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Phase Transitions at Finite Temperature and Dimensional Reduction for Fermions and Bosons

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Abstract

In a recent Letter we discussed the fact that large-N expansions and computer simulations indicate that the universality class of the finite temperature chiral symmetry restoration transition in the 3D Gross-Neveu model is mean field theory. This was seen to be a counterexample to the standard 'sigma model' scenario which predicts the 2D Ising model universality class. In this article we present more evidence, both theoretical and numerical, that this result is correct. We develop a physical picture for our results and discuss the width of the scaling region (Ginzburg criterion), $1/N$ corrections, and differences between the dynamics of BCS superconductors and Gross-Neveu models. Lattices as large as 12×72^2 are simulated for both the $N = 12$ and $N = 4$ cases and the numerical evidence for mean field scaling is quite compelling. We point out that the amplitude ratio for the model's susceptibility is a particularly good observable for distinguishing between the dimensional reduction and the mean field scenerios, because this universal quantity differs by almost a factor of 20 in the two cases. The simulations are done close to the critical point in both the symmetric and broken phases, and correlation lengths of order 10 are measured. The critical indices β_{mag} and δ also pick out mean field behavior. We trace the breakdown of the standard scenario (dimensional reduction and universality) to the composite character of the mesons in the model. We point out that our results should be generic for theories with dynamical symmetry breaking, such as Quantum Chromodynamics.

We also simulated the $O(2)$ model on 8×16^3 lattices to establish that our methods give the results of dimensional reduction in purely bosonic cases where its theoretical basis is firm. We also show that Z_2 Nambu Jona-Lasinio models simulated on 8×16^3 lattices give mean field rather than three dimensional Ising model indices.

1 Introduction

Considerable work has been done on the physics of the finite temperature chiral transition in QCD and related models[1][2][3]. Although, there is little disagreement about the existence and order of the transition, no quantitative work, simulations or otherwise, has been done that decisively determines the univerality class of the QCD transition with two flavors. At present, lattice simulations can not distinguish between $O(2)$ and $O(4)$ fits of the universal features of the transition[1]. In fact, they can not even rule out any $O(N)$ fit decisively. Universality arguments [4] have been invoked in order to reduce the problem to a manageable form, and the analysis of lattice data has used these frameworks. Until recently, universality arguments were the most appealing ones available, and due to their beauty and simplicity were by and large accepted as true. In essence, they can be phrased as follows. At finite temperature phase transitions, the correlation length diverges in the transition region and low energy physics is expected to be dominated by soft modes. The contribution of non-zero modes should be suppressed. Consequently, fermions, which satisfy antiperiodic boundary conditions, and do not have zero modes, are expected to decouple from the scalar sector, together with the other non-zero modes. The theory effectively reduces to a lower dimensional scalar field theory. This is an important point since it renders the treatment of such complicated theories as QCD to that of the three dimensional σ model. This argument is based on the universality hypothesis and it is hard to see how it can fail.

In ref.[5] we presented analytic and numerical evidence that this scenario does not hold in the Gross-Neveu model. We argued that dimensional reduction does not necessarily hold when chiral symmetry breaking is accompanied by composite mesons. We analyzed the three dimensional Gross-Neveu model at finite temperature where the global symmetry is Z_2 and universality arguments imply that the finite temperature restoration transition should be in the universality class of the two-dimensional Ising model. Using the large-N expansion and computer simulations, we found no hint of such behavior, but observed mean-field scaling. Several aspects of the analysis presented in [5] are, however, subject

to criticism. We address those criticisms in this paper and present new analyses and simulations. The criticisms are:

1. *The large-N analysis could be misleading.* After all, this happens in a rather dramatic way in the two-dimensional Gross-Neveu model and could be happening above two dimensions as well [6]. Thus, it is conceivable that the leading order expansion gives incorrect exponents and the correct ones are recovered only when N is finite. Our simulations, presented in [5], were performed for $N = 12$ species, and the coincidence of the large-N and lattice results might be caused by the fact that the data are not sufficiently accurate to see the $1/N$ corrections. This is certainly possible and, for that reason, we studied and simulated a low-N theory, namely $N = 4$. We found no change in our conclusions – the scaling remains mean-field with a high degree of confidence. Large-N results are summarized in Sec. 4. $N = 12$ simulations are given in Sec. 6. The results for $N = 4$ are given in Sec. 7.

