{ for } \nu-r'\geq 3
\end{aligned} \tag{3.18}
$$

.

Furthermore if we recall

$$
\begin{aligned}
L &= n_1 + a_1 + \sum_{i=2}^{\nu}(2i-3)(n_i+a_i) + m_\nu \\
L &= n_1 + N_2 + m_2 + \tilde{a}_1 \\
m_i &= 2\sum_{l=i+1}^{\nu}(N_l+\tilde{a}_l) + m_\nu; \quad i\geq 2
\end{aligned} \tag{3.19}
$$

and use (3.14), we may verify the following identities for $\phi(\vec{n},\vec{A})$

$$
\begin{aligned}
&\phi(\vec{n}) + n_1 + m_2 = \phi(\vec{n}-\vec{e}_2) + L - \frac{1}{2} + \alpha \\
&\phi(\vec{n}) + m_2 = \phi(\vec{n}-\vec{e}_1-\vec{e}_2) + L - 1 + \beta \\
&[\phi+m_l](\vec{n}-\vec{e}_{l-1}+\vec{e}_l+\vec{e}_{\nu-r'-1}-\vec{e}_{\nu-r'}) = \tilde{\phi}(\vec{n}) + m_{l-1} - 1, \quad \text{for } 3\leq l\leq \nu-r' \\
&\text{with } \ \tilde{\phi}(\vec{n}) \equiv \phi(\vec{n}+\vec{e}_{\nu-r'-1}-\vec{e}_{\nu-r'}).
\end{aligned} \tag{3.20}
$$

Then from (3.18) and (3.20) we obtain the following expressions:

$$F_{r'}(L) = q^{\phi(\vec{n})}\begin{Bmatrix}0\\0\end{Bmatrix}_{\mathcal{D}_{r',s'}} \tag{3.21}$$

$$F_{r'-1}(L-1) = q^{\phi(\vec{n})}\begin{Bmatrix}\vec{E}_{2,\nu-r'}\\0\end{Bmatrix}_{\mathcal{D}_{r',s'}} - \mathcal{B} \tag{3.22}$$

$$q^{L-\frac{1}{2}+\alpha}F_{r'}(L-1) = q^{\phi(\vec{n})+n_1+m_2}\begin{Bmatrix}\vec{E}_{1,2}\\-\vec{e}_2\end{Bmatrix}_{\mathcal{D}_{r',s'}} \tag{3.23}$$

$$q^{L-1+\beta}F_{r'}(L-2) = q^{\phi(\vec{n})+m_2}\begin{Bmatrix}\vec{E}_{1,2} \\ \vec{E}_{1,2}\end{Bmatrix}_{\mathcal{D}_{r',s'}} \tag{3.24}$$

$$F_{r'}(L-2) = q^{\tilde{\phi}(\vec{n})}\begin{Bmatrix}2\vec{E}_{2,\nu-r'-1} + \vec{e}_{\nu-r'-1} - \vec{e}_{\nu-r'} \\ \vec{e}_{\nu-r'-1} - \vec{e}_{\nu-r'}\end{Bmatrix}_{\mathcal{D}'_{r',s'}} \tag{3.25}$$

$$F_{r'+1}(L-1) = q^{\tilde{\phi}(\vec{n})}\begin{Bmatrix}\vec{E}_{2,\nu-r'-1} + \vec{e}_{\nu-r'-1} - \vec{e}_{\nu-r'} \\ \vec{e}_{\nu-r'-1} - \vec{e}_{\nu-r'}\end{Bmatrix}_{\mathcal{D}'_{r',s'}} + \mathcal{B} \tag{3.26}$$

where

$$\mathcal{B} \equiv q^{\phi(\vec{n})}\theta(s'>r')\begin{Bmatrix}\vec{E}_{2,\nu-r'-1} \\ 0\end{Bmatrix}_{\tilde{\mathcal{D}}_{r',s'}} \tag{3.27}$$

and $\theta(s'>r') = 1$ if $s'>r'$ and 0 otherwise. We note that the term $\mathcal{B}$ arises because in general $\mathcal{D}_{r',s'} \neq \mathcal{D}_{r'\pm1,s'}$.

The method we use to prove (3.9)–(3.11) is the telescopic expansion technique of [22] which is based on the following two identities which follow from (3.1):

1)Telescopic expansion from right to left

$$\begin{Bmatrix}\vec{P} \\ \vec{Q}\end{Bmatrix} = \begin{Bmatrix}\vec{P}+\vec{E}_{l,k} \\ \vec{Q}\end{Bmatrix} + \sum_{i=l}^{k} q^{m_i+P_i-Q_i}\begin{Bmatrix}\vec{P}+\vec{E}_{l,i} \\ \vec{Q}-\vec{e}_i\end{Bmatrix}; \tag{3.28}$$

2)Telescopic expansion from left to right

$$\begin{Bmatrix}\vec{P} \\ \vec{Q}\end{Bmatrix} = \begin{Bmatrix}\vec{P}+\vec{E}_{l,k} \\ \vec{Q}\end{Bmatrix} + \sum_{i=l}^{k} q^{m_i+P_i-Q_i}\begin{Bmatrix}\vec{P}+\vec{E}_{i,k} \\ \vec{Q}-\vec{e}_i\end{Bmatrix}. \tag{3.29}$$

The proof of (3.9) will follow from (3.10) with the definition $F_{-1}(L) \equiv F_0(L)$. To prove (3.10) we begin by applying the right to left telescopic expansion (3.28) to $F_r(L)$ to obtain

$$F_{r'}(L) = q^{\phi(\vec{n})}\begin{Bmatrix}\vec{E}_{2,\nu-r'} \\ 0\end{Bmatrix}_{\mathcal{D}_{r',s'}} + \sum_{l=2}^{\nu-r'} q^{\phi(\vec{n})+m_l}\begin{Bmatrix}\vec{E}_{2,l} \\ -\vec{e}_l\end{Bmatrix}_{\mathcal{D}_{r',s'}} \tag{3.30}$$

and then further expand the term in the sum with $l=2$ using (3.2) to get

$$\begin{aligned}F_{r'}(L) = {} & q^{\phi(\vec{n})}\begin{Bmatrix}\vec{E}_{2,\nu-r'} \\ 0\end{Bmatrix}_{\mathcal{D}_{r',s'}} - \mathcal{B} \\ & + q^{\phi(\vec{n})+n_1+m_2}\begin{Bmatrix}\vec{E}_{1,2} \\ -\vec{e}_2\end{Bmatrix}_{\mathcal{D}_{r',s'}} + q^{\phi(\vec{n})+m_2}\begin{Bmatrix}\vec{E}_{1,2} \\ \vec{E}_{1,2}\end{Bmatrix}_{\mathcal{D}_{r',s'}} + Z\end{aligned} \tag{3.31}$$

where

$$Z \equiv \sum_{l=3}^{\nu-r'} q^{\phi(\vec{n})+m_l} \begin{Bmatrix} \vec{E}_{2,l} \\ -\vec{e}_l \end{Bmatrix}_{\mathcal{D}_{r',s'}} + \mathcal{B}. \tag{3.32}$$

Then making use of (3.22)–(3.24) we find

$$F_{r'}(L) - F_{r'-1}(L-1) - q^{L-\frac{1}{2}+\alpha} F_{r'}(L-1) - q^{L-1+\beta} F_{r'}(L-2) = Z. \tag{3.33}$$

We then change the summation variables in the l^{th} term in the expansion of Z as

$$\vec{n} \to \vec{n} - \vec{e}_{l-1} + \vec{e}_l + \vec{e}_{\nu-r'-1} - \vec{e}_{\nu-r'} \qquad \text{for } 3 \le l \le \nu - r' \tag{3.34}$$

where we note that this change depends on l and sends the domain $\mathcal{D}_{r',s'}$ to $\mathcal{D}'_{r',s'}$. Thus, making use of (3.20), we obtain

$$Z = \sum_{l=2}^{\nu-r'-1} q^{\tilde{\phi}(\vec{n})+m_l-1} \begin{Bmatrix} \vec{E}_{2,\nu-r'-1} + \vec{E}_{l,\nu-r'-1} + \vec{e}_{\nu-r'-1} - \vec{e}_{\nu-r'} \\ -\vec{e}_l + \vec{e}_{\nu-r'-1} - \vec{e}_{\nu-r'} \end{Bmatrix}_{\mathcal{D}'_{r',s'}} + \mathcal{B}. \tag{3.35}$$

To complete the proof we expand $F_{r'+1}(L-1) - \mathcal{B}$ given by (3.26) using the left to right telescopic expansion (3.29) as

$$\begin{aligned} F_{r'+1}(L-1) &= q^{\tilde{\phi}(\vec{n})} \begin{Bmatrix} \vec{E}_{2,\nu-r'-1} + \vec{e}_{\nu-r'-1} - \vec{e}_{\nu-r'} \\ \vec{e}_{\nu-r'-1} - \vec{e}_{\nu-r'} \end{Bmatrix}_{\mathcal{D}'_{r',s'}} + \mathcal{B} \\ &= q^{\tilde{\phi}(\vec{n})} \begin{Bmatrix} 2\vec{E}_{2,\nu-r'-1} + \vec{e}_{\nu-r'-1} - \vec{e}_{\nu-r'} \\ \vec{e}_{\nu-r'-1} - \vec{e}_{\nu-r'} \end{Bmatrix}_{\mathcal{D}'_{r',s'}} \\ &+ \sum_{l=2}^{\nu-r'-1} q^{\tilde{\phi}(\vec{n})+m_l-1} \begin{Bmatrix} \vec{E}_{2,\nu-r'-1} + \vec{E}_{l,\nu-r'-1} + \vec{e}_{\nu-r'-1} - \vec{e}_{\nu-r'} \\ -\vec{e}_l + \vec{e}_{\nu-r'-1} - \vec{e}_{\nu-r'} \end{Bmatrix}_{\mathcal{D}'_{r',s'}} + \mathcal{B}. \end{aligned} \tag{3.36}$$

Thus comparing the right hand side of (3.36) with (3.25) and (3.35) we obtain

$$F_{r'+1}(L-1) - F_{r'}(L-2) = Z \tag{3.37}$$

and hence, the desired result (3.10) follows from comparing (3.33) and (3.37).

