equations in integrable models

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Abstract

The definitions of the main notions related to the quantum inverse scattering methods are given. The Yang-Baxter equation and reflection equations are derived as consistency conditions for the factorizable scattering on the whole line and on the half-line using the Zamolodchikov-Faddeev algebra. Due to the vertex-IRF model correspondence the face model analogue of the ZF-algebra and the IRF reflection equation are written down as well as the Z_2-graded and colored algebra forms of the YBE and RE.

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1 Introduction.

Integrable and/or solvable models have been always quite important for theoretical physics. One of their attractive features is the direct possibility to go beyond perturbation theory giving a solid background for theoretical hypothesis and constructions. Another valuable characteristic is related to the combination of different mathematical methods required to solve some specific model thus giving rise to mutual interrelations among formally separated fields of mathematics. The start of the recent activity in the quantum integrable models (see reviews [1, 2, 3, 4, 5, 6, 7, 8] and Refs. therein) was definitely related with the development of the soliton theory, although the influence of the preceding pioneering contributions is out of discussion (see [10, 11, 12, 13] and Refs. therein). This is a large field of research and even recent monographs [11, 12, 13, 14, 15, 16, 17, 18] do not overlap much in between.

There is variety of quantum integrable models and quite a few interrelations among them. In these lectures we restrict ourselves to those models the integrability of which is related more or less directly to the Yang-Baxter equation and to reflection equations. More of that, we will discuss mostly the algebraic properties of these equations, their solutions and corresponding integrable models putting aside very elaborated analytical techniques and/or problems (of great physical significance) such as the thermodynamic limits of finite size systems, massless and/or conformal field theory limits, continuous limits of the lattice models, correlation functions and form factors, critical exponents etc. The main notions related to the Yang-Baxter equation (YBE) will be introduced: factorizable scattering on the line, the Zamolodchikov-Faddeev algebra, the fusion procedure or the bound state scattering, the Yang-Baxter or transition matrix algebra, integrals of motion. The factorizable scattering on a half-line gives rise to the reflection equation (RE) and a boundary operator. Most of the applications of the YBE can be extended to the RE case with appropriate modifications. Due to the vertex-face model correspondence of statistical mechanics we introduce an interaction round face analogue of the Zamolodchikov-Faddeev algebra and the corresponding reflection equation for the boundary weights. The super or Z_2-graded analogue of these constructions with the generalizations to the color algebras will also be given as well as some particular integrable systems with finite degrees of freedom. The interrelations among few forms of the Bethe Ansatz: coordinate, algebraic, analytical and functional will be mentioned as well.

More particular applications of the Bethe Ansatz technique to specific integrable models with direct physical meaning (a high energy limit of the QCD, the Azbel - Hofstadter problem) and of the reflection equation formalism to the Chern-Simons theory and quantized moduli spaces of flat connections can be found in other lectures of this School.

2 Yang-Baxter equation and Zamolodchikov-Faddeev algebra

Let us start the presentation which will be full of definitions and technicalities from a specific integrable model to have in mind an example. This is the one-dimensional Bose gas consisting of n sort particles with the Dirac delta-function two particle potential. The field operators $\psi_a(x), \psi_b^\dagger(y)$ satisfy the canonical commutation relations, $a, b = 1, 2, \cdots, n$

$$[\psi_a(x), \psi_b^\dagger(y)] = \delta_{ab}\delta(x-y) \quad , \quad [\psi_a(x), \psi_l(y)] = 0. \tag{1}$$

The field Hamiltonian is

$$H = \int dx(\partial_x \psi_a^\dagger(x)\partial_x \psi_a(x) + c\psi_a^\dagger(x)\psi_b^\dagger(x)\psi_b(x)\psi_a(x)) \tag{2}$$

The translational invariance (on the whole line or with the periodic boundary conditions, circle) and the internal $U(n)$ invariance are obvious. The corresponding symmetry generators are

$$P = -i\int dx\psi_a^\dagger(x)\partial_x\psi_a(x), \quad U_{ab} = \int dx\psi_c^\dagger(x)(u_{ab})_{cd}\psi_d(x). \tag{3}$$

where u_{ab} are n^2 generators of the unitary Lie algebra $u(n)$. One of these operators is the particle number operator

$$N = \int dx\psi_a^\dagger(x)\psi_a(x)\ . \tag{4}$$

The Fock space of states is the direct sum of the number operator N eigenspaces

$$\mathcal{H}_F = \sum_{M=0}^{\infty} \mathcal{H}_M \tag{5}$$

The common eigenfunctions Ψ_M of operators N, P, and H (2)

$$\Psi_M = \int d^M x\Psi(1, \cdots, M|\lambda_1, \cdots, \lambda_M)\prod_{i=1}^{M}\psi_{a_i}^\dagger(x_i)|0\rangle \tag{6}$$

are constructed as appropriate linear combinations of the one particle eigenstates (the plane waves). The numbers $j = 1, 2, \cdots M$ in the argument of Ψ (6) refer to both coordinate x_j and isotopic a_j indices. The coefficients in the linear combination depend on the one particle parameters (momenta) λ_j and on the isotopic indices a_j

$$\Psi(\{j\}|\{\lambda_j\}) = \sum_{\sigma\in\mathcal{S}_M} A_\sigma(\{a_j\}, \{\lambda_n\})exp(i\sum_{m=1}^{M}\lambda_{\sigma m}x_m) \tag{7}$$

where σ are all elements of the permutation group $\mathcal{S}_M$.