2. *The scaling region could be too narrow at large-N.* One might argue that the true scaling behavior is not mean-field as we observe, but is undetected in our simulations because the scaling region is inaccessible to the lattices that we work on. In other words, larger lattices and larger correlation lengths are needed to see the two dimensional (2d) Ising scaling predicted by dimensional reduction in the three dimensional (3d) case. There are two aspects of this objection. First, for the Gross-Neveu model at zero temperature, Landau behavior is not marked by mean-field exponents. Because of the presence of the fermions, there are long-range interactions that change the universality class and the scaling is different from local bosonic models with the same symmetry. This statement is uncontroversial since it is well known that the scaling laws of the Gross-Neveu model[14], in any dimension below four, have nothing to do with the Ising model. The discussion of this problem is presented in Sec. 5. Thus, the 'canonical' belief that the scaling dynamics and widths of critical regions here is the same as that of BCS superconductors is not true – the mean-field scaling observed at finite temperature is not Landau behavior that should be followed by non-Gaussian behavior once the true scaling region is reached.

Nevertheless, to clarify these points further, we first wanted to see if there might exist some intrinsic difficulties in observing dimensional reduction in lattice simulations. For that purpose, we simulated a scalar $O(2)$ model which we know undergoes dimensional reduction and we found an unambiguous and clear confirmation for its presence already on lattices of modest sizes, considerably smaller than we used for the Gross-Neveu model. We present these findings in Sec. 3.

3. *Is it possible to imagine a physical picture that would be consistent with the failure of dimensional reduction?* When the relevant length scale diverges, the dimensional reduction scenerio [4] states that the non-zero modes should decouple and the resulting theory can be described in terms of light, pointlike mesons. Thus, a simple, familiar physical picture accompanies the dimensional reduction scenerio. In Sec. 2. we show that our scenerio, where dimensional reduction does not occur, is also accompanied by a consistent physical picture. The loophole to the 'standard' scenerio in the case of fermions is that the mesons are composite and that, because of chiral symmetry, their size scales as the correlation length, e.g. $R_\pi \sim 1/f_\pi$, and diverges as the critical point is approached. Thus, the meson size can not be neglected on physical length scales and the fermionic substructure is always apparent. This type of behavior is nothing new – it has been observed and commented on in the context of quantum mechanics and model field theories [8] many times in the literature. In other words, the effective theory near the chiral phase transition can not be described in terms of locally interacting bosonic variables only. We also point out some physical differences between the chiral restoration and superconducting transitions at finite temperature. For superconductors the size of a Cooper pair increases as the critical temperature is approached. Unlike the Gross-Neveu model, where fermion pair condensation occurs as well, in superconductors, the range of interaction is determined by the size of Cooper pairs. Therefore, increasing the size of a pair promotes the importance of long-range interactions and, as a consequence, the theory's scaling laws are rather well described by mean field theory. This occurs over a wide range of temperatures, even very close to the critical point. This is the reason for having a narrow scaling

region at the superconducting transition – the long range interactions promote mean field behavior and only extremely close to the critical point are the critical fluctuations, which are expected in an interacting three dimensional system, sufficiently strong to be detected experimentally. As the critical point is approached to extraordinary accuracy in BCS superconductors, the increase in the size of Cooper pairs leads to a regime where the size of pairs is too large to support superconductivity – the relevant physics occurs on the scale where pairing is not visible and the system returns abruptly to the normal state. The transition to the normal state is, in fact, weakly first order, although, due to the long range interactions, the scaling appears to be well approximated by mean field theory over a wide range of temperatures. In the Gross-Neveu model[9] the situation is completely different: the range of interaction does not scale like the length scale associated with the condensate – long range interactions exist already at zero temperature and are essential for its non-Gaussian behavior. Without the long range interactions, the scaling in the zero temperature large-N Gross-Neveu model would follow canonical mean field theory. Thus, although the pairs' size diverges as the critical point is approached, it is the screening of the long range interactions caused by finite temperature effects that is responsible for the emergence of mean field behavior at the chiral restoration transition of the Gross-Neveu model.