It remains to prove (3.11). To do this we expand $F_{\nu-2}(L)$ as

$$F_{\nu-2}(L) = q^{\phi(\vec{n})} \begin{Bmatrix} \vec{E}_{2,2} \\ 0 \end{Bmatrix}_{\mathcal{D}_{\nu-2,s'}} + q^{\phi(\vec{n})+n_1+m_2} \begin{Bmatrix} \vec{E}_{1,2} \\ -\vec{e}_2 \end{Bmatrix}_{\mathcal{D}_{\nu-2,s'}} + q^{\phi(\vec{n})+m_2} \begin{Bmatrix} \vec{E}_{1,2} \\ \vec{E}_{1,2} \end{Bmatrix}_{\mathcal{D}_{\nu-2,s'}} \tag{3.38}$$

from which (3.11) follows upon using (3.22)–(3.24).

The proof of the equation (3.13) for $\nu = 2$ is completely analogous to the proof of (3.11) and will be omitted.

We close this section with a few remarks. The major new feature of this derivation which did not occur in [22] is the occurrence of the extra terms (2.17) in the allowed range of solutions $\mathcal{D}_{r',s'}$ of the constraint equations (2.1). These terms are forced upon us by the necessity of using the recursion relation (3.1) for the case $m = n = 0$ and is what requires us to keep track of the three different domains of definitions $\mathcal{D}_{r',s'}$, $\mathcal{D}'_{r',s'}$ and $\tilde{\mathcal{D}}_{r',s'}$ and the resulting boundary terms $\mathcal{B}$. This complicates the presentation, but since none of these terms make an explicit contribution to the equations we advise the reader to ignore them on first reading. Clearly, the method used can be extended to the general case where $\tilde{\vec{a}}^{(s')}$ and $\vec{A}$ are subject only to (3.14). We plan to discuss this in a separate publication.

4. Proof of Bosonic Recursion Relations

Our proof that the bosonic forms (2.26)–(2.28) satisfy the recursion relations (2.35) relies on various identities satisfied by the q-trinomial coefficients. Some of these have appeared previously in the literature [45]–[47] and some occur in this proof for the first time. For clarity we will first list all the identities we shall require and relegate the proofs of the new ones to the Appendix. We will then use these identities to verify the bosonic form of the recursion relations. The three distinct cases will be considered in separate subsections for $\nu \geq 3$. The special case $\nu = 2$ is easily treated by the same methods but the proof will be omitted.

4.1. Identities of q-trinomials

In the course of our proofs we will need several identities satisfied by the q-trinomials. These identities are of three types:

A) Pascal triangle identities which are nontrivial for $q = 1$:

$$\begin{aligned} T_{-n}(L, A; q^{\frac{1}{2}}) =& T_{-n}(L-1, A+1; q^{\frac{1}{2}}) + T_{-n}(L-1, A-1; q^{\frac{1}{2}}) \\ &+ q^{L-\frac{1-n}{2}} T_{-n}(L-1, A; q^{\frac{1}{2}}) + (q^{L-1}-1) T_{-n}(L-2, A; q^{\frac{1}{2}}), \end{aligned} \tag{4.1}$$

$$\begin{aligned} &T_1(L, A; q^{\frac{1}{2}}) - T_1(L-1, A; q^{\frac{1}{2}}) \\ &\quad = q^{\frac{L+A}{2}} T_0(L-1, A+1; q^{\frac{1}{2}}) + q^{\frac{L-A}{2}} T_0(L-1, A-1; q^{\frac{1}{2}}), \end{aligned} \tag{4.2}$$

B) Identities derivable from the Pascal triangle identities (for $q = 1$):

$$
\begin{aligned}
&T_0(L, A; q^{\frac{1}{2}}) - T_0(L-1, A-1; q^{\frac{1}{2}}) \\
&\quad = q^{A+\frac{1}{2}}\big[T_0(L, A+1; q^{\frac{1}{2}}) - T_0(L-1, A+2; q^{\frac{1}{2}})\big],
\end{aligned}
\tag{4.3}
$$

$$
\begin{aligned}
&T_1(L, A; q^{\frac{1}{2}}) - T_1(L, A+1; q^{\frac{1}{2}}) \\
&\quad = q^{\frac{L-A}{2}} T_0(L-1, A-1; q^{\frac{1}{2}}) - q^{\frac{L+A+1}{2}} T_0(L-1, A+2; q^{\frac{1}{2}}),
\end{aligned}
\tag{4.4}
$$

$$
\begin{aligned}
&T_1(L+1, A; q^{\frac{1}{2}}) + T_1(L, A; q^{\frac{1}{2}}) \\
&\quad = T_{-1}(L, A+1; q^{\frac{1}{2}}) + T_{-1}(L, A-1; q^{\frac{1}{2}}) + 2T_{-1}(L, A; q^{\frac{1}{2}})
\end{aligned}
\tag{4.5}
$$

C) Identities which become tautologies when $q = 1$:

$$
\begin{aligned}
&T_1(L, A; q^{\frac{1}{2}}) - T_1(L, A+1; q^{\frac{1}{2}}) \\
&\quad = q^{\frac{L-A}{2}} T_0(L, A; q^{\frac{1}{2}}) - q^{\frac{L+A+1}{2}} T_0(L, A+1; q^{\frac{1}{2}}),
\end{aligned}
\tag{4.6}
$$

$$
\begin{aligned}
&T_{-1}(L, A; q^{\frac{1}{2}}) - T_{-1}(L-1, A\pm 1; q^{\frac{1}{2}}) \\
&\qquad = q^{\frac{L\mp A}{2}} T_0(L, A; q^{\frac{1}{2}}) - q^L T_{-1}(L-1, A\pm 1; q^{\frac{1}{2}})
\end{aligned}
\tag{4.7}
$$

$$
\begin{aligned}
&q^{\frac{L\pm A}{2}} T_0(L, A; q^{\frac{1}{2}}) - T_1(L, A; q^{\frac{1}{2}}) \\
&\qquad = (q^L - 1)[T_{-1}(L-1, A; q^{\frac{1}{2}}) + T_{-1}(L-1, A\mp 1; q^{\frac{1}{2}})]
\end{aligned}
\tag{4.8}
$$

$$
\begin{aligned}
&T_0(L, A; q^{\frac{1}{2}}) - T_0(L, A+2; q^{\frac{1}{2}}) \\
&\quad = q^{\frac{L-A}{2}} T_1(L, A; q^{\frac{1}{2}}) - q^{\frac{L+2+A}{2}} T_1(L, A+2; q^{\frac{1}{2}})
\end{aligned}
\tag{4.9}
$$

The identities (4.1) with $n = 0$ and (4.3) are needed for the proof in the NS sector. Identity (4.1) is proven in the appendix and (4.3) follows by combining eqns. (2.26) and (2.29) of [45].

The identities (4.1) with $n = -1$, (4.2),(4.4) and (4.6) are needed for the R^- Ramond sector. Identity (4.2) is eqn.(2.16) of [45], identity (4.6) is eqn. (2.20) of [45], and identity (4.4) follows from combining (4.3) and (4.6).

The identities (4.1) with $n = 1$, (4.7)–(4.9) are needed for the R^+ Ramond sector. Identity (4.7) is (2.23) of [45] with $B = A + 1$, identity (4.8) is obtained by combining (2.23) and (2.24) of [45] both with $B = A + 1$ and identity (4.9) will be proven in the appendix. Finally, identity (4.5) is needed to establish an $R^+ - R^-$ connection and to derive (1.22). Identity (4.5) will also be proven in the appendix.

4.2. Proof of the generic equations for all sectors

We separate the recursion relations (2.35) into two classes: the equations for $h_0, \cdots, h_{\nu-3}$ which we call generic and the last equation of (2.35) (or equivalently (2.38)) which we call the closing equation. The proof of the generic equations is identical for the three separate cases of NS and $R^{\pm}$. In all cases the generic equation follows immediately from the identity (4.1) and the fact that the bosonic polynomials (2.26), (2.27), (2.28) are linear combinations of T_{-n} with n given by (2.36). The identity (4.1) guarantees that the generic recursion relation holds for each term separately in the sum over j. Consequently, these generic equations do not determine the factors $(-1)^j q^{\nu j^2+(s'+\frac{1-|n|}{2})j}$ which appear in (2.26)–(2.28). These factors are determined by the closing equation and for this the three cases need to be considered separately.

To keep notations manageable we will write $T_n(L,A)$ instead of $T_n(L,A;q^{\frac{1}{2}})$ throughout the rest of this paper.

4.3. Proof of the closing equation for the Neveu-Schwarz sector

To verify the closing equation (2.38) for the NS bosonic polynomials (2.26) we consider

$$I_{NS}(L) = B^{(\nu,0)}_{\nu-1,s'}(L,q) - B^{(\nu,0)}_{\nu-2,s'}(L-1,q) \tag{4.10}$$

and substitute (2.26) to find

$$\begin{aligned} I_{NS}(L) = \sum_{j=-\infty}^{\infty} (-1)^j q^{\nu j^2+(s'+\frac{1}{2})j} \Big(& T_0(L, 2\nu j + s' - \nu + 1) + T_0(L, 2\nu j + s' + \nu) \\ & - T_0(L-1, 2\nu j + s' - \nu + 2) - T_0(L-1, 2\nu j + s' + \nu - 1)\Big). \end{aligned} \tag{4.11}$$

This does not vanish term by term under the summation sign. However, if we first send $j \to -j$ in the first and third terms inside of $\big(\cdots\big)$ and use (2.24) we have

$$\begin{aligned} I_{NS}(L) = \sum_{j=-\infty}^{\infty} (-1)^j q^{\nu j^2} \Big(& q^{(s'+\frac{1}{2})j}\big[T_0(L, 2\nu j + \nu + s') - T_0(L-1, 2\nu j + \nu + s' - 1)\big] \\ & + q^{-(s'+\frac{1}{2})j}\big[T_0(L, 2\nu j + \nu - s' - 1) - T_0(L-1, 2\nu j + \nu - s' - 2)\big]\Big). \end{aligned} \tag{4.12}$$

In this sum the terms with j and $-j-1$ cancel by use of (4.3). Thus we have completed the verification that the NS bosonic polynomials (2.26) satisfy the recursion relations (2.35) with $n = 0$.