Such a form of the Hamiltonian eigenfunctions is known as the **coordinate Bethe Ansatz.** The conditions of the wave function continuity and its appropriate derivative jumps on the hyperplanes $x_j = x_{j+1}$ (the sewing conditions) define the coefficients $A_\sigma(\{a_j\}|\{\lambda_k\})$. The coefficients A_σ and $A_{\sigma'}$ with $\sigma' = \sigma_j\sigma$ where σ_j is the transposition of the indices $j, j+1$, are related by the two particle S-matrix [20]:

$$A_{\sigma'} = S(\lambda_j - \lambda_{j+1})A_\sigma \ , \tag{8}$$

where for the model chosen (2)

$$S(\lambda) = (\lambda + ic\mathcal{P})/(\lambda - ic) \ , \tag{9}$$

and $\mathcal{P}$ is the permutation operator in $C^n \otimes C^n$. The periodicity condition for the system on finite interval $(0, L)$ results in the **Bethe equations** for the set of M momenta λ_j

$$exp(i\lambda_j L) = -\prod_k S_{jk}(\lambda_j - \lambda_k) \ , \tag{10}$$

where in the ordered product $k = j+1, ..., M-1, M, 1, ..., j-1$. Hence the meaning of the RHS is the scattering matrix of the j-th particle on the other $(M-1)$ particles. This would be just a phase factor for the scalar particles $(S(\lambda) = (\lambda + ic)/(\lambda - ic))$, but for the n component case one has to diagonalize the complicated scattering matrix to arrive to a system of scalar equations. All quantities in the RHS are particular values at $\lambda = \lambda_j$ of the **transfer matrix** $(k = 1, 2, ..., M)$

$$t(\lambda; \{\lambda_m\}) = tr_a T(\lambda; \{\lambda_m\}) \equiv tr_a \prod_k S_{ak}(\lambda - \lambda_k) \tag{11}$$

of the inhomogeneous $GL(n)$-spin magnet of M sites with the transition or monodromy matrix $T(\lambda; \{\lambda_m\})$. The trace in the expression for the transfer matrix $t(\lambda; \{\lambda_m\})$ is taken over the **auxiliary space** $V_a = C^n$, while $t(\lambda; \{\lambda_m\})$ is an operator (matrix) in the space $\prod_{k=1}^M (C^n)_k$. The important property of the commutativity of the transfer matrix for different values of the spectral parameter

$$[t(\lambda; \{\lambda_m\}), t(\mu; \{\lambda_m\})] = 0$$

follows easily from the fundamental commutation relation for the transition matrix $T(\lambda; \{\lambda_m\})$ (see (15)).

Hence the RHS of (10) $t(\lambda_j; \{\lambda_m\})$ can be diagonalize simultaneously. As result the **hierarchy of the Bethe Ansatze** appears (or the nested Bethe Ansatz) and the complete parametrization of $\Psi(\{j\}|\{\lambda_j\})$ has n sets of "quasi-momenta" including λ_j (see below).

The consistency condition of this system is the **Yang-Baxter equation** (YBE) for the S-matrix $S(\lambda)$:

$$S_{jk}(\lambda_j-\lambda_k)S_{jl}(\lambda_j-\lambda_l)S_{kl}(\lambda_k-\lambda_l)=$$

$$=S_{kl}(\lambda_k-\lambda_l)S_{jl}(\lambda_j-\lambda_l)S_{jk}(\lambda_j-\lambda_k). \qquad (12)$$

One uses very often in the general setting the notation R (the R-**matrix**) for the YBE solution.

The complete scattering matrix $S(\{\lambda_k\})$ of the M particle is given by the ratio of the coefficients of the incoming wave ($\lambda_1 < ... < \lambda_M$) and the outgoing wave

$$exp(i\sum_j \lambda_j x_j), \quad exp(i\sum_j \lambda_{M-j+1}x_j)\ .$$

This ratio is factorized into the ordered product of $M(M-1)/2$ two particle S-matrices (9) (the **factorizable scattering** on the line).

It was proposed for an algebraic description of the factorizable scattering in the general case to introduce a set of (annihilation) operators $Z_a(\lambda)$ [19] satisfying the commutation relations (the **Zamolodchikov algebra**)

$$Z_a(\lambda)Z_b(\nu)=S_{ab,cd}(\lambda-\nu)Z_d(\nu)Z_c(\lambda), \qquad (13)$$

where $S(\lambda-\nu)$ is an $n^2\times n^2$ matrix. Using the associativity property of this algebra and changing the order of the product $Z_{a_1}(\lambda_1)Z_{a_2}(\lambda_2)Z_{a_3}(\lambda_3)$ in two possible ways one arrives to the consistency condition (12). Extending the Zamolidchikov algebra (13) by adding n more (creation) conjugated operators $Z_a^\dagger(\nu)$ one gets the **Zamolodchikov-Faddeev algebra** (ZF-algebra)

$$Z_a(\lambda)Z_b^\dagger(\nu)=\delta_{ab}\delta(\lambda-\nu)+Z_c^\dagger(\nu)\hat{S}_{ac,bd}(\lambda-\nu)Z_d(\lambda). \qquad (14)$$

It is useful to write down the ZF-algebra in a compact matrix form by introducing the n component column $A(\lambda)=(Z_1(\lambda),...,Z_n(\lambda))^t$ and the n component row $A^\dagger(\nu)=(Z_1^\dagger(\nu),...,Z_n^\dagger(\nu))$. Then the defining relations of the ZF-algebra are

$$A(\lambda)\otimes A(\nu)\equiv A_1(\lambda)A_2(\nu)=S_{12}(\lambda-\nu)A_2(\nu)A_1(\lambda),$$

$$A_1^\dagger(\lambda)A_2^\dagger(\nu)=A_2^\dagger(\nu)A_1^\dagger(\lambda)S_{21}^\dagger(\nu-\lambda),$$

$$A_1(\lambda)\otimes A_1^\dagger(\nu)=I_1\delta(\lambda-\nu)+A_2^\dagger(\nu)\hat{S}_{12}(\nu-\lambda)A_2(\lambda),$$

where subscripts refer to the corresponding isotopic spaces $C^n\otimes C^n\equiv V_1V_2$ and $S_{21}=\mathcal{P}S_{12}\mathcal{P}$, $\hat{S}_{12}=\mathcal{P}S_{12}$. Due to the unitarity property of the S-matrix: $S_{12}(\lambda-\nu)S_{21}(\nu-\lambda)=I_{12}$ and the YBE one can construct the Fock space representation $\mathcal{H}_F$ of the ZF-algebra using the generalizing symmetrizing operators which include $\hat{S}$ instead of the permutation operators [37].