Our main results concerning critical exponents of the finite temperature, chiral symmetry restoration transition in Gross-Neveu models read: $\beta_{mag} = 1/2, \delta = 3, \gamma = 1$. This should be contrasted with the 2d Ising values $\beta_{mag} = 1/8, \delta = 15, \gamma = 7/4$ that, according to the dimensional reduction scenerio, should describe the finite temperature transition of the 3d σ model. Note that the most dramatic difference is between δ's – they differ by a factor of 5 and it is hard to imagine that the lattice simulations could mistake one for the other. In addition, we measured the universal amplitude ratios[10]. The quantity that we found useful in this context is the susceptibility amplitude ratio C_+/C_- defined by $\chi_\pm^{-1} = C_\pm |t|^\gamma$, where C_+ and C_- refer to the broken and symmetric phases, respectively. The ratio C_+/C_- is universal. In mean field theory $C_+/C_- = 2$, while for the $2d$

Ising model $C_+/C_- = 37.69$. This quantity is especially suitable to test for the absence of dimensional reduction since the amplitude ratios in the two scenerios differ by nearly a factor of 20, and the chance for getting an ambiguous answer here is very small. Our simulations for the amplitude ratio are very close to the mean field value and any attempt to accomodate higher values failed. It will also prove valuable to estimate the correlation length ξ in our simulations. It diverges at the critical point with the exponent ν, $\xi^{-1} = D_\pm |t|^\nu$, where D_+ and D_- refer to the broken and symmetric phases respectively. The ratio D_+/D_- is also universal. In mean field theory $D_+/D_- = 2^{1/2}$, while for the $2d$ Ising model $D_+/D_- = 2$. Since we will have explicit calculations of the correlation length in the broken phase, these universal relations will allow us to calculate the correlation length in the symmetric phase in both scenerios.

2 Physical picture

The main appeal of dimensional reduction is the simplification it introduces by assuming that the all non-zero modes decouple. This is the source of major simplifications since the fermions, being antiperiodic, should completely disappear from the low energy physics. Where can this reasoning fail? First, one should be aware that universality arguments do not hold for systems with long range interactions. This is especially relevant for the Gross-Neveu model where long range interactions play a crucial part. Nevertheless, it is not necessary to invoke renormalization group arguments in order to see that fermions should not decouple at the chiral transition. The argument can be made much simpler using quantum mechanics only. For that purpose, we digress in this section and consider some simple quantum mechanical models that illustrate the role fermi statistics play in situations where the overlap of wave-functions becomes substantial. The aim of this section is to show that decoupling of the fermions, in fact, is *not* consistent with the physics of chiral symmetry restoration. In the vicinity of the phase transition, compositeness of the light mesons has some interesting implications for low-energy physics.

As an example, we consider a quantum mechanical system of relativistic fermions in a

box. This model has been taken from the context of nuclear physics, and adapted for our purposes with minimum modifications[11]. All the arguments present standard material in many-body physics and we merely paraphrase them below. We construct the two-body mesonic states as,

$$|p>\equiv C^\dagger(p)|0>=\int_k f_p(k)b^\dagger(k+p)d^\dagger(k)|0> \tag{1}$$

where $b^\dagger, d^\dagger$ are fermion and antifermion creation operators respectively. The spin indices have been suppressed for simplicity. The wave function, $f_p(k)$, satisfies the normalization condition $\int_k |f_p(k)|^2 = 1$. Using the anticomuting properties of fermions, we obtain the expressions for the canonical comutation relations for the two body operators,

$$[C(p), C(p')] = 0, \qquad [C(p), C^\dagger(p')] = \delta_{p,p'} - \Delta(p,p') \tag{2}$$

In general, $\Delta \neq 0$. It is a measure of the deviation from the purely bosonic behavior of the mesons. For $p = p'$, $\Delta(p,p)$ has a simple physical interpretation.