4.4. Proof of the closing equation for the R^- Ramond sector

To verify the closing equation (2.38) for the R^- polynomials (2.28) we consider

$$I_{R^-}(L) = B^{(\nu,-1)}_{\nu-1,s'}(L,q) - B^{(\nu,-1)}_{\nu-2,s'}(L-1,q) \tag{4.13}$$

and substitute (2.28) to find

$$\begin{aligned} I_{R^-}(L) = &\sum_{j=-\infty}^{\infty} (-1)^j q^{\nu j^2 + s' j} \\ &\times \Big(\sum_{i=-(\nu-1)}^{\nu-1} (-1)^{\nu-1+i} T_1(L, 2\nu j + s' + i) \\ &- \sum_{i=-(\nu-2)}^{\nu-2} (-1)^{\nu-2+i} T_1(L-1, 2\nu j + s' + i) \Big). \end{aligned} \tag{4.14}$$

We now transform the summand of (4.14) for each j by adding and subtracting $T_1(L-1, 2\nu j + s' - 1 + \nu)$ and regrouping terms to obtain

$$\begin{aligned} &\sum_{i=0}^{\nu-2} \big[T_1(L, 2\nu j + s' + 1 - \nu + 2i) - T_1(L, 2\nu j + s' + 2 - \nu + 2i) \big] \\ &+ \big[T_1(L, 2\nu + s' - 1 + \nu) - T_1(L-1, 2\nu j + s' - 1 + \nu) \big] \\ &- \sum_{i=0}^{\nu-2} \big[T_1(L-1, 2\nu j + s' + 2 - \nu + 2i) - T_1(L-1, 2\nu j + s' + 3 - \nu + 2i) \big]. \end{aligned} \tag{4.15}$$

Then we use (4.4) on the first line, (4.2) on the second line and (4.6) on the third line and note that all terms cancel in pairs except

$$q^{\frac{L-1+\nu}{2}} \Big(q^{\frac{2\nu j + s'}{2}} T_0(L-1, 2\nu j + s' + \nu) + q^{-\frac{2\nu j + s'}{2}} T_0(L-1, 2\nu j + s' - \nu) \Big). \tag{4.16}$$

Thus we have

$$\begin{aligned} I_{R^-}(L) = &q^{\frac{L-1+\nu}{2}} \sum_{j=-\infty}^{\infty} (-1)^j q^{\nu j^2 + s' j} \\ &\times \big[q^{\frac{2\nu j + s'}{2}} T_0(L-1, 2\nu j + s' + \nu) + q^{-\frac{2\nu j + s'}{2}} T_0(L-1, 2\nu j + s' - \nu) \big]. \end{aligned} \tag{4.17}$$

which is seen to vanish if we replace j by $j+1$ in the second of two terms in $\big[\cdots\big]$. Thus we have completed the verification that the R^- bosonic polynomials (2.28) satisfy the recursion relations (2.35) with $n = -1$.

4.5. Proof of the closing equation for the R^+ Ramond sector, $R^+ - R^-$ relations

To verify the closing equation (2.38) for the R^+ polynomials (2.27) we consider

$$I_{R^+}(L) = B^{(\nu,1)}_{\nu-1,s'}(L,q) - B^{(\nu,1)}_{\nu-2,s'}(L-1,q) \tag{4.18}$$

and substitute (2.27) to find

$$\begin{aligned} I_{R^+}(L) = \sum_{j=-\infty}^{\infty} (-1)^j q^{\nu j^2+s'j} \Big(& [T_{-1}(L,2\nu j+s'-\nu+1) - T_{-1}(L-1,2\nu j+s'-\nu+2)] \\ & +[T_{-1}(L,2\nu j+s'+\nu) - T_{-1}(L-1,2\nu j+s'+\nu-1)] \\ & +[T_{-1}(L,2\nu j+s'-\nu) - T_{-1}(L-1,2\nu j+s'-\nu+1)] \\ & +[T_{-1}(L,2\nu j+s'+\nu-1) - T_{-1}(L-1,2\nu j+s'+\nu-2)]\Big). \end{aligned} \tag{4.19}$$

We now use (4.7) on each of the four terms inside of $\big(\cdots\big)$ to obtain after regrouping

$$\begin{aligned} I_{R^+}(L) = & \sum_{j=-\infty}^{\infty} (-1)^j q^{\nu j^2+s'j} \times \\ & \Big([q^{\frac{L-(s'-\nu+1+2\nu j)}{2}} T_0(L,2\nu j+s'-\nu+1) + q^{\frac{L+(s'+\nu-1+2\nu j)}{2}} T_0(L,2\nu j+s'+\nu-1)] \\ & - q^L[T_{-1}(L-1,2\nu j+s'-\nu+2) + T_{-1}(L-1,2\nu j+s'-\nu+1)] \\ & - q^L[T_{-1}(L-1,2\nu j+s'+\nu-1) + T_{-1}(L-1,2\nu j+s'+\nu-2)] \\ & + [q^{\frac{L+s'+\nu+2\nu j}{2}} T_0(L,2\nu j+s'+\nu) + q^{\frac{L-(s'-\nu+2\nu j)}{2}} T_0(L,2\nu j+s'-\nu)]\Big). \end{aligned} \tag{4.20}$$

The expression in the last set of the square brackets is seen to vanish if we take $j \to j+1$ in the second of the two terms in $[\cdots]$. Then, if we multiply both sides of (4.20) by (q^L-1) and use (4.8) on the contents of the second and third sets of square brackets, we obtain

$$\begin{aligned} (q^L-1)I_{R^+}(L) = & \sum_{j=-\infty}^{\infty} q^{\nu j^2+s'j} \times \\ & \Big(-q^{\frac{L+s'+\nu-1+2\nu j}{2}} T_0(L,2\nu j+s'+\nu-1) - q^{\frac{L-(s'-\nu+1+2\nu j)}{2}} T_0(L,2\nu j+s'-\nu+1) \\ & + q^L T_1(L,2\nu j+s'+\nu-1) + q^L T_1(L,2\nu j+s'-\nu+1)\Big). \end{aligned} \tag{4.21}$$

We now let $j \to j-1$ in the first and third terms in $\left(\cdots\right)$ to obtain the expression

$$
\begin{aligned}
(q^L - 1) I_{R^+}(L) = & \sum_{j-\infty}^{\infty} (-1)^j q^{\nu j(j-1)+s'j+\frac{L+\nu-s'-1}{2}} \\
& \times \Bigg(T_0(L, 2\nu j + s' - \nu - 1) - T_0(L, s' - \nu + 1) \\
& - q^{\frac{L-(s'-\nu-1+2\nu j)}{2}} T_1(L, 2\nu j + s' - \nu - 1) \\
& + q^{\frac{L+(s'-\nu+1+2\nu j)}{2}} T_1(L, 2\nu j + s' - \nu + 1) \Bigg)
\end{aligned}
\tag{4.22}
$$

which vanishes term by term under the summation sign due to (4.9). Thus we have completed the verification that the R^+ bosonic polynomials (2.27) satisfy the recursion relations (2.35) with $n = 1$.

We conclude this section by noting the following intriguing identities

$$
\begin{aligned}
& F^{(\nu,-1)}_{r',s'}(L+1, q) + F^{(\nu,-1)}_{r',s'}(L, q) = 2\big[B^{(\nu,1)}_{r',s'}(L, q) + B^{(\nu,1)}_{r',s'+1}(L, q)\big]; \quad s' \neq \nu - 1 \\
& F^{(\nu,-1)}_{r',\nu-1}(L+1, q) + F^{(\nu,-1)}_{r',\nu-1}(L, q) = 2B^{(\nu,1)}_{r',\nu-1}(L, q),
\end{aligned}
\tag{4.23}
$$

which can be easily proven with the help of (2.34) and (4.5).
These identities reveal the intimate connection between R^+ and R^- representations of the Ramond sector characters.

5. The Indices

In this section we turn to the objects $\tilde{F}^{(\nu,n)}_{s'}(q)$ and $\tilde{B}^{(\nu,n)}_{s'}(q)$ and prove the properties discussed in the introduction. To this end we introduce the polynomials $\tilde{F}^{(\nu,n)}_{r',s'}(L, q)$ as

$$
\tilde{F}^{(\nu,n)}_{r',s'}(L, q) = \sum_{\mathcal{D}_{r',s'}} (-1)^{m_1} q^{Qf + Lf_{n,s'}} \prod_{j=1}^{\nu} \begin{bmatrix} n_j + m_j \\ n_j \end{bmatrix}_q; \qquad n = 0, \pm 1, \quad r' = 0, 1, \cdots, \nu - 2
\tag{5.1}
$$

where Qf, $Lf_{n,s'}$, and $\mathcal{D}_{r',s'}$ are defined in (1.8), (1.9) and (2.17). One can easily establish

$$
\lim_{L\to\infty} \tilde{F}^{(\nu,n)}_{r',s'}(L, q) = \tilde{F}^{(\nu,n)}_{s'}(q)
\tag{5.2}
$$

and

$$
\tilde{F}^{(\nu,0)}_{r',s'}(L, q) = F^{(\nu,0)}_{r',s'}(L, qe^{2\pi i})
\tag{5.3}
$$

which hold for all r'.