To solve the non-linear Schrödinger equation (NS) (the Heisenberg equation of motion for $\psi_a(x,t)$ with the Hamiltonian (2)) in the framework of the quantum inverse scattering method (QISM) [1 - 7] one has to find the corresponding **auxiliary linear problem**

$$\frac{d}{dx}T(\lambda,x) = L(\lambda,x)T(\lambda,x)\,.$$

For the model in question L depends on the spectral parameter λ and the original dynamical local variables (1)

$$L(\lambda,x) = \lambda J + \sum_a (c\psi_a(x)e_{a,n+1} + \psi_a^\dagger(x)e_{n+1,a})$$

where J is $(n+1)\times(n+1)$ diagonal matrix:$J =$ diag $(I_n,-1)$ and e_{ij} are the $(n+1)\times(n+1)$ basis matrices. Hence the auxiliary linear space here is C^{n+1}. This is the **L-operator** of the QISM or the classical soliton theory [13]. Solution to the operator valued matrix first order equation with normal ordering with respect to the local fields (1) and the vacuum $|0\rangle$

$$T(\lambda,x) =: exp(\int^x L(\lambda,y)dy) :$$

defines the **transition or monodromy matrix** $T(\lambda,x)$. Its entries are the new variables of the model (the quantum scattering data (QSD)).

For the NS case it is natural to represent $T(\lambda,x)$ in the block form

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where A is an $n\times n$ matrix, B and C are n component vectors, D is a scalar. The quadratic commutation relations of the new variables are defined by the fundamental relation [1, 2, 3]

$$R(\lambda-\nu)T_1(\lambda)T_2(\nu) = T_2(\nu)T_1(\lambda)R(\lambda-\nu) \qquad (15)$$

where the standard QISM notations [1-7] are used $T_1 = T\otimes I$, $T_2 = I\otimes T$ to embed matrices in C^n into $C^n\otimes C^n$. From the structure of the L-operator one can conclude that the n elements of the row $C(\lambda)$ act on the vacuum as some creation operators. This is the starting point to construct the eigenfunctions of the transfer matrix $t(\mu) = trT(\mu) = trA + D$ algebraically:

$$\Psi_M = \prod_{j=1}^{M} C_{a_j}(\lambda_j)0\rangle$$

in the framework of the **algebraic Bethe Ansatz** [1, 2, 3].

The mentioned above commutativity of the transfer matrices $t(\mu) = trT(\mu)$ follows from (15) taking the trace of $T_1T_2 = R_{12}^{-1}T_2T_1R_{12}$ over both spaces $C^n\otimes C^n$. The trace form of the integrals of motion generating function $t(\mu)$ for continuous as NS (2) or chain models as (11) gives rise to the periodic boundary conditions. For the M site spin Hamiltonian $H = (\sum_{n=1}^{M-1} h_{n,n+1}) + h_{M,1}$ the M-th site spin interacts with the first one. The treatment of the non-periodic boundary conditions in the algebraic framework of the QISM requires the RE [29, 32, 34, 31] (for the coordinate Bethe Ansatz see [12, 28]).

Using block decomposition for the R-matrix one can rewrite the compact form of the fundamental relations (15) in terms of the blocks A, B, C, and D. Restricting to the case of finite number of particle and after appropriate limit to the whole line one gets the ZF-algebra realization in terms of the QSD [50]

$$Z(\lambda) = (A)^{-1}(\lambda)B(\lambda), \quad Z^{\dagger}(\nu) = C(\nu)D^{-1}(\nu)\,.$$

The R-matrix of (15) satisfies the YBE

$$R_{12}(\lambda-\mu)R_{13}(\lambda-\nu)R_{23}(\mu-\nu) = R_{23}(\mu-\nu)R_{13}(\lambda-\nu)R_{12}(\lambda-\mu)\,.$$

For the NS the R-matrix has the same structure as (9) (but it is $(n+1)^2 \times (n+1)^2$ matrix).

In the general situation the R-matrix depends on the spectral parameter u and some other parameters $R(u;\eta,...)$. Although there is no complete mathematical theory of the Yang-Baxter equation, variety of solutions are known as well as different fields of their applications. In particular, many solutions are related to the simple Lie (super) algebras. They are classified by the Lie algebra, its irreducible representations Λ_j and the spectral parameter dependence: rational, trigonometric and elliptic ones [3, 9, 11, 48]. The $sl(2)$ spin 1/2 R-matrix related to the XXX-magnet is used in few lectures of these Proceedings. The recent development relates the spectral parameter dependent R-matrices with the affine Lie (super-)algebras [16]. There are also solutions to the YBE with the spectral parameter on the algebraic curves of higher genus (the Potts models). The R-matrices acting in infinite dimensional spaces can be found in papers [58].

Some particular properties of the YBE solution $R(u;\eta,...)$, which are important for different applications (but not always valid for a given solution) are: regularity

$$R(0) = \phi(\eta)\mathcal{P}\ ,$$

P-symmetry

$$\mathcal{P}R_{12}(u)\mathcal{P} \equiv R_{21}(u) = R_{12}(u)\ ,$$

T-symmetry

$$R_{12}^{T}(u) = R_{12}(u)\,,$$

unitarity

$$R_{12}(u)R_{21}(-u) = \rho(u)I\ ,$$

crossing symmetry

$$R_{12}^{t_1}(u) R_{12}^{t_1}(-u-\eta) = \xi(u) I \, ,$$

quasiclassical property

$$R(u;\eta) = I + \eta r(u) + \mathcal{O}(\eta^2) \, ,$$

where $r(u)$ is the classical r-matrix [3, 13] and $\phi(u)$, $\rho(u)$, $\xi(u)$ are some functions related to the R-matrix normalization. Many R-matrices have only PT-symmetry: $R_{12}^T(u) = R_{21}(u)$. The regularity is used to extract from $t(u)$ of the lattice models the integrals of motion which are local in terms of initial spin variables when the R-matrix itself is the L-operator. The quasiclassical property gives rise to the direct connection of the quantum model to the corresponding classical one [3, 13].