$$\Delta(p,p) = \int_k |f_p(k)|^2 (n(k+p) + \bar{n}(k)) \tag{3}$$

where $n(p), \bar{n}(p)$ are number operators for fermions and antifermions. The operator $n(k+p) + \bar{n}(k)$ counts the number of fermions plus number of antifermions with momentum difference p and is equal to the number of mesons with momentum p. The weight factor normalizes it to the total number of states in the box. Therefore, Δ is the ratio of the number of mesons to the total number of states in the system. If the number of composite states is small, $\Delta \approx 0$ and the mesons behave approximately as bosons. Their structure can be neglected and there is little error in describing the system as a gas of pointlike bosons. If, on the other hand, the meson density is high, i.e. the number of meson states is comparable to the total number of states, there is a large number of overlapping fermions and the composite structure of the mesons is revealed. This is reflected in the nonvanishing value of Δ. The bosonic description is inadequate in this regime.

As seen by this example, even if the theory possesses no fundamental pointlike fermion excitations (like QCD, for example), the fermionic nature of its constituents gives rise to two regimes: 1) when the density of composite mesons is low, the theory is well described by an equivalent bosonic model; 2) when the mesons are dense, the bosonic description fails; the fermions become essential degrees of freedom irrespective of how heavy they are.

To make a connection with the chiral restoration transition, we recall the effect of finite temperatures on fermions and mesons. First, since they are the lightest particles in the system, composite mesons are easily excited by thermal effects. Fermions, although light in the critical region, do not appear so in the Euclidean theory. Due to antiperiodic boundary conditions, their Matsubara frequencies are odd and they do not possess a zero mode. Thus, on the energy scale of the scalar sector, they appear as heavy and might be expected to decouple at low energies. The zero mode component of the scalar sector would then give the dominant contribution in the scaling region and the effective theory would be a dimensionally reduced scalar model. This is the reasoning behind the standard scenario[4].

The problem that we are addressing in this paper concerns the adequacy of the effective scalar theory. In terms of the meson degrees of freedom the applicability of the effective scalar theory can be phrased in terms of how dense the mesonic gas is near the critical temperature. There are two possibilities. The first one corresponds to the standard scenario where the meson size does not change appreciably as the temprerature increases from zero. In this case in the critical regime where the thermal correlation length is large, the system can be treated as a dilute mesonic gas and an effective scalar theory could apply. In the alternative realization, the meson size increases significantly near the critical temperature. If this is the case, then the effective scalar theory has to break down even in the broken phase.

We briefly examine which of the two alternatives is likely to occur in the case of the chiral transition. Heating the system causes an increase in the thermal motion of the particles. As the temperature increases, the contituent fermions inside a meson will

wiggle at an increasing rate. One expects that the total energy of the composite will follow the increase in the zero-point energy of constituents. In the phase without spontaneous symmetry breaking, this results in a well known linear increase of the meson masses with temperature. How can this be reversed in the broken phase where the symmetry restoration requires that the meson masses gradually decrease to zero as the system is heated? The only way to neutralize an increase in zero point energy due to a temperature increase is to allow the radius of the meson to increase as well. Any mechanism that would prevent this from happening would increase the energy of the system. Furthermore, the size of composites should diverge as the correlation length, since if it didn't, the energy of the system would not scale properly – introducing a composite of finite extent into the critical system would increase the total energy by a finite amount that does not vanish at the critical point.

This being the case, the transition region can be described as a system of highly overlapping composites. In this regime, the violation of the bosonic character of the mesons is maximal. The scale on which quantum fluctuations that cause the binding to occur is the same as the thermal correlation length. Thus, the classical description in terms of an effective scalar model is not adequate in this regime. The effective scalar theory fails to capture the relevant degrees of freedom even in the broken phase, and this might, in principle, occur well before the critical temperature is reached.

These arguments do not depend on whether the fermions are confined or not. The effect of having fermion constituents, confined or not, implies that by populating the system with composite mesons certain energy levels become inaccessible. As a consequence, at low energies the effective mesonic interaction is less operative and the binding energy is reduced. Thus, weaker binding occurs at higher densities. Consequently, the mesons are fluffier as their density increases. Allowing fermions to be thermally excited as well, as happens in the absence of confinement, does not alter this picture qualitatively. On the contrary, their presence combines constructively with Pauli blocking caused by the mesons.