It is straightforward to repeat the analysis carried out in secs. 3 and 4 to prove recursion relations for $\tilde{F}^{(\nu,n)}_{r',s'}(L,q)$

$$\tilde{F}^{(\nu,n)}_{0,s'}(L,q)=\tilde{F}^{(\nu,n)}_{1,s'}(L-1,q)+(1-q^{L-\frac{1-n}{2}})\tilde{F}^{(\nu,n)}_{0,s'}(L-1,q)+(q^{L-1}-1)\tilde{F}^{(\nu,n)}_{0,s'}(L-2,q), \tag{5.4}$$

$$\begin{aligned}\tilde{F}^{(\nu,n)}_{r',s'}(L,q)=&\tilde{F}^{(\nu,n)}_{r'-1,s'}(L-1,q)+\tilde{F}^{(\nu,n)}_{r'+1,s'}(L-1,q)\\&-q^{L-\frac{1-n}{2}}\tilde{F}^{(\nu,n)}_{r',s'}(L-1,q)+(q^{L-1}-1)\tilde{F}^{(\nu,n)}_{r',s'}(L-2,q)\text{ for }1\le r'\le\nu-3,\end{aligned} \tag{5.5}$$

$$\tilde{F}^{(\nu,n)}_{\nu-2,s'}(L,q)=\tilde{F}^{(\nu,n)}_{\nu-3,s'}(L-1,q)-q^{L-\frac{1-n}{2}}\tilde{F}^{(\nu,n)}_{\nu-2,s'}(L-1,q)+q^{L-1}\tilde{F}^{(\nu,n)}_{\nu-2,s'}(L-2,q); \tag{5.6}$$

with $\nu\ge 3$,

$$\tilde{F}^{(2,n)}_{0,s'}(L,q)=(1-q^{L-\frac{1-n}{2}})\tilde{F}^{(2,n)}_{0,s'}(L-1,q)+q^{L-1}\tilde{F}^{(2,n)}_{0,s'}(L-2,q) \tag{5.7}$$

and identities

$$\tilde{F}^{(\nu,0)}_{r',s'}(L,q)=\tilde{B}^{(\nu,0)}_{r',s'}(L,q) \tag{5.8}$$

$$\tilde{B}^{(\nu,1)}_{r',s'}(L,q)=\tilde{F}^{(\nu,1)}_{r',s'}(L,q)-\tilde{F}^{(\nu,1)}_{r',s'-1}(L,q),\qquad s'\neq 0 \tag{5.9}$$

where

$$\begin{aligned}\tilde{B}^{(\nu,0)}_{r',s'}(L,q)=(-1)^{L+s'+r'}\sum_{j=-\infty}^{\infty}q^{\nu j^2+(s'+\frac{1}{2})j}\Big(&T_0(L,2\nu j+s'-r')\\&-T_0(L,2\nu j+s'+1+r')\Big)\end{aligned} \tag{5.10}$$

and

$$\begin{aligned}\tilde{B}^{(\nu,1)}_{r',s'}(L,q)=(-1)^{L+r'+s'+1}&\sum_{j=-\infty}^{\infty}q^{\nu j^2+s'j}\\&\times\Big(T_{-1}(L,2\nu j+s'+r'+1)-T_{-1}(L,2\nu j+s'-r')\\&+T_{-1}(L,2\nu j+s'+r')-T_{-1}(L,2\nu j+s'-r'-1)\Big)\\&.\end{aligned} \tag{5.11}$$

Identity (5.10) could have been proven directly by simply replacing $q^{\frac{1}{2}}$ with $-q^{\frac{1}{2}}$ in (2.32) and then using (2.25) and (5.3). To avoid confusion we want to stress that $\tilde{B}^{(\nu,1)}_0(q)$ defined by (1.20) is not $L\to\infty$ limit of $\tilde{B}^{(\nu,1)}_{r',0}(L,q)$.

If we let $L \to \infty$ in (5.9)and apply the limiting formulas (2.29) with $n = 1$ and (5.2) we obtain

$$\tilde{F}_{s'}^{(\nu,1)}(q) = \tilde{F}_{s'-1}^{(\nu,1)}(q), \qquad s' \neq 0, \tag{5.12}$$

i.e. $\tilde{F}_{s'}^{(\nu,1)}(q)$ does not depend on s'. In fact,

$$\tilde{F}_{s'}^{(\nu,1)}(q) = 1 \tag{5.13}$$

as stated in the introduction (1.19). To see this, we rearrange (5.4)- (5.7) in the following fashion (suppressing the argument q for compactness):

1) For $\nu = 2$

$$\tilde{F}_{0,s'}^{(2,1)}(L) + q^L \tilde{F}_{0,s'}^{(2,1)}(L-1) = \tilde{F}_{0,s'}^{(2,1)}(L-1) + q^{L-1} \tilde{F}_{0,s'}^{(2,1)}(L-2) \tag{5.14}$$

2) For $\nu \geq 3$

$$\begin{gathered}
\tilde{F}_{0,s'}^{(\nu,1)}(L) + q^L \tilde{F}_{0,s'}^{(\nu,1)}(L-1) = \tilde{F}_{0,s'}^{(\nu,1)}(L-1) + \tilde{F}_{1,s'}^{(\nu,1)}(L-1) + (q^{L-1}-1)\tilde{F}_{0,s'}^{(\nu,1)}(L-2), \\
\tilde{F}_{r',s'}^{(\nu,1)}(L) + q^L \tilde{F}_{r',s'}^{(\nu,1)}(L-1) - \tilde{F}_{r'-1,s'}^{(\nu,1)}(L-1) = \tilde{F}_{r'+1,s'}^{(\nu,1)}(L-1) + (q^{L-1}-1)\tilde{F}_{r',s'}^{(\nu,1)}(L-2) \\
\text{for } 1 \leq r' \leq \nu-3, \\
\tilde{F}_{\nu-2,s'}^{(\nu,1)}(L) + q^L \tilde{F}_{\nu-2,s'}^{(\nu,1)}(L-1) - \tilde{F}_{\nu-3,s'}^{(\nu,1)}(L-1) = q^{L-1} \tilde{F}_{\nu-2,s'}^{(\nu,1)}(L-2).
\end{gathered} \tag{5.15}$$

We add together the $\nu - 1$ equations to find

$$\begin{aligned}
&\sum_{r'=0}^{\nu-2} \big[\tilde{F}_{r',s'}^{(\nu,1)}(L,q) + q^L \tilde{F}_{r',s'}^{(\nu,1)}(L-1,q)\big] - \sum_{r'=0}^{\nu-3} \tilde{F}_{r',s'}^{(\nu,1)}(L-1,q) \\
&\quad = \sum_{r'=0}^{\nu-2} \big[\tilde{F}_{r',s'}^{(\nu,1)}(L-1,q) + q^{L-1} \tilde{F}_{r',s'}^{(\nu,1)}(L-2,q)\big] - \sum_{r'=0}^{\nu-3} \tilde{F}_{r',s'}^{(\nu,1)}(L-2,q).
\end{aligned} \tag{5.16}$$

The above is of the form $I(L) = I(L-1)$. Thus both sides are separately equal to a constant independent of L which by evaluation for small L is found to be 1 and hence

$$\sum_{r'=0}^{\nu-2} \big[\tilde{F}_{r',s'}^{(\nu,1)}(L,q) + q^L \tilde{F}_{r',s'}^{(\nu,1)}(L-1,q)\big] - \sum_{r'=0}^{\nu-3} \tilde{F}_{r',s'}^{(\nu,1)}(L-1,q) = 1. \tag{5.17}$$

Taking (5.2) into account we may send $L \to \infty$ in (5.17) to derive

$$\tilde{F}_{s'}^{(\nu,1)}(q) = \lim_{L\to\infty} \tilde{F}_{r',s'}^{(\nu,1)}(L,q) = 1 \tag{5.18}$$

which proves (1.19) of the introduction.

When $\nu = 2$ there is yet another bosonic companion of $\tilde{F}^{(2,1)}_{0,s'}(L,q), \;\; s' = 0,1$. Indeed, in this case (5.17) becomes a simple first order difference equation

$$\tilde{F}^{(2,1)}_{0,s'}(L,q) + q^L \tilde{F}^{(2,1)}_{0,s'}(L-1,q) = 1. \tag{5.19}$$

By direct evaluation, one finds boundary conditions for (5.19)

$$\tilde{F}^{(2,1)}_{0,s'}(s',q) = 1, \qquad s' = 0,1. \tag{5.20}$$

It is now trivial to solve (5.19) and (5.20) to obtain

$$\tilde{F}^{(2,1)}_{0,s'}(L,q) = \sum_{j=0}^{L-s'} (-1)^j q^{Lj - \frac{j(j-1)}{2}}. \tag{5.21}$$

From this the limit (5.18) is immediate.
Next let us consider $\tilde{F}^{(2,-1)}_{0,s'}(L,q), \; s' = 0,1$. If we define

$$X_{s'}(L,q) = \tilde{F}^{(2,-1)}_{0,s'}(L,q) - \tilde{F}^{(2,-1)}_{0,s'}(L-1,q), \tag{5.22}$$

then the second order difference equation (5.7) can be rewritten in the first order form

$$X_{s'}(L,q) = -q^{L-1} X_{s'}(L-1,q). \tag{5.23}$$

This is easily solved to get

$$X_{s'}(L,q) = (-1)^{L-1} q^{\frac{L(L-1)}{2}} X_{s'}(1,q) \tag{5.24}$$

where $X_0(1,q) = -X_1(1,q) = -1$. Then, since

$$\tilde{F}^{(2,-1)}_{0,0}(0,q) = \tilde{F}^{(2,-1)}_{0,1}(1,q) = 1 \tag{5.25}$$

we obtain from (5.22) and (5.24)

$$\begin{aligned} \tilde{F}^{(2,-1)}_{0,0}(L,q) &= 1 + \sum_{j=1}^{L} (-1)^j q^{\frac{j(j-1)}{2}} \\ \tilde{F}^{(2,-1)}_{0,1}(L,q) &= 1 - \sum_{j=2}^{L} (-1)^j q^{\frac{j(j-1)}{2}}. \end{aligned} \tag{5.26}$$

From this we note that

$$\tilde{F}^{(2,-1)}_{0,1}(L,q)+\tilde{F}^{(2,-1)}_{0,0}(L,q)=1. \tag{5.27}$$

The equality with the false theta functions (1.22) is easily established by letting $L\to\infty$ in (5.26) and (5.27).

In fact, equation (5.27) can be generalized to

$$1=\sum_{s'=0}^{\nu-1}\tilde{F}^{(\nu,-1)}_{r',s'}(L,q). \tag{5.28}$$

To prove (5.28) it is sufficient to notice that a constant is always a solution to (5.4)–(5.6) with $n=-1$ and that (5.28) holds true for $L=0,1$. Letting $L\to\infty$ in (5.28) one recovers (1.21), (1.24) for $s'=0$.