It is easy to see that the product of the R-matrices $R_{13}(u) R_{23}(v)$ and $R_{1'3}(w)$ are intertwined (like (15)) by the matrix $R_{11'}(u-w) R_{21'}(v-w)$. If the original R-matrix $R_{12}(x)$ is degenerated into a projector at $x = \eta$ then one can project the above product to the corresponding subspace for the fixed difference of the spectral parameters $u - v = \eta$. For the Yang solution (9) at $\lambda = \pm ic$ one has the (anti)symmetrizer P_+, P_- . This is the **fusion procedure** [3] to get new R-matrices from the known ones. It has a direct physical interpretation as construction of the bound state S-matrix [19, 21]. Using the fusion procedure the L-operators in the higher dimensional irreducible representations can be obtained giving rise to the ZF operators for the bound states and to the integrable lattice models of higher spins such as the spin s XXZ-model. The connection of the R-matrices and integrable models with the simple Lie (super-)algebras is reflected in the structure of the Bethe equations: they include r sets of "quasimomenta" , where r is the Lie algebra rank, and the Cartan matrix [37, 47].

Omitting the spectral parameter dependence in the YBE one gets still very interesting equation solutions of which (the **constant R-matrices**) can be considered as structure constants of the quantum groups and the quantum algebras. Then the quadratic relations (15) (without the spectral parameter) are the defining relations on the n^2 generators T_{ab} of the corresponding quantum group. The constant R-matrices $\hat{R}_{ij}$ have also direct relation to the braid group (BG), for one of the defining relations of the BG generators $\sigma_i \, \sigma_{i+1} \, \sigma_i = \sigma_{i+1} \, \sigma_i \, \sigma_{i+1}$ coincides with the YBE for $\hat{R}_{ij} = \mathcal{P}_{\rangle|} \mathcal{R}_{\rangle|}$:

$$\hat{R}_{i-1i} \; \hat{R}_{ii+1} \; \hat{R}_{i-1i} = \hat{R}_{ii+1} \; \hat{R}_{i-1i} \; \hat{R}_{ii+1} \; .$$

3 Reflection equations and their covariance

Let us consider factorizable scattering of particles with internal degrees of freedom on a half-line [22]. Then even one-particle process in nontrivial (a reflection from the wall) and it is described by an $n \times n$ matrix $K(u)$ the **reflection matrix** or the boundary S-matrix.

For an algebraic description it is useful to add to the ZF-algebra a formal boundary operator B with relation [24]

$$Z_a(u)B = K_{ab}(u)Z_b(-u)B\,.$$

Then the two particle factorizability gives rise additionally to the YBE for the two body S-matrix (12) to the **reflection equation** (RE)

$$S_{12}(u-v)K_1(u)S_{21}(u+v)K_2(v) = \\ K_2(v)S_{12}(u+v)K_1(u)S_{21}(u-v)\,. \tag{16}$$

There are natural properties of the reflection matrix $K(u)$ as in the preceding Sec. of $R(u;\eta,...)$: $K(0) = I$ (regularity); $K(u)K(-u) = I$ (unitarity); T-symmetry $K^T(u) = K(u)$; the crossing symmetry is more elaborated and it involves the S-matrix itself [24]. The constructions of the quantum group invariant spin systems [30] uses the RE [31, 32, 42, 54]. Recent field theoretical applications of the RE can be found in [23-27, 36].

It is interesting to point out that as a consequence of the RE (16) the quadratic combination

$$\varphi(u) = \sum_{a,b} Z_a(-u)K_{ab}Z_b(u)$$

is a "local" field : $[\varphi(u),\ \varphi(w)] = 0$ [24, 52]. This property was used to construct the metric tensor field for the quantum Liouville theory [52] with a slightly different RE and to propose a form of the boundary operator $B = exp(\int_0^\infty \varphi(u)du)$ in the dual Hamiltonian picture of the boundary conformal field theory [24].

Although for many R-matrices solutions to the reflection equations were found (cf [22, 25, 32, 33]) there is no direct relation of them to the Lie algebra theory. In particular, many of them do not depend on the quasiclassical parameter η (in the fundamental representation for the R-matrix).

Let us give two examples of the RE solutions. Due to the $GL(n)$ symmetry of the Yang solution (9): $[R, M \otimes M], M \in GL(n)$ the corresponding K-matrix can be transformed $K \to K' = MKM^{-1}$ with arbitrary M and the solution to the RE is

$$K(u) = \xi I + uC, \quad C^2 = I.$$

In the elliptic case (the eight-vertex R-matrix [11]) the solution $K(u)$ is [22, 25, 33]

$$K(u) = \begin{pmatrix} x(u) & y(u) \\ z(u) & w(u) \end{pmatrix},$$

$$
\begin{aligned}
x(u) =& \quad sn(\xi + u) \\
y(n) =& \quad \mu\, sn2u(\lambda(1 - ksn^2u) + 1 + ksn^2u)(1 - k^2 sn^2\xi sn^2 u)^{-1}\,, \\
w(u) =& \quad sn(\xi - u)\,, \\
z(n) =& \quad \mu\, sn2u(\lambda - 1 - (\lambda + 1)ksn^2u)(1 - k^2 sn^2\xi sn^2 u)^{-1}\,,
\end{aligned}
$$

where $snu \equiv sn(u;k)$ is the Jacobi elliptic function of modulus $0 < k < 1$, ξ, λ, μ are parameters.