3 Dimensional reduction in a scalar model

–$O(N)$ Scalar Theory

To illustrate the idea of dimensional reduction and its relevance for the phase transitions at finite temperature, we start with the simplest possible example of an N-component scalar theory and consider the large-N limit[12] for simplicity. The potential of the model has the standard form: $V(\phi) = 1/2\mu^2\vec{\phi}^2 + \lambda/4(\vec{\phi}^2)^2$. Complications due to Goldstone bosons can be avoided by by working in the symmetric phase. To leading order in $1/N$, the corrections to the propagator are given by the single tadpole contribution. Since the tadpole is momentum independent, it effects only the susceptibility and not the wave function renormalization. The susceptibility, $\chi = \int_x < \phi(x)\phi(0) >_c$, is the zero-momentum projection of the correlation function and at zero temperature it is given by

$$\chi^{-1} = \mu^2 + \lambda \int_q \frac{1}{q^2 + \chi^{-1}} \tag{4}$$

where $\int_q = \int d^dq/(2\pi)^d$, and we absorb the combinatorial factor into λ. Defining the critical curvature μ_c^2 as the point where the susceptibility diverges ($\mu_c^2 + \lambda \int_q 1/q^2 = 0$), the expression for the inverse susceptibility can be recast into

$$\chi^{-1}\left(1 + \lambda \int_q \frac{1}{q^2(q^2 + \chi^{-1})}\right) = \mu^2 - \mu_c^2 \tag{5}$$

The extraction of the critical index γ reduces to counting powers of the infrared (IR) singularities on the left hand side (LHS) of eq.(5). Above four dimensions, both terms are IR finite and the scaling is mean field ($\gamma = 1$). This is supplemented by logarithmic corrections in four dimensions. Below four dimensions the second term in eq.(5) dominates the scaling region – the integral diverges as $\chi^{(4-d)/2}$. This gives the zero-temperature susceptibility exponent $\gamma = 2/(d-2)$ [12].

The finite temperature susceptibility is obtained from eq.(4) after replacing the frequency integral with the Matsubara sum. For a given value of μ^2 we define the critical

temperature, T_c, by

$$\mu^2 + \lambda T_c \sum_n \int_{\vec{q}} 1/(\omega_{nc}^2 + \vec{q}^2) = 0 \tag{6}$$

where $\omega_{nc} = 2\pi n T_c$. The momentum integrals are now performed over $d-1$ dimensional space. This equation defines the critical surface in the space spanned by μ^2, λ, and T_c which reads

$$\mu^2 - \mu_c^2 + a\lambda T_c^{d-2} = 0 \tag{7}$$

where a is some positive constant. For fixed value of the curvature, λ decreases as the temperature grows. Conversely, for a fixed value of the coupling, the temperature tends to flatten the curvature. This effect is controlled by the strength of the coupling and is reduced at small λ. One would expect that perturbation theory becomes progressively more reliable as the system is heated. Inserting eq.(6) into the expression for the inverse susceptibility results in

$$\chi^{-1}\left(1 + \lambda T_c \sum_n \int_{\vec{q}} \frac{1}{(\omega_{nc}^2 + \vec{q}^2)(\omega_n^2 + \vec{q}^2 + \chi^{-1})}\right) \tag{8}$$

$$= \lambda T_c \sum_n \int_{\vec{q}} \frac{\vec{q}^2(T/T_c - 1) + \omega_n^2(T_c/T - 1)}{(\omega_{nc}^2 + \vec{q}^2)(\omega_n^2 + \vec{q}^2 + \chi^{-1})} \tag{9}$$

Separating the $n = 0$ mode ($\omega_0 = 0$) from the rest of the sum, we get the leading singular behavior

$$\chi^{-1}\left(1 + \lambda T_c \int_{\vec{q}} \frac{1}{\vec{q}^2(\vec{q}^2 + \chi^{-1})} + \sum_{n\neq 0} \ldots\right) = \lambda T_c \int_{\vec{q}} \frac{T/T_c - 1}{\vec{q}^2 + \chi^{-1}} + \sum_{n\neq 0} \ldots \tag{10}$$

The $n = 0$ piece dominates the scaling region. It resembles the zero-temperature expression, eq.(5), except that now, the integrals are performed in $d-1$ dimensions, instead of d. The power counting is the same as before and it yields the thermal exponent

$\gamma_T = 2/(d-3)$. This is the same as the zero-temperature γ in $d-1$ dimensions [12]. Other critical exponents show the same type of behavior as γ.