To verify (1.21)–(1.24) in general, we will need the following analogue of (4.23)

$$\tilde{F}^{(\nu,-1)}_{r',s'}(L+1,q)-\tilde{F}^{(\nu,-1)}_{r',s'}(L,q)=\tilde{B}^{(\nu,1)}_{r',s'+1}(L,q)-\tilde{B}^{(\nu,1)}_{r',s'}(L,q). \tag{5.29}$$

which is proven by observing that lhs and rhs satisfy the same equations (5.4), (5.5) with $n=1$ and that (5.29) holds true for $L=0,1$. Then we use (5.29) along with

$$\tilde{F}^{(\nu,-1)}_{r',s'}(L=0,q)=\delta_{r',s'} \tag{5.30}$$

to find the bosonic companion of $\tilde{F}^{(\nu,-1)}_{r',s'}(L,q)$

$$\tilde{F}^{(\nu,-1)}_{r',s'}(L+1,q)=\sum_{l=0}^{L}\big\{\tilde{B}^{(\nu,1)}_{r',s'+1}(l,q)-\tilde{B}^{(\nu,1)}_{r',s'}(l,q)\big\}+\delta_{r',s'}. \tag{5.31}$$

To proceed further, we set $r'=0$ and send L to infinity in (5.31) and use (5.11) to find

$$\begin{aligned}\tilde{F}^{(\nu,-1)}_{s'}(q)=\lim_{L\to\infty}\tilde{F}^{(\nu,-1)}_{0,s'}(L,q)=(-1)^{s'}\Bigg\{&\sum_{j=-\infty}^{\infty}q^{\nu j^2+s'j}g(2\nu j+s',q)\\&+\sum_{j=-\infty}^{\infty}q^{\nu j^2+(s'+1)j}g(2\nu j+s'+1,q)\Bigg\}+\delta_{0,s'}\end{aligned} \tag{5.32}$$

where

$$g(j,q)=\sum_{l=0}^{\infty}(-1)^l\big[T_{-1}(l,j+1)-T_{-1}(l,j-1)\big]. \tag{5.33}$$

The function $g(j,q)$ has the two important properties:

$$g(-j,q)=-g(j,q); \qquad j\in Z \tag{5.34}$$

and

$$g(j,q)+g(j+1,q)=-\delta_{j,0}; \qquad j\in Z,\ j\geq 0. \tag{5.35}$$

Formula (5.34) is a simple consequence of (2.24) and formula (5.35) is proven in the appendix. Clearly, equations (5.34) and (5.35) specify $g(j)$ uniquely as

$$g(j)=\begin{cases}(-1)^j\mathrm{sign}(j); & j\neq 0\\ 0; & j=0.\end{cases} \tag{5.36}$$

Combining (5.36) and (5.32) we obtain

$$\tilde{F}^{(\nu,-1)}_{s'}(q)=I^{(\nu)}_{s'}(q)-I^{(\nu)}_{s'+1}(q) \tag{5.37}$$

with $I^{(\nu)}_{s'}(q)$ defined by (1.23). Thus, we completed the proof of (1.22).

To the best of our knowledge, q-trinomial representation (5.31) of the "truncated" false theta function has never appeared in the literature before.

We conclude this section with a derivation of the $q\to 1^-$ limit (1.25) of the false theta function (1.23) $I^{(\nu)}_{s'}(q)$ given in the introduction. To this end we rewrite the sum in (1.23) as

$$I^{(\nu)}_{s'}(q)=1+\sum_{j=1}^{\infty}e^{-\nu x^2(j)}\left(e^{-s'x(j)(|\ln q|)^{\frac{1}{2}}}-e^{s'x(j)(|\ln q|)^{\frac{1}{2}}}\right) \tag{5.38}$$

where

$$x(j)=j(|\ln q|)^{\frac{1}{2}}. \tag{5.39}$$

As $q\to 1^-$ the rhs of (5.38) is dominated by large j terms and, as a result, can be approximated by an integral

$$I^{(\nu)}_{s'}(q)\sim 1+\int_0^{\infty}\frac{dx}{(|\ln q|)^{\frac{1}{2}}}e^{-\nu x^2}(e^{-s'x(|\ln q|)^{\frac{1}{2}}}-e^{s'x(|\ln q|)^{\frac{1}{2}}}). \tag{5.40}$$

Expanding

$$e^{-s'x(|\ln q|)^{\frac{1}{2}}}-e^{s'x(|\ln q|)^{\frac{1}{2}}}=-2s'x(\ln q|)^{\frac{1}{2}}+O(\ln q) \tag{5.41}$$

we find the limit

$$\begin{aligned}\lim_{q\to 1}I^{(\nu)}_{s'}(q)&=1-2s'\int_0^{\infty}xe^{-\nu x^2}dx\\ &=1-\frac{s'}{\nu}\end{aligned}. \tag{5.42}$$

The formula above is the result (1.25) we set out to obtain.

6. Duality $q \to q^{-1}$

The bosonic and fermionic polynomials given in sec. 2 reduce to the characters of the $SM(2,4\nu)$ superconformal model as $L \to \infty$ when $q<1$. However, when $q>1$ it is also possible to take the $L \to \infty$ limit after removing a suitable power q. We show here that this leads to the linear combinations of the characters of the minimal model $M(2\nu-1,4\nu)$ where we recall that for all models $M(p,p')$ the bosonic form of the characters (normalized to one at $q=0$) is [55]

$$\chi^{(p,p')}_{r,s}(q) = \chi^{(p,p')}_{p-r,p'-s}(q) = \frac{1}{(q)_\infty} \sum_{j=-\infty}^{\infty} (q^{j(jpp'+rp'-sp)} - q^{(jp+r)(jp'+s)}). \tag{6.1}$$

We study the region $q>1$ by making the dual transformation $q \to q^{-1}$ in the bosonic/fermionic polynomials. It is worth mentioning that this operation has a direct physical meaning: it transforms particles into holes and vice-versa.

We use the definition (2.22) to express the dual polynomials in terms of $\binom{L,A-n;q}{A}_2$:

1) in NS as

$$q^{\frac{L^2}{2}} B^{(\nu,0)}_{r',s'}(L,q^{-1}) = \sum_{j=-\infty}^{\infty} (-1)^j q^{-\nu j^2-(s'+\frac{1}{2})j} \left(q^{\frac{(2\nu j+s'-r')^2}{2}} \binom{L, 2\nu j+s'-r'; q}{2\nu j+s'-r'}_2 + q^{\frac{(2\nu j+s'+1+r')^2}{2}} \binom{L, 2\nu j+s'+r'+1; q}{2\nu j+r'+1}_2 \right) \tag{6.2}$$

2) in R^- as

$$q^{\frac{L(L-1)}{2}} B^{(\nu,-1)}_{r',s'}(L,q^{-1}) = \sum_{j=-\infty}^{\infty} (-1)^j q^{-\nu j^2-s'j} \sum_{i=-r'}^{r'} (-1)^{r'+i} q^{\frac{(2\nu j+s'+i)(2\nu j+s'+i-1)}{2}} \binom{L, 2\nu j+s'+i-1; q}{2\nu j+s'+i}_2 . \tag{6.3}$$

3) in R^+ as

$$q^{\frac{L(L+1)}{2}} B^{(\nu,1)}_{r',s'}(L,q^{-1}) = \frac{1}{2} \sum_{j=-\infty}^{\infty} (-1)^j q^{-\nu j^2-s'j} \Bigg(q^{\frac{(2\nu j+s'-r'-1)(2\nu j+s'-r')}{2}} \Big[\binom{L, 2\nu j+s'-r'; q}{2\nu j+s'-r'-1}_2 + q^{2\nu j+s'-r'} \binom{L, 2\nu j+s'-r'+1; q}{2\nu j+s'-r'}_2 \Big] + q^{\frac{(2\nu j+s'+r')(2\nu j+s'+r'+1)}{2}} \Big[\binom{L, 2\nu j+s'+r'+1; q}{2\nu j+s'+r'}_2 + q^{2\nu j+s'+r'+1} \binom{L, 2\nu j+s'+r'+2; q}{2\nu j+s'+r'+1}_2 \Big] \Bigg) \tag{6.4}$$

In this form we may now let $L\to\infty$ by using two limiting results of [45]

$$\lim_{L\to\infty}\binom{L,A;q}{A}_2=\frac{1}{(q)_\infty} \tag{6.5}$$

$$\lim_{L\to\infty}\binom{L,A-1;q}{A}_2=\frac{1+q^A}{(q)_\infty} \tag{6.6}$$

and the asymptotic formula which may be derived from (6.5) and (2.23) of [45]

$$\lim_{L\to\infty}\left[\binom{L,A+1;q}{A}_2+q^{A+1}\binom{L,A+2;q}{A+1}_2\right]=\frac{1}{(q)_\infty} \tag{6.7}$$

to obtain for $n=0,\pm 1$

$$\begin{aligned}(1+\theta(n>0))\lim_{L\to\infty}&q^{\frac{L(L+n)}{2}}B^{(\nu,n)}_{r',s'}(L,q^{-1})\\ &=q^{\frac{(s'-r')(s'-r'-|n|)}{2}}\chi^{(2\nu-1,4\nu)}_{\nu-r'-1,2\nu-2s'-1+|n|}(q)\\ &+q^{\frac{(s'+r'+1)(s'+r'+1-|n|)}{2}}\chi^{(2\nu-1,4\nu)}_{\nu+r',2\nu-2s'-1+|n|}(q).\end{aligned} \tag{6.8}$$

The equation (6.8) demonstrates that in the limit $L\to\infty$ the model $SM(2,4\nu)$ is related to the model $M(2\nu-1,4\nu)$ by the dual transformation $q\to\frac{1}{q}$. This latter nonunitary minimal model is a special case of the models $M(p,p')$ studied in [36]. It is of interest to note that while dual polynomials (6.2)–(6.4) yield $M(2\nu-1,4\nu)$ characters in the limit $L\to\infty$, the dual polynomials themselves are not the same as those of [56] and [36]. This emphasizes the fact that there are many different polynomial expressions which yield the same character in the $L\to\infty$ limit. Indeed the polynomials of this paper and those of [36] must be different because in [36] the $M(2\nu-1,4\nu)$ polynomials transform into the $M(2\nu+1,4\nu)$ polynomials while the polynomials (6.2)–(6.4) transform into $SM(2,4\nu)$ ones.