The RE (16) has an important **covariance property**: if $T(u)$ and $K(u)$ satisfy the relations (15), (16) then $K'(u) = T(u)K(u)T(-u)^{-1}$ is also the RE solution provided that the entries of $K(u)$ and $T(u)$ commute $[K_{ab}(u), T_{cd}(u)] = 0$. The proof follows easily by the substitution of $K'(u)$ into the RE and using few times different forms of the fundamental relation (15) e.g.

$$
T_2^{-1}(-v)R_{12}(u+v)T_1(u) = T_1(u)R_{12}(u+v)T_2^{-1}(-v).
$$

This property gives rise to the **Sklyanin monodromy matrix** [29, 34]

$$
\mathcal{T}(u) = T(u)K(u)T(-u)^{-1}. \tag{17}
$$

If the matrix $T(u)$ is constructed as an ordered product of N independent L-operators then $\mathcal{T}(\sqcap)$ can be interpreted as the monodromy matrix of N site lattice model with a boundary condition described by the matrix $K(u)$ or a boundary interaction described by the operator valued entries of the matrix $K(u)$. To extract the corresponding Hamiltonian and other integrals of motion the transfer matrix is constructed using a special trace

$$
\tau(u) = trK_+(u)T(u)K(u)T(-u)^{-1} = trK_+(u)\mathcal{T}(u)\,. \tag{18}
$$

An extra K-matrix $K_+(u)$ is any solution of a "conjugated" RE [29, 32, 34, 51] defined in such way to guarantee the commutativity $[\tau(u),\ \tau(w)] = 0$. In the regular case

$$
R_{an}(0) \sim \mathcal{P}_{an}\,, \quad K(0) \sim I\,, \quad h_{n,n+1} = (\partial/\partial u \hat{R}_{n,n+1})(0)\,,
$$

the Hamiltonian is [29]

$$
H = \sum_{n=1}^{M-1} h_{n,n+1} + K_1'(0) + (tr_0 K_+(0) h_{M,0})/trK_+(0)\,. \tag{19}
$$

The structure of the **fusion procedure for the reflection equation** is similar to the R-matrix case, but the projected combination ($R_{12}(\eta) \sim$ projector)

$$
K_1(u)R_{21}(2u-\eta)K_2(u-\eta)
$$

includes the additional R-matrix in between the K-matrices [32, 51].

More general RE with four different R-matrices (with or without the spectral parameter)

$$R^{(1)}_{12}\ K_1\ R^{(2)}_{12}\ K_2\ = K_2\ R^{(3)}_{12}\ K_1\ R^{(4)}_{12}\ .$$

which are related among themselves by some consistency conditions similar to (12) and (15), can be found in different papers these days (see [55, 51] and Refs therein). In particular, this kind of RE was used for the quantum group invariant spin models with topological interaction [42, 43]. Some generalizations of the RE are related to the Coxeter groups [44].

The constant RE is also of interest as it was in the YBE case. The corresponding RE-algebras with the K-matrix entries considered as the generators, are related to the quantum group homogeneous spaces while the c-number solutions K can be considered as representations of a special generator of the $\mathrm{BG}^{(1)}$ in the solid handlebody [51] with the defining relation

$$\hat{R}_{12}\ K_1\ \hat{R}_{12}\ K_1\ = K_1\ \hat{R}_{12}\ K_1\ \hat{R}_{12}\ .$$

In solvable models of statistical mechanics the R-matrix defines the Boltzmann weights $R_{\alpha\beta;\gamma\delta}(u;\eta,...)$ of the vertex models, where α,β,γ and δ are spin variables on the four edges round a vertex. The models on the dual lattice are known as the interaction round face models (IRF). The corresponding Boltzmann weights $w(a,b,c,d|u)$ satisfy the **star-triangular equation (relation)** (STR) or Baxter relation [11]

$$\sum_g w(a,b,g,f|u)w(f,g,d,e|u+v)w(g,b,c,d|v) =$$

$$= \sum_g w(f,a,g,e|v)w(a,b,c,g|u+v)w(g,c,d,e|u)$$

The four sites surrounding a face are ordered anticklockwise from the southwest corner and u is a complex (spectral) parameter.

Let us introduce a face analogue of the ZF-algebra: Its generators are parametrized by two indices $Z_{ab}(u)$ and satisfy the relations (see e.g. [57])

$$Z_{ab}(u)Z_{bc}(v) = \sum_d w(a,b,c,d|u-v)Z_{ad}(v)Z_{dc}(u)$$

with the only one summation over d in the RHS. As in the previous case the scattering of a one particle $Z_{da}(u)$ on N others $Z_{b_i c_i}(v_i)$ will give a transition matrix

$$T(u,v_1,\cdots,v_N) = \prod_{j=1}^{N} w(b_j,b_{j+1},c_{j+1}\ ,c_j|u-v_j).$$

by analogy with the YBE and ZF algebra constructions one can formulate the following problems: 1) what are the conjugated (annihilation) operators and their commutation relations with $Z_{ab}(u)$? 2) how to translate the fusion procedure to the STR case? 3) what is an IRF analogue to the RE ?

Let us start from the last question and introduce a formal operator B of the boundary which satisfies relations with $Z_{ab}(u)$:

$$Z_{ab}(u)B = Q(a,s,b;u)Z_{as}(-u)B\,,$$

where $Q(a,s,b;n)$ is a face analogue of the reflection matrix K. To derive an STR analogue of the reflection equation the two ways of transforming the product of three operators may be considered

$$Z_{ab}(u)Z_{bc}(v)B \longrightarrow Z_{ak}(-u)Z_{kd}(-v)B$$

The resulting relation for the "scattering" matrix $w(\cdots)$ and the reflection matrix $Q(\cdots)$ is [2]

$$w(a,b,c,g|u-v)Q(g,c,f|u)w(a,g,f,k|v+u)Q(k,d,f|v) =$$

$$= Q(b,m,c|v)w(a,b,m,s|u+v)Q(s,d,m|u)w(a,s,d,k|u-v)\,,$$

with summation over g,f in the LHS and over m,s in the RHS .