Neglecting the non-zero modes, which is justifiable in the scaling region, the expression in eq.(10) resembles that of the $d-1$ theory with effective coupling $\lambda_{eff} = \lambda T_c$. This is in the spirit of dimensional regularization since the dimensions of quartic couplings in scalar theories in d and $d-1$ dimensions differ by one unit. As long as $d > 3$, the system is perturbative around the critical line – the effective coupling λT is small compared to $\lambda T \sim T_c^{3-d}$ even at high temperature.

This large-N exercise illustrates the mechanism of dimensional reduction and was chosen here for its simplicity. A technically more complicated case of finite-N models can be carried out using the ϵ-expansion, for example. The arguments presented above are based on counting only the light degrees of freedom and are thus generic. It would be difficult to imagine how they can fail in the context of scalar models. We undertake a numerical study of the finite temperature transition in the four-dimensional $O(2)$ model, believing that dimensional reduction occurs and consider under what conditions and degree of confidence we can establish it numerically. This exercise is performed for several reasons. Before going on with the study of fermionic models, we want to determine some criteria for establishing dimensional reduction numerically by simulating the model when the results are known. For that purpose we choose to work with the simplest scalar theory. In particular, we want to know the lattice sizes required, the statistics needed etc..

–Computer simulations of the $O(2)$ σ-model

We simulated a 8×16^3 lattice using the O(2) model code we used in a recent study of scalar electrodynamics [13]. The code is a relatively conservative one which combines the Metropolis and over-relaxation methods. It is discussed in greater detail in ref. 13. It was adequate for our exploratory studies here, but will be replaced by a cluster algorithm in future, more accurate studies. As we shall see, the evidence for dimensional reduction was very clear, so it served our purposes. The Euclidean action of the model reads,

$$S = -\beta \sum_{x,\mu} (\phi_x^* \phi_{x+\mu} + c.c.) \tag{11}$$

where β is the coupling constant and $\phi_x = exp(i\alpha(x))$ is a phase factor at each site of the lattice. At each coupling β, we typically ran 10^6 sweeps of the algorithm and measured the mean magnetization σ and its susceptibility χ. These measurements were first done without any external magnetic field in the system because we want to obtain the critical coupling β_c and the critical index β_{mag} for the magnetization transition. Since the mean magnetization would average to zero in such a computer experiment without a symmetry-breaking field, we actually measured the magnitude of the magnetization and used this as our 'effective' order parameter. This is a standard procedure in such cases, and is reliable in the broken symmetry phase where the mean magnetization is substantial. It is not correct too close to the critical point and in the symmetric phase. If one were simulating the Ising model, for example, then vacuum tunneling on the finite lattice would restore its global Z_2 symmetry. These tunneling effects diminish on larger lattices and they determine the range of couplings that can be reliably simulated on a fixed lattice size. Our data for the O(2) model on the 8×16^3 lattice is accumulated in Table 1. The error bars recorded there account for the statistical uncertainties in each 10^6 sweep run and were estimated by the usual binning procedures which account for the correlations in the raw data sets. We found that we could not reliably measure the order parameter of the model for couplings below .1525 due to tunneling effects. The magnetization data was fit to the form $\sigma = A(\beta - \beta_c)^{\beta_{mag}}$ for couplings between .17 and .155. The data and the fit are shown in Fig.1. The simple powerlaw fit is very good and results in the predictions $A = 1.97(7)$, $\beta_c = .1519(1)$ and $\beta_{mag} = .34(2)$. The confidence level for the fit is .50. The best measurements of the critical indices of the three dimensional O(2) model yield $\beta_{mag} = .346(2)$. So, our relatively modest simulation is in good agreement with the dimensional reduction hypothesis and certainly distinguishes mean field exponents where $\beta_{mag} = .50$.