Curiously enough, the $SM(2,8)$ model is, in fact, self-dual

$$\lim_{L\to\infty}q^{\frac{L^2}{2}}B^{(2,0)}_{0,s'}(L,q^{-1})=q^{\frac{s'}{2}}\left(\chi^{(3,8)}_{1,3-2s'}(q)+q^{\frac{1}{2}+s'}\chi^{(3,8)}_{2,3-2s'}(q)\right)=q^{\frac{s'}{2}}\hat{\chi}^{(2,8)}_{1,3-2s'}(q) \tag{6.9}$$

$$\begin{aligned}\lim_{L\to\infty}q^{\frac{L(L\pm1)}{2}}B^{(2,\pm1)}_{0,s'}(L,q^{-1})&=\frac{2}{3\pm1}q^{\frac{s'-1}{2}}\left(\chi^{(3,8)}_{1,4-2s'}(q)+q\chi^{(3,8)}_{2,4-2s'}(q)\right)\\ &=\frac{2}{3\pm1}q^{\frac{s'-1}{2}}(1+q\delta_{s',0})\hat{\chi}^{(2,8)}_{1,2+2s'}(q)\end{aligned} \tag{6.10}$$

where $s' = 0, 1$.

To complete the study of q-duality we transform the fermionic sums using the relation

$$\begin{bmatrix} n+m \\ m \end{bmatrix}_{q^{-1}} = q^{-mn} \begin{bmatrix} n+m \\ m \end{bmatrix}_q . \tag{6.11}$$

We then obtain fermionic sums with a quadratic form matrix of the type discussed in [36]. In particular, we consider $\nu \times \nu$ matrix $\mathbf{B}$ defined by its matrix elements

$$(\mathbf{B})_{j,k} = \begin{cases} 2 & \text{for } j=k=1 \\ \delta_{k,2} & \text{for } j=1,\ 2 \le k \le \nu \\ \delta_{j,2} & \text{for } k=1,\ 2 \le j \le \nu \\ \frac{1}{2}\delta_{j,2}\delta_{k,2} + \delta_{j,k} - \frac{1}{2}\delta_{j,k+1} - \frac{1}{2}\delta_{j,k-1} - \frac{1}{2}\delta_{j,\nu}\delta_{k,\nu} & \text{otherwise.} \end{cases} \tag{6.12}$$

We also define

$$\tilde{\mathbf{m}}^t = (n_1, m_2, m_3, \cdots, m_\nu) \tag{6.13}$$

and the $\nu - 1$-dimensional vector $\vec{v}^{(k)}$

$$(\vec{v}^{(k)})_i = k\theta(1 \le i \le \nu - k - 1) + (\nu - 1 - i)\theta(k > 0)\theta(\nu - k - 1 < i \le \nu - 1) \tag{6.14}$$

where $k = 0, 1, \cdots, \nu - 1$. Then, the q-duality transform of the fermionic polynomials (2.10) can be expressed as

$$q^{\frac{L(L+n)}{2}} F^{(\nu,n)}_{r',s'}(L, q^{-1}) = \sum_{\mathcal{D}_{r',s'}} q^{\Phi_n(\tilde{\mathbf{m}}, r', s')} \prod_{i=1}^{\nu} \begin{bmatrix} n_i + m_i \\ n_i \end{bmatrix}_q \tag{6.15}$$

where n, $\mathcal{D}_{r',s'}$ are given by (2.36), (2.17) and $\Phi_n(\tilde{\mathbf{m}}, r', s')$ is defined as

$$\Phi_n(\tilde{\mathbf{m}}, r', s') = \frac{1}{2}\tilde{\mathbf{m}}\mathbf{B}\tilde{\mathbf{m}} + L_n(\tilde{\mathbf{m}}, s') + C_n(r', s') \tag{6.16}$$

$$2L_n(\tilde{\mathbf{m}}, s') = \tilde{m}_\nu - \tilde{m}_{\nu - s'} + \tilde{m}_1\delta_{s',\nu-1} + (2\tilde{m}_1 + \tilde{m}_2)(n + \delta_{s',\nu-1}) \tag{6.17}$$

$$4C_n(r', s') = s' - r' + (1 + 2n)\delta_{s',\nu-1}. \tag{6.18}$$

We now let $L \to \infty$ to obtain the following

$$\lim_{L\to\infty} q^{\frac{L(L+n)}{2}} F^{(\nu,n)}_{r',s'}(L, q^{-1}) = \sum_{k=0}^{\min[r',s']} \sum_{\tilde{\mathbf{m}}-\text{restrictions}[k]} \frac{q^{\Phi_n(\tilde{\mathbf{m}}, r', s')}}{(q)_{\tilde{m}_1}(q)_{\tilde{m}_2}}$$
$$\times \left\{ \delta_{k,\nu-2} + \theta(\nu - 3 \ge k) \prod_{i=3}^{\nu} \begin{bmatrix} ((1-\mathbf{B})\tilde{\mathbf{m}})_i - a_i^{(r')} - a_i^{(s')} \\ \tilde{m}_i \end{bmatrix}_{c,q} \right\} \tag{6.19}$$

where the inhomogeneous vectors $a_i^{(s')}$ and $a_i^{(r')}$ are given by (2.12) the restrictions[k] on the summation variables $\tilde{\mathbf{m}}$ are

$$\tilde{m}_i - \tilde{m}_\nu = \left(\vec{v}^{(s')} + \vec{v}^{(r')}\right)_{i-1} (\mathrm{mod}\ 2); \qquad i = 2, 3, \cdots, \nu - k - 1; \quad k \neq \nu - 2 \tag{6.20}$$

$$\tilde{m}_{\nu-j} = -2(k-j); \qquad j = 0, 1, 2, \cdots, k \neq 0. \tag{6.21}$$

and the symbol $\begin{bmatrix} A \\ B \end{bmatrix}_{c,q}$ in (6.19) stands for the conventional q-binomial coefficient (i.e. it vanishes if either A or B takes on negative values). Remarkably, it turns out that the formula (6.19) can be simplified as

$$\begin{aligned} \lim_{L\to\infty} q^{\frac{L(L+n)}{2}} F_{r',s'}^{(\nu,n)}(L, q^{-1}) = & \sum_{\tilde{\mathbf{m}}-\mathrm{restrictions}[0]} \frac{q^{\Phi_n(\tilde{\mathbf{m}}, r', s')}}{(q)_{\tilde{m}_1} (q)_{\tilde{m}_2}} \\ & \times \prod_{i=3}^{\nu} \begin{bmatrix} ((1-\mathbf{B})\tilde{\mathbf{m}})_i - a_i^{(r')} - a_i^{(s')} \\ \tilde{m}_i \end{bmatrix}_q . \end{aligned} \tag{6.22}$$

Combining results (6.15), (6.22) and (2.32)–(2.34) one derives the following Fermi/Bose $M(2\nu-1, 4\nu)$ character identities

$$\begin{aligned} \lim_{L\to\infty} q^{\frac{L^2}{2}} F_{r',s'}^{(\nu,0)}(L, q^{-1}) = & q^{\frac{(s'-r')^2}{2}} \chi_{\nu-r'-1, 2\nu-2s'-1}^{(2\nu-1,4\nu)}(q) \\ & + q^{\frac{(s'+r'+1)^2}{2}} \chi_{\nu+r', 2\nu-2s'-1}^{(2\nu-1,4\nu)}(q) \end{aligned} \tag{6.23}$$

$$\begin{aligned} \lim_{L\to\infty} q^{\frac{L(L-1)}{2}} F_{r',\nu-1}^{(\nu,-1)}(L, q^{-1}) = & q^{\frac{(\nu-r'-1)(\nu-r'-2)}{2}} \chi_{\nu-r'-1, 2}^{(2\nu-1,4\nu)}(q) \\ & + q^{\frac{(\nu+r')(\nu+r'-1)}{2}} \chi_{\nu+r', 2}^{(2\nu-1,4\nu)}(q) \end{aligned} \tag{6.24}$$

$$\begin{aligned} \lim_{L\to\infty} q^{\frac{L(L-1)}{2}} F_{r',s'}^{(\nu,-1)}(L, q^{-1}) = & q^{\frac{(s'-r')(s'-r'-1)}{2}} \chi_{\nu-r'-1, 2\nu-2s'}^{(2\nu-1,4\nu)}(q) \\ & + q^{\frac{(s'+r')(s'+r'+1)}{2}} \chi_{\nu+r', 2\nu-2s'}^{(2\nu-1,4\nu)}(q) \\ & + q^{\frac{(s'-r'+1)(s'-r')}{2}} \chi_{\nu-r'-1, 2\nu-2s'-2}^{(2\nu-1,4\nu)}(q) \\ & + q^{\frac{(s'+r'+1)(s'+r'+2)}{2}} \chi_{\nu+r', 2\nu-2s'-2}^{(2\nu-1,4\nu)}(q), \qquad s' \neq \nu - 1. \end{aligned} \tag{6.25}$$

$$\begin{aligned} & \lim_{L\to\infty} q^{\frac{L(L+1)}{2}} \left(F_{r',s'}^{(\nu,1)}(L, q^{-1}) + F_{r',s'-1+\delta_{s',0}}^{(\nu,1)}(L, q^{-1}) \right) \\ & \quad = q^{\frac{(s'+r')(s'+r'+1)}{2}} \chi_{\nu+r', 2\nu-2s'}^{(2\nu-1,4\nu)}(q) + q^{\frac{(s'-r')(s'-r'-1)}{2}} \chi_{\nu-r'-1, 2\nu-2s'}^{(2\nu-1,4\nu)}(q). \end{aligned} \tag{6.26}$$

We would like to point out that the eqns. (6.23) and (6.24) are consistent with the results obtained in [36] whenever r' or s' is equal to 0. In the general case identities (6.23) and (6.24) are new. They demonstrate how two quantum groups describing braiding properties of the conformal blocks "interact" on the character level. Identities (6.25) and (6.26) are also new. It is of interest to ascertain whether or not these new identities can be obtained by means of Bailey Lattice technique [57], [24].

We conclude this section with the following observation. It appears that there exist RG flows connecting dual regimes of the same model. In particular, it was proposed in [58] that dual regimes $Z_{\nu-1}$ and $M(\nu,\nu+1)$ of the ABF model [59] are RG connected as

$$Z_{\nu-1}+\psi_1\bar{\psi}_1+\psi_1^{\dagger}\bar{\psi}_1^{\dagger}\longrightarrow M(\nu,\nu+1). \tag{6.27}$$

Recently, the duality $M(p,p')\iff M(p'-p,p')$ established in [56], [36] was given the following RG interpretation in [60](see also [61])

$$M(p,p')+\phi_{2,1}\rightarrow M(p'-p,p') \tag{6.28}$$

It is thus plausible that one can find an appropriate operator which would generate a RG flow connecting $SM(2,4\nu)$ and $M(2\nu-1,4\nu)$.