To relate vertex models with the IRF models the Baxter intertwining vectors come to play [11]

$$\sum_{\beta,\nu} R_{\beta\nu;\lambda\nu}\phi_{ll'}(\beta)z_{m'l'}(\nu) = \sum_{m} w(m,m',l,l')z_{ml}(\lambda)\phi_{mm'}(\alpha)\,.$$

These vectors remind a combination of the operators of the ZF-algebras of the YBE and STR types. These vectors can be used to relate the reflection matrices of the vertex and IRF models:

$$\bar{\phi}_{ab}(\beta)K_{\beta\nu}(u)\phi_{bs}(\nu) = Q(a,s,b;u)\,,$$

where the spectral parameter dependence of the vectors is omitted.

4 Integrable models with anticommuting variables (fermions)

One of the first integrable models of the quantum field theory-massless Thirring model had attracted a lot of attention of theoretical physicists during decades

$$H = \int dx(\bar{\psi}\gamma\partial\psi + g(\bar{\psi}\psi)^2)\,.$$

[2]M.D. Gould informed me that the IRF analogue of the RE was considered by P.A. Pearce.

It contains Fermi fields $\psi_\alpha(x,t)$, $\psi^+_\beta(x,t)$ satisfying anticommutation relations

$$\{\psi_\alpha(x,t), \psi^+_\beta(y,t)\} = \delta_{\alpha\beta}\delta(x-y) \, .$$

The addition of the mass term $m\bar{\psi}\psi$ to the Hamiltonian gives rise to the very rich dynamical properties of the massive Thirring model. In particular, it is dual to the famous sine-Gordon model with the change of the strong coupling regime to the weak coupling one.

The famous δ-potential Bose gas model [20] (another name is quantum nonlinear Schrödinger equation) (2) was also generalized to the multicomponent fermionic case in the framework of the coordinate Bethe Ansatz [45] (see below).

Taking into account the "superization" procedure of the theoretical physics in the seventies, let us follow this pattern and extend most of the preceding equations and constructions to the case of the anticommuting variables. Mathematically it is related with Z_2-graded vector space, Z_2-graded algebras or as in the theoretical physics text: super-spaces, super-algebras, super-analysis...

The Z_2-graded Zamolodchikov algebra is generated by the $(m+n)$ operators $Z_i(u)$ satisfying

$$Z_i(u)Z_j(v) = (-)^{p(i)p(j)} R_{ij;kl}\,(u,v) Z_l(v) Z_k(u) \, . \tag{20}$$

where $i,j = 1,2,\cdots,m+n, p(i)$ is a parity function: $p(i) = 0$ $(i = 1,\cdots,m)$ and $p(i) = 1$ $(i = m+1,\cdots,m+n)$, hence one has m boson and n fermion operators. The coefficients in (20) are c-numbers and the R-matrix is even:

$$p(R_{ij;kl}) = p(i) + p(j) + p(k) + p(l) = 0 \, . \tag{21}$$

Due to the Z_2 valuedness of the function $p(i)$, and the R-matrix structure (21) one can change sign factor in (20) to $(-)^{p(k)p(l)}$. The super-associativity requirement for $Z_a(u)Z_b(v)Z_c(w)$ results in the graded Yang-Baxter equation (grYBE)

$$R_{ab;a'b'}\,(u,v) R_{a'c;jc'}\,(u,w) R_{b'c';hl}\,(v,w)\,(-)^{p(b')(p(j)+p(a'))} = \tag{22}$$

$$= R_{bc;b'c'}\,(v,w) R_{ac';a'l}\,(u,w) R_{a'b';jk}\,(u,v) (-)^{p(b')(p(a)+p(a'))} \, ,$$

where the condition (21) was used to reduce the factors in the LHS and RHS:

$$(-)^{p(a)p(l)+p(a')p(c)+p(b')p(c')}; \quad (-)^{p(b)p(c)+p(a)p(c')+p(a')p(b')} \quad .$$

Recalling these sign factors one can write the grYBE in the same matrix form as previously but now in the Z_2-graded tensor product of three super-spaces $C^{m|n}$

$$R_{12}\,(u,v) R_{13}\,(u,w) R_{23}\,(v,w) = R_{23}\,(v,w) R_{13}\,(u,w) R_{12}\,(u,v) \, . \tag{23}$$

Considering the third space as an unspecified quantum space (or substituting instead of the $(m+n)\times(m+n)$ blocks in R_{13} and R_{23} formal entries) one arrives to the graded FRT-relation

$$R_{12}\,(u,v)T_1\,(u)T_2\,(v) = T_2\,(v)T_1\,(u)R_{12}\,(u,v) \quad . \tag{24}$$

The latter relation has the same sign factors in the component form as (22). These factors can be interpreted as result of the Z_2-graded tensor product of two even matrices F and G ($p(F_{ac}) = p(a) + p(c)$)

$$(F\otimes G)_{ab;cd} = (-)^{p(b)(p(a)+p(c))}F_{ac}G_{bd}\ . \tag{25}$$

Hence, one gets additional sign factors for T_1 and the factor free T_2 (to consider more tensor factors the choice of T_1 factor free is more convenient)

$$(T_1)_{ab;cd} = (T\otimes I)_{ab;cd} = (-)^{p(b)(p(a)+p(c))}T_{ac}\delta_{bd}\ , \tag{26}$$

$$(T_2)_{ab;cd} = (I\otimes T)_{ab;cd} = \delta_{ac}T_{bd}$$

An additional sign factor appears also in the matrix of the permutation (flip) operator

$$\mathcal{P}(v\otimes w) = (-)^{p(v)p(w)}w\otimes v, \quad v,w\in C^{m|n}$$

$$(\mathcal{P})_{ab;cd} = (-)^{p(a)p(b)}\delta_{ad}\delta_{bc}\ . \tag{27}$$

To arrive later to the graded reflection equation it is convenient to write (26) in a matrix form with the usual tensor product

$$(T_1)_{gr} = \Gamma(T\otimes I)\Gamma, \tag{28}$$

where $(\Gamma)_{ab;cd} = (-)^{p(b)p(a)}\delta_{ac}\delta_{bd}$, hence $\Gamma^2 = I$ (which is not the case of the color algebras). The Z_2-graded algebraic structures were analyzed also in [35] and many R-matrices were found [48].