Next we consider the susceptibility which should diverge in the critical region with a

powerlaw singularity controlled by the index γ. On a finite lattice it is conventional to parametrize such data with the form $1/\chi = A(\beta - \beta_c)^\gamma + C$. The data in Table 1 fit this simple powerlaw hypothesis over the coupling range .17 to .155 (as shown in Fig.2) and a fit with confidence level .41 predicted the parameters $A = 71.5(3.7)$, $C = .029(2)$ and $\gamma = 1.22(1)$ for the choice $\beta_c = .1519$ found in the magnetization data. We are particularly interested in the universal index γ here and we found that it was not sensitive to the precise value of β_c chosen in the fitting form. Our predicted value for γ is close to the best estimate of γ for the three dimensional O(2) model, $\gamma = 1.32(2)$, and is rather far from the mean field presiction of $\gamma = 1.00$. The three dimensional O(2) model result of $\gamma = 1.32(2)$ is based on a thorough finite size scaling analysis and its uncertainty reflects that, unlike our relatively modest calculation which extracts an estimate for γ from one lattice size and just records the uncertainty in a single fit. A thorough finite size analysis is planned, but our purpose here is just to indicate that a single simulation on a 8×16^3 lattice gives good evidence for the dimensional reduction scenerio. In particular, we find no need for enormous lattices. In addition, the critical region appears to be easily accessible and naive powerlaw fits work well. Although we did not measure the system's correlation length (inverse of the scalar mass) with enough precision to extract the critical index ν, we did confirm that the correlation length is just a few lattice spacings throughout the region of parameter space where we extracted the critical indices of the finite temperature transition. It was *not* necessary to work so close to the transition itself that the correlation length would be large compared to the temporal extent of the lattice. In other words, we confirmed the well-known result in statistical mechanics and computer simulation physics, that once the correlation length is comparable to the temporal extent, then the finite temperature critical indices control the simulation data. The importance of this comment lies in the fact that formal derivations of the dimensional reduction scenerio require that the zero mode dominate the Matsubara sum and this occurs when the scalar mass is small compared to the temperature. Numerical simulations suggest that the result of the dimensional reduction scenerio – that the critical indices of the

finite temperature transition are those of the three dimensional O(2) model – is more robust than its formal derivation. Apparently the higher Matsubara frequencies that are simply ignored in the formal derivations do not effect the critical singularities but can be absorbed into noncritical, nonuniversal aspects of the transition. We suspect that this fact applies to other models besides those studied here.

Next we turned to our most discerning measurement - the index δ. Recall that at the critical point β_c, the mean magnetization should depend on an external symmetry breaking field m as $\sigma = Am^{1/\delta}$. Here we are using the notation 'm' where magnetic field 'h' would be more conventional. We do this because we will be turning to chiral symmetry below and there the symmetry breaking field is m, the bare mass of the fermion. We simulated the O(2) model on the 8×16^3 lattice at our best estimate of $\beta_c = .1519$. We considered symmetry breaking fields m ranging from .002 to .05 as shown in Table 2. Again, 10^6 sweeps of the algorithm were done at each parameter setting. On a modest 8×16^3 lattice, we were not able to sensibly simulate smaller m values due to vacuum tunneling. Fits to the data over the m range of .002 - .010 were very good, producing confidence levels of .92 and the parameters $A = .773(2)$ and $\delta = 4.7(2)$. The fit is shown in Fig.3 and 4 as σ vs. m and $ln(\sigma)$ vs. $ln(m)$. The value of δ for the three dimensional O(2) model is $\delta = 4.808(7)$, so we have further, quite decisive, evidence for the dimensional reduction scenerio. The mean field value of $\delta = 3.00$ is easily rejected.

In summary, conventional simulations of the O(2) model on a 8×16^3 lattice are in good agreement with the dimensional reduction scenerio expected of a purely bosonic theory. Powerlaw fits work well and the critical region, as parametrized in conventional fashion, is accessible even on a lattice of modest size and asymmetry. The critical indices predicted by dimensional reduction are found over a region of couplings where the correlation length is just several lattice spacings. Huge lattices and huge correlation lengths exceeding the temporal extent of the lattice are not necessary when measuring the critical indices of interest to the accuracy discussed in this article.