7. On the Combinatorial bases

It is well known that each side of a Rogers-Ramanujan type identity can be interpreted as a generating function for a certain set of restricted partitions [54]. In particular, let $B_{\nu,s'}(N)$ denote the number of partitions of N into parts $\neq 2 (\mathrm{mod}\ 4)$ and $\neq 0,\pm(2\nu-1-2s') (\mathrm{mod}\ 4\nu)$, and $F_{\nu,s'}(N)$ denote the number of partitions of N of the form

$$N=\sum_{i=1}^{\infty} i f_i \tag{7.1}$$

where

$$f_1+f_2\leq \nu-s'-1, \qquad f_{2i-1}\leq 1, \qquad f_{2i}+f_{2i+1}+f_{2i+2}\leq \nu-1. \tag{7.2}$$

Then $F_{s'}^{(\nu,0)}(q^2)\big(B_{s'}^{(\nu,0)}(q^2)\big)$ is a generating function for $F_{\nu,s'}(N)\big(B_{\nu,s'}(N)\big)$. Moreover, according to [39], [40], eqn. (1.11) implies

$$F_{\nu,s'}(N)=B_{\nu,s'}(N). \tag{7.3}$$

By analogy with the analysis given in [62], Melzer [37] proposed a representation theoretical interpretation of (7.3) which we rephrase as follows.
Let $\mid \hat{\Delta}^{(2,4\nu)}_{1,2\nu-2s'-1} >$ be the highest-weight state of conformal dimension $\hat{\Delta}^{(2,4\nu)}_{1,2\nu-2s'-1}$ in the Verma module of NS sector of $SM(2,4\nu)$. Then the set of states

$$W_m^{f_m}\cdots W_2^{f_2}W_1^{f_1} \mid \hat{\Delta}^{(2,4\nu)}_{1,2\nu-2s'-1} > \tag{7.4}$$

form a basis for the irreducible highest-weight representation. Here

$$W_i = \begin{cases} L_{\frac{-i}{2}}, & i \equiv \text{even} \\ G_{\frac{-i}{2}}, & i \equiv \text{odd}. \end{cases} \tag{7.5}$$

L_i, G_i are the standard generators of the $N=1$ super-Virasoro algebra and f_i are the same as in (7.2).

Motivated by the partition identities due to Burge (theorems 1 and 2 in [44]) we would like to propose a different basis construction for $SM(2,4\nu)$.

Let us introduce a set of states

$$G^{f_m}_{-\frac{m}{2}}\cdots G^{f_2}_{-\frac{2}{2}}G^{f_1}_{-\frac{1}{2}} \mid \hat{\Delta}^{(2,4\nu)}_{1,2\nu-2s'-1+n} > \tag{7.6}$$

where $n=0(1)$ corresponds to the $NS(R)$ sector and f_i are defined as

$$f_1 \le 2(\nu - s' - n), \qquad f_{2i-n} \equiv 0(\text{mod}2), \qquad f_i + f_{i+1} \le 2(\nu-1) \tag{7.7}$$

with $s' \neq 0$ for $n=1$.
We conjecture that the above set forms an irreducible basis. Note that our proposal includes both NS and R-sectors.
To prove the above conjectures, it is sufficient to show that sets (7.4) and (7.6) are linearly independent. Thus, one is led to the important open question of finding an analog of the Kac determinant for the restricted partitions.

8. Concluding remarks

We expect that it would be straightforward to apply the techniques and methods developed here to study general $N=1$ $SM(p,p')$ character identities. Extension to other $SU(2)$ cosets would call for higher spin analogs of q-trinomial coefficients whose properties are at present poorly understood. It is important to find partition theoretical and configuration sum interpretations of (2.32)–(2.34). We believe that this interpretation would provide important clues leading to Boltzman weights for new integrable models which would have $SM(2,4\nu)$ and $M(2\nu-1,4\nu)$ as dual regimes. It would also be interesting to find a q-trinomial generalization of Bailey's lemma. We hope to address these challenging questions in our future publications.

Acknowledgments. We are pleased to thank G. Andrews, W. Eholzer, A. Honecker, V. Rittenberg, M. Rösgen, and O. Warnaar for many stimulating discussions. We are also grateful to G. Andrews for an advance copy of [63]. One of us (BMM) wishes to thank Prof. J. Maillard for his hospitality at Laboratoire de Physique Théorique et des Hautes Energies located at Université de Paris 6-7 where part of this work was done. This work was partially supported by NSF grant DMR 9404747 and by Deutsche Forschungs Gemeinschaft.

Appendix A. Proofs of Identities for q-trinomial Coefficients

In this appendix we prove the identities (4.1), (4.5), (4.9) of q-trinomial coefficients and the limiting formulas (2.29), (5.35). We follow the notation of [45]–[47].

The proof of (4.1) is practically the same as that of (2.3) of [47]. We first use (2.22) to rewrite (4.1) as

$$\begin{aligned}\binom{L, A-n; q}{A}_2 = q^{L-A}\binom{L-1, A-1-n; q}{A-1}_2 + q^{L+A-n}\binom{m-1, A-n+1; q}{A+1}_2 \\ + \binom{L-1, A-n; q}{A}_2 + q^{L-1-n}(1-q^{L-1})\binom{L-2, A-n; q}{A}_2.\end{aligned} \tag{A.1}$$

Then, recalling the definitions (2.21)–(2.23) and (1.2) one can easily derive

$$q^{L-A} t_n(L-1, A-1; j) = \frac{q^{L-A-j} - q^L}{1-q^L} t_n(L, A; j) \tag{A.2}$$

$$q^{L+A-n} t_n(L-1, A+1; j-1) = \frac{q^{L-j} - q^L}{1-q^L} t_n(L, A; j) \tag{A.3}$$

$$t_n(L-1, A; j) = \frac{1-q^{L-2j-A}}{1-q^L} t_n(L, A; j) \tag{A.4}$$

$$q^{L-n-1}(1-q^{L-1}) t_n(L-2, A; j-1) = \frac{q^{L-2j-A}}{1-q^L}(1-q^j)(1-q^{(j+A)}) t_n(L, A; j). \tag{A.5}$$

Then it is clear that

$$
\begin{aligned}
&q^{L-A}t_n(L-1,A-1;j)+q^{L+A-n}t_n(L-1.A+1;j-1)\\
&\quad+t_n(L-1,A;j)+q^{L-n-1}(1-q^{L-1})t_n(L-2,A;j-1)\\
&=\Big(\frac{q^{L-A-j}-q^L}{1-q^L}+\frac{q^{L-j}-q^L}{1-q^L}\\
&\quad+\frac{1-q^{L-2j-A}}{1-q^L}+\frac{q^{L-2j-A}(1-q^j)(1-q^{j+A})}{1-q^L}\Big)t_n(L,A;j)\\
&=t_n(L,A;j)
\end{aligned}
\tag{A.6}
$$

from which (A.1) follows by summing over j.

We now prove (4.5). To this end we note that both sides of (4.5) satisfy the same equation (4.1) with $n=1$. To conclude the proof one needs to verify that (4.5) holds true for $L=0,1$. This can be easily done by the direct inspection.

Let us now consider (4.9). This identity can be obtained from the slightly more general identity

$$
\begin{aligned}
&T_{n-1}(L,A;q^{\frac12})-T_{n-1}(L,A-2;q^{\frac12})\\
&\qquad=q^{\frac{L+A}{2}}T_n(L,A;q^{\frac12})-q^{\frac{L+2-A}{2}}T_n(L,A-2;q^{\frac12})
\end{aligned}
\tag{A.7}
$$

by setting $n=1$ and letting $A\to -A$. We first use (2.22) to rewrite (A.7) as

$$
\begin{aligned}
&\begin{pmatrix}L,A-n;q\\A\end{pmatrix}_2-q^A\begin{pmatrix}L,A-n+1;q\\A\end{pmatrix}_2\\
&\qquad=q^{1+n-A}\left[\begin{pmatrix}L,A-2-n;q\\A-2\end{pmatrix}_2-\begin{pmatrix}L,A-1-n;q\\A-2\end{pmatrix}_2\right].
\end{aligned}
\tag{A.8}
$$

Then, noting the identities

$$
t_n(L,A;j)-q^At_{n-1}(L,A;j)=\frac{q^{j(j+A-n)}(q)_L}{(q)_j(q)_{j+A-1}(q)_{L-2j-A}}
\tag{A.9}
$$

and

$$
\begin{aligned}
&q^{1+n-A}\Big[t_n(L,A-2;j+1)-t_{n-1}(L,A-2;j+1)\Big]\\
&\qquad=\frac{q^{j(j+A-n)}(q)_L}{(q)_j(q)_{j+A-1}(q)_{L-2j-A}}
\end{aligned}
\tag{A.10}
$$

we see that

$$
t_n(L,A;j)-q^At_{n-1}(L,A;j)=q^{1+n-A}\Big[t_n(L,A-2;j+1)-t_{n-1}(L,A-2;j+1)\Big]. \tag{A.11}
$$

Since

$$t_n(L, A-2; 0) - t_{n-1}(L, A-2; 0) = 0, \tag{A.12}$$

the desired result (A.8) follows by summing (A.11) over j.

To prove the limiting formula (2.29) we slightly extend the analysis given in [45]. Let us use the elementary relation

$$(q^{-1})_m = (-1)^m q^{\frac{-m(m+1)}{2}} (q)_m \tag{A.13}$$

in the definitions (2.21)–(2.22) to write

$$\begin{aligned} T_n(L, A; q^{\frac{1}{2}}) &= \sum_{l\geq 0} \frac{q^{2l^2-nl}(q)_L}{(q)_{\frac{(L-A-2l)}{2}}(q)_{\frac{(L+A-2l)}{2}}(q)_{2l}} \qquad \text{for } L-A \text{ even} \\ &= q^{\frac{(1-n)}{2}} \sum_{l\geq 0} \frac{q^{2l^2+(2-n)l}(q)_L}{(q)_{\frac{(L-A-2l-1)}{2}}(q)_{\frac{(L+A-2l-1)}{2}}(q)_{2l+1}} \text{ for } L-A \text{ odd.} \end{aligned} \tag{A.14}$$

It is now trivial to take the limit

$$\begin{aligned} \lim_{L\to\infty} T_n(L, A; q^{\frac{1}{2}}) &= \frac{1}{(q)_\infty} \sum_{j\geq 0,\text{even}} \frac{q^{\frac{j(j-n)}{2}}}{(q)_j} \quad \text{for } L-A \text{ even} \\ &= \frac{1}{(q)_\infty} \sum_{j\geq 0 \text{ odd}} \frac{q^{\frac{j(j-n)}{2}}}{(q)_j} \quad \text{for } L-A \text{ odd} \end{aligned} \tag{A.15}$$

from which, using the identity (2.20) of [54]

$$\sum_{j=0}^{\infty} \frac{t^j q^{\frac{j(j-1)}{2}}}{(q)_j} = \prod_{j=0}^{\infty}(1+tq^j) \tag{A.16}$$

with $t = \pm q^{\frac{(1-n)}{2}}$, the desired result (2.29) is obtained.

Finally, let us prove (5.35). Recalling (5.33) we can express lhs of identity (5.35) as

$$g(j,q) + g(j+1,q) = \lim_{L\to\infty} G(L,j,q); \quad j \in Z,\ j \geq 0 \tag{A.17}$$

where

$$\begin{aligned} G(L,j,q) = \sum_{l=0}^{L}(-1)^l \times \Big\{ &\big[T_{-1}(l,j+2) + T_{-1}(l,j) + 2T_{-1}(l,j+1)\big] - \\ &\big[T_{-1}(l,j+1) + T_{-1}(l,j-1) + 2T_{-1}(l,j)\big]\Big\}. \end{aligned} \tag{A.18}$$

Taking (4.5) into account, we find

$$G(L,j,q) = (-1)^{L+1}\big[T_1(L+1,j) - T_1(L+1,j+1)\big] - \delta_{j,0}. \tag{A.19}$$

Combining (A.17), (A.19) and (2.30) we obtain equation

$$g(j,q) + g(j+1,q) = \lim_{L\to\infty} G(L,j,q) = -\delta_{j,0} \tag{A.20}$$

which proves (5.35).

References

[1] A. Cappelli, C. Itzykson and J.-B. Zuber, Nucl. Phys. B280 (1987) 445.

[2] J. Lepowsky and S. Milne, Adv. in Math. 29 (1978) 271.

[3] A.J. Feingold and J. Lepowsky, Adv. in Math. 29 (1978) 271.

[4] J. Lepowsky and R.L. Wilson, Proc. Nat. Acad. Sci. USA (1981) 72254.

[5] J. Lepowsky and M. Primc, *Structure of the standard modules for the affine Lie algebra* $A_1^{(1)}$, Contemporary Mathematics, Vol. 46 (AMS, Providence, 1985).

[6] R. Kedem and B.M. McCoy, J. Stat. Phys. 71 (1993),883.

[7] R. Kedem, T.R. Klassen, B.M. McCoy and E. Melzer, Phys. Letts. B304(1993) 263.

[8] R. Kedem, T.R. Klassen, B.M. McCoy and E. Melzer, Phys. Letts. B 307 (1993) 68.

[9] S. Dasmahapatra, R. Kedem, T.R. Klassen, B.M. McCoy and E. Melzer, Int. J. Mod. Phys. B (1993) 3617.

[10] S. Dasmahapatra, R. Kedem, B.M. McCoy and E. Melzer, J. Stat. Phys. 74 (1994) 239.

[11] R. Kedem, B.M. McCoy, and E. Melzer, The sums of Rogers, Schur and Ramanujan and the Bose-Fermi correspondence in $1+1$ dimensional quantum field theory, hepth 9304056.

[12] M. Terhoeven, Lift of dilogarithm to partition identities, BONN-HE-92-36 (1992), hepth 9211120.

[13] A. Kuniba, T. Nakanishi and J. Suzuki, Mod. Phys. Lett. A8 (1993) 1835.

[14] J. Kellendonk and A. Recknagel, Phys. Lett. 298B (1993) 329.

[15] S. Dasmahapatra, String hypothesis and characters of coset CFT's, hepth 9305024.

[16] B. Feigin and A. Stoyanovsky, Quasi-particles models for the representations of Lie algebras and geometry of flag manifold, preprint RIMS 942, hepth 9308079 (1993)

[17] J. Kellendonk, M. Rösgen and R. Varnhagen, Int. J. Mod. Phys. A9 (1994) 1009.

[18] M. Terhoeven, Mod. Phys. Lett. A9 (1994) 133.

[19] E. Melzer, Int. J. Mod. Phys. A9 (1994) 1115.

[20] E. Melzer, Lett. Math. Phys. 31 (1994) 233.

[21] S. Dasmahapatra, Int. J. Mod. Phys. A10 (1995) 875.

[22] A. Berkovich, Nucl. Phys. B431 (1994) 315.

[23] O. Foda and Y-H. Quano, Polynomial Identities of the Rogers Ramanujan Type, hepth 9407191.

[24] O. Foda and Y-H. Quano, Virasoro character Identities from the Andrews-Bailey Construction, hepth 9408086.

[25] O. Foda and S.O. Warnaar, A bijection which implies Melzer's polynomial identities: the $\chi_{1,1}^{(p,p+1)}$ case, hepth 9501088.

[26] S.O. Warnaar, Fermionic solution of the Andrews-Baxter-Forrester model I: Unification of TBA and CTM methods, hepth 9501134.

[27] S.O. Warnaar and P.A. Pearce, J. Phys. A27 (1994) L891.

[28] S.O. Warnaar and P.A. Pearce, A-D-E Polynomial Identities and the Rogers-Ramanujan Identities, hepth 9411009.

[29] P. Bouwknegt, A. Ludwig and K. Schoutens, Phys. Lett. B 338 (1994) 488

[30] P. Bouwknegt, A. Ludwig and K. Schoutens, Spinon basis for $(\widehat{sl}_2)_k$ integrable highest weight modules and new character formulas, to appear in the Proceedings of "Statistical Mechanics and Quantum Field Theory," USC, May 16-31,1994.

[31] D. Gepner, Lattice models and generalized Rogers-Ramanujan identities, Phys. Lett. B348 (1995) 377.

[32] E. Baver and D. Gepner, Fermionic sum representations for the Virasoro characters of the unitary superconformal minimal models, hepth 9502118.

[33] G. Georgiev, Combinatorial constructions of modules for infinite-dimensional dimensional Lie algebras, I Principal subspace, hepth 9412054; II Parafermionic Space, hepth 9504024.

[34] A. Nakayashiki and Y. Yamada, Crystallizing the spinon basis, hepth 9504052.

[35] A. Nakayashiki and Y. Yamada, Crystalline spinon basis for RSOS models, hepth 9505083.

[36] A. Berkovich and B.M. McCoy, Letts. in Math. Phys. (in press) hepth 9412030.

[37] E. Melzer, Supersymmetric Analogs of the Gordon-Andrews Identities and Related TBA Systems, hepth 9412154

[38] B.L. Feigin and D.B. Fuchs, Funct. Anal. Appl. 16 (1982) 114; P. Goddard, A. Kent and D. Olive, Comm. Math. Phys. 103 (1986) 105; A. Meurman and A. Rocha-Caridi, Comm. Math. Phys. 107 (1986) 263.

[39] G.E. Andrews, in *The theory and applications of special functions*, ed. R. Askey (Academic Press, New York, 1975)

[40] D.M. Bressoud, Quart. J. Math. Oxford (2) 31 (1980) 385.

[41] L. Slater, Proc. Lond. Math. Soc. (2) 54 (1952) 147.

[42] H. Göllnitz, J. Reine Angew. Math. 225 (1967) 154.

[43] B. Gordon, Duke. Math. J. 31 (1965) 741.

[44] W.H. Burge, Europ. J. Combinatorics 3 (1982) 195.

[45] G.E. Andrews and R.J. Baxter, J. Stat. Phys. 47 (1987) 297.

[46] G.E. Andrews, J. Am. Math. Soc. 3 (1990) 653.

[47] G.E. Andrews, Contemporary Mathematics 166 (1994) 141.

[48] E. Witten, Nucl. Phys. B202 (1982) 253.

[49] W. Eholzer and R. Hübel, Nucl. Phys. B414 (1994) 348.

[50] L.J. Rogers, Proc. London Math. Soc. (2), 16 (1917) 315.

[51] G.E. Andrews, *Partitions: yesterday and today*, New Zealand Math. Soc., Wellington, 1979.

[52] M. Takahashi and M. Suzuki, Prog. of Theo. Phys. 48 (1972) 2187.

[53] P. DiFrancesco, H. Saleur and J.-B. Zuber, Nucl. Phys. 300 (1988) 393.

[54] G.E. Andrews, *Theory of Partitions*, Addison-Wesley, Reading Mass. (1976).

[55] A. Rocha-Caridi,in *Vertex Operators in mathematics and physics,* ed. J. Lepowsky, S. Mandelstam and I.M. Singer,(Springer, Berlin, 1985).

[56] P.J. Forrester and R.J. Baxter, J. Stat. Phys. 38 (1985) 435.

[57] D.M. Bressoud, in *Ramanujan Revisited*, pp. 57-67, G.E. Andrews et al eds., Academic Press, 1988.

[58] V.A. Fateev and Al.B. Zamolodchikov, Phys. Lett. B271 (1991) 91.

[59] G.E. Andrews, R.J. Baxter and P.J. Forrester, J. Stat. Phys. 35 (1984) 193.

[60] R. Tateo, New functional dilogarithm identities and sine-Gordon Y-systems, hepth 9505022.

[61] F. Ravanini, M. Staniskov and R. Tateo, Integrable perturbations of CFT with complex parameter: the $M_{3/5}$ model and its generalizations, hepth 9411085.

[62] B.L. Feigin, T. Nakanishi and H. Ooguri, Int. J. Mod. Phys. A7, Suppl. 1A (1992) 217.

[63] G.E. Andrews, Rogers-Ramanujan polynomials for modulus 6, (preprint).