As in the first lecture the simplest integrable models correspond to the rational solutions of the grYBE. The $GL(m|n)$ symmetric R-matrix is (the graded Yang solution)

$$R(u-v) = (u-v) + \eta\mathcal{P}. \tag{29}$$

The corresponding models (L-operators) are the $GL(m|n)$ isotropic graded magnets and the super-matrix nonlinear Schrödinger equation [37, 53]

$$L_{gm}(u) = u + \eta\sum_{i,j} e_{ij}\otimes s_{ji}(-)^{p(j)}, \tag{30}$$

$$L_{NS}(x,u) = uJ + \eta\sum(e_{ab}\psi_{ba}(x) \pm e_{ba}\psi^{\dagger}_{ab}(x)), \tag{31}$$

where s_{ij} are the generators of the super-algebra $gl(m|n)$, $i,j = 1,2,\cdots,m+n; \psi(x), \psi^{\dagger}_{ab}(x)$ are bose or Fermi fields according to $p(a)+p(b) = 0$ or 1 and $1 \le b \le N$, $N+1 \le a \le m+n$, J is the block diagonal matrix $(I_N - I_{(m+n-N)})$.

Among integrable models related to the Z_2-graded case one can mention $osp(1|2)$ - non-linear Schrödinger equation [37], which is described by the rational limit of the $osp(1|2)$ R-matrix and the supersymmetric sine-Gordon model with the trigonometric $sl(2|1)$ R-matrix [3]. Although the structure of the $osp(1|2)$ R-matrix is similar to the one of the $sl(2)$ spin 1 R-matrix the solution of the corresponding $osp(1|2)$-magnet (or NS) is obtained [37] by the **analytic Bethe Ansatz**. Due to the arguments of analyticity, crossing symmetry and the bare vacuum the form of the transfer matrix $t(u)$ eigenvalue is

$$\Lambda(u,\{v_j\}) = (u(u-d))^N \prod_{k=1}^{M} S_1(u-v_k-1/2)S_{-1}(u-v_k-1)-$$
$$-((u-1)(u-d))^N \prod_{k=1}^{M} S_1(u-v_k) - (u(u-d+1))^N \prod_{k=1}^{M} S_{-1}(u-v_k-d)\,, \tag{32}$$

where $S_l(u) = (u+l/2)/(u-l/2)$ and $d = 3/2$. The regularity condition of $\Lambda(u,\{v_j\})$ in u (the Manakov's principle): $Res\Lambda(u,\{v_j\}) = 0$ at $u = v_k$ gives the Bethe equations for the set of the quasimomenta $\{v_j\}$

$$(S_1(v_k))^N = \prod_{j=1}^{M} S_2(v_k-v_j)S_{-1}(v_k-v_j).$$

Hence the M super-particle eigenstates are parametrized by one set of the quasimomenta according to the Lie super-algebra rank one.

Another way to define the form of the reflection equation different from the factorizability requirement is related with the covariance properties of the K-matrix: if K satisfies the RE then the same is true for the transformed $K' = TKT^{-1}$. It follows from (24), (28)

$$R_{12}\,(u,v)\Gamma K_1\,(u)\Gamma R_{21}\,(v,-u)K_2\,(v) = \tag{33}$$

$$= K_2\,(v)R_{12}\,(u,-v)\Gamma K_1\,(u)\Gamma R_{21}\,(-v,-u).$$

Further generalizations of the YBE and the corresponding RE [37] are related to the colored algebras [59] and/or more complicated commutation relations among entries of the T-matrices and the K-matrices. However the structure of these equations can be easily obtained following the standard ZF algebra pattern. In particular, if the multiplicative factor $\omega(a,b)$ of the color ZF algebra is nondegenerate [59], where $a,b \in \mathcal{A}$ and $\mathcal{A}$ is an abelian grading group, then the graded FRT-relation is

$$R_{12}\,(u,v)\Gamma^{-1}T_1\,(u)\Gamma T_2\,(v) = T_2\,(v)\Gamma^{-1}T_1\,(u)\Gamma R_{12}\,(u,v) \tag{34}$$

where $(\Gamma)_{ab;cd} = \omega(a,b)\delta_{ac}\delta_{bd}$ and we identify for simplicity the matrix indices with the grading group elements (which is not always the case as e.g. the Z_2-grading). Now Γ is not necessary unipotent. The corresponding RE follows from the covariance arguments

$$R_{12}\,(u,v)\Gamma^{-1}K_1\,(u)\Gamma R_{21}\,(v,-u)K_2\,(v) = \tag{35}$$

$$= K_2\,(v)R_{12}\,(u,-v)\Gamma^{-1}K_1\,(u)\Gamma R_{21}\,(-v,-u).$$

To extract the commuting functionals of the $\mathcal{A}$-graded monodromy matrix T or the corresponding Sklyanin's matrix $\mathcal{T}$ with the entries as the homogeneous elements of the grading the $\mathcal{A}$-graded trace [59] has to be used

$$t(u) = tr_{\mathcal{A}}T(u) = \sum \omega(a,a)T_{aa}(u)\ .$$

This trace can be considered as a particular example of the quantum trace [51] with Γ as the R-matrix. It is also possible to extend these constructions further by using instead of the multiplicative factor $\omega(a,b)$ and/or Γ an appropriate R-matrix, however the main problem for the moment is to find an interesting example the solution of which requires the mentioned above equations.