4 Large-N Analysis of the Gross-Neveu Model at Finite Temperature

Large-N studies of the four-fermi models proved successful in understanding their fixed point structure and nonperturbative renormalization. At zero temperature the critical coupling that separates the two phases is identified as an ultra-violet (UV) fixed point around which an interacting continuum limit can be constructed. The emerging critical exponents are non-Gaussian and obey hyperscaling [14]. The validity of the large-N expansion has been shown to hold using explicit expansions up to $1/N^2$ [15]. These studies find strong support in lattice simulations of the model with both $N = 2$ [16] and $N = 12$ [7]. Recently, a rigorous proof in 3d has been constructed. Thermodynamics of the Gross-Neveu model has been studied extensively and the results of the leading order calculations both in three and four dimensions produced mean field exponents [17]. These results are in conflict with dimensional reduction [4] since the model possesses Z_2 symmetry and its thermodynamics would be expected to lie in the universality class of a dimensionally reduced Ising model. However, this result should not be that surprising since at zero temperature the symmetry is Ising as well, but the critical exponents of Gross-Neveu model are far from the Ising values. This discrepancy can be attributed to the presence of massless fermions in the model which induce long range interactions that cause violations of universality. One could argue, on the other hand, that at finite temperature long range interactions are absent due to Debye screening and fermions receive, even in the symmetric phase, thermal (chirally invariant) masses equal to their lowest Matsubara frequency $\omega_0 = \pi T$.

This apparent departure of the large-N result from the expected scenario of dimensional reduction remained by and large unnoticed by the theory community chiefly because of the belief that large-N expansions are misleading at leading order and that corrections were needed to get the correct physics. This is well known to be the case in two dimensions where the failure of the large-N limit is most dramatic – not only does it predict erroneously a finite, instead of zero, restoration temperature, but it also fails to capture

the dynamics of kinks and their condensation. In ref.[7] we carried out numerical simulations of the 3d Gross-Neveu model with the aim of verifying or refuting the predictions of the large-N expansion. We pointed out that, in fact, above two dimensions, the large-N expansion can be trusted and that the existing results seem to provide a good guide to the thermodynamics of the model. Results of the lattice simulations that $\delta = 3$ and $\beta_{mag} = 1/2$ seem robust because dimensional reduction predicts 15 and 1/8 for the two exponents, respectively.

This controversial result comes very naturally in the large-N expansion and is attributed to the compositeness of the scalar mesons. However, some of its aspects like the precise values of the exponents, their sensitivity to the change in dimensionality, the width of the critical region and, therefore, the credibility of the data can be subjected to greater scrutiny. In this section, we extend the analysis of the thermodynamics of the three dimensional Gross-Neveu model outlined in ref.[7] and extend it to the four dimensional model as well. One of the reasons for doing the four dimensional model is to see what ingredients are relevant for the failure of dimensional reduction. In particular, the difference between three and four dimensional models is that the two have different relevant operators. For example, at the fixed point where the dimension of the σ-field is 1 independent of d, σ^4 in four dimensions is marginal, while in $3d$ it is irrelevant. In four dimensions, therefore, there is a quartic term and dimensional reduction can not be ruled out even if one accepts that it does not occur in the three dimensional model. Also, it is of interest to compare the scaling regions and their widths in the $O(2)$ and Gross-Neveu models and confront the fermion model with data of comparable (or better) quality than that taken in our numerical study of the scalar theory where dimensional reduction was established.

As we announced in the introduction, we expect the departure from dimensional reduction in the fermionic models on physical grounds. Our arguments are based on the expectation that the effective description provided by the bosonic theory fails in the vicinity of the restoration transition since the size of the composite mesons scales as the

thermal correlation length and at low-energies scalars are not the only relevant degrees of freedom. This scaling of the meson size, we argued in the introduction, is implied by the symmetry restoration nature of the transition. To put things into perspective, we recall that according to standard arguments, fermions are expected to decouple near the phase transition and the effective theory is described in terms of mesonic degrees of freedom. The reason for fermionic decoupling is that their Matsubara frequencies are odd integer multiples