5 Integrable models with finite degrees of freedom.

The intensive development of the YBE and the quantum group theory was strongly influenced by the conformal field theory. The latter one as most of the field theoretical integrable models has the two dimensional space-time. However, the rich structure of the YB-algebra and RE-algebra or the quadratic R-matrix algebras with the spectral parameter dependence permits to include into this formalism variety of known integrable models with finite degrees of freedom (e.g. [38-40, 41, 46]) and to find new ones, which are physically interesting systems in the space of the three (and more) dimensions.

Let us consider as an example the Kowalewski-Chaplygin-Goryachev top (KCG top). The Hamiltonian of this model is [46].

$$H = \frac{1}{2}(J_1^2 + J_2^2 + 2J_3^2) + c_1x_1 + c_2x_2 + c_3(x_1^2 - x_2^2) + c_4x_1x_2 + c_5/x_3^2, \tag{36}$$

where $c_i\,,\, i = 1,\cdots,5$ are arbitrary constants and J_i, x_i are the angular momenta and coordinates. This system is integrable provided the constraint:

$$l = \sum_{j=1}^{3} x_jJ_j = 0. \tag{37}$$

One gets the famous Kowalewski's top for $c_3 = c_4 = c_5 = 0$. The corresponding auxiliary linear problem (the L-operator) is related to the simplest R-matrix: the Yang solution (9) for $sl(2)$ with $c = \kappa$. The L-operator has quadratic dependence on the spectral parameter u

$$L(u) = \left(\begin{array}{cc} y_0 u^2 + y_2 u + y_1 & y_4^+ u + y_6^+ \\ y_4^- u + y_6^- & y_3 \end{array} \right) = \left(\begin{array}{cc} a & b \\ c & d \end{array} \right)(u) \tag{38}$$

where $y_4^{\pm} = y_4 \pm y_5$ and the same for $y_6^{\pm}$; $y_0, ..., y_7$ are eight dynamical variables of the model.

The corresponding YB-algebra, generated by four entries $a(u), ..., d(u)$ of the L-operator, has its centre generated by the q-determinant of the L-operator (38) [5, 3]:

$$\begin{array}{rl} det_q L(u) = & a(u + i\kappa/2)d(u - i\kappa/2) - b(u + i\kappa/2)c(u - i\kappa/2) \\ & = Q_1 u^2 + Q_2 u - Q_3 + \kappa^2/4Q_1 \quad , \end{array} \tag{39}$$

where $Q_0 = y_0$, $Q_1 = y_0 y_3 - y_4^2 - y_5^2$, $Q_2 = y_2 y_3 - 2y_4 y_6 - 2y_5 y_7$, $Q_3 = y_6^2 + y_7^2 - \frac{1}{2}\{y_1, y_2\} + \frac{1}{2}\kappa^2 y_0 y_3$, $\{y_i, y_j\} = y_i y_j + y_j y_i$.

The YB-algebra for the entries of $T(u) \equiv L(u)$ results in quadratic commutation relations for the dynamical variables y_k. The nontrivial problem is to realize the latter ones in terms of physically significant variables. One of the realization is given [7] by the momenta p_i and coordinates q_i, $i = 1, 2$, of the two site Toda lattice, so that the L-operator (38) is the product of two elementary ones ($i = 1,\ 2$)

$$L_i(u) = \left(\begin{array}{cc} u - p_i & -e^{q_i} \\ e^{-q_i} & 0 \end{array} \right) \quad . \tag{40}$$

The realization we are looking for defined by the generators J_k, x_k, $k = 1, 2, 3$ of the Lie algebra e(3), provided that $l = \sum_{k=1}^{3} x_k J_k = 0$ [46]

$$\begin{array}{ll} y_0 = 1 & y_4 = ibx_1 \\ y_1 = -(J_1^2 + J_2^2 + \frac{1}{4} + 2\alpha/x_3^2) & y_5 = ibx_2 \\ y_2 = -2J_3 & y_6 = -\frac{1}{2}ib\{x_3, J_1\} \\ y_3 = b^2 x_3^2 & y_7 = -\frac{1}{2}ib\{x_3, J_2\} \quad , \end{array} \tag{41}$$

where α and b are constants related to the YB-algebra central elements. The integrals of motion are generated by the trace of (38)

$$t(u) = trL(u) = a(u) + d(u) = u^2 - 2J_3 u - 2H - \frac{1}{4} \quad , \tag{42}$$

where $H = \frac{1}{2}(J_1^2 + J_2^2 - b^2 x_3^2) + \alpha/x_3^2$ is the Hamiltonian of the Newmann's systems. The solution of this system by the separation of variables (from J_k, x_k to the new ones) in the framework of the QISM is achieved using the **functional Bethe Ansatz** [7, 41]: the introduction of the new variables as the operator roots $\hat{u}_i$, $i = 1, 2$, of the entry $c(u)$ (38) and the conjugated variables $\hat{m}_i$ as the values of $a(u)$ and $d(u)$ at these roots. Then, the eigenvalue equation for the transfer matrix $t(u)$ (two degrees of freedom for the Newmann's case) is reduced to the one-dimensional problems. To embed the KCG top into such approach one has to use the RE and the Sklyanin monodromy matrix.

The c-number solutions to the RE with the $sl(2)$ R-matrix have the form

$$K_a(u) = \alpha_a \sigma_0 + u(\beta_a^- \sigma_+ + \beta_a^+ \sigma_-), \tag{43}$$

where $a = \pm$, $(\beta_-^-, \beta_-^+, \beta_+^-, \beta_+^+) = (1, -\beta_1, \beta_2, -1)$. Using the L-operator (38) one can construct according to the general recipe [29] the monodromy matrix

$$\mathcal{T}(u) = L(u) K_-(u - ix/2) \sigma_2 L^T(-u) \sigma_2 \tag{44}$$

and the correspondent generating function $\tau(u)$ of the integrals of motion

$$\tau(u) = tr K_+(u + ix/2) \mathcal{T}(u) \quad . \tag{45}$$

The latter one gives rise to the KCG top Hamiltonian (36).

Another wide class of models embedded recently into this scheme [49] includes the so called quasi-solvable models; some of them are physically relevant.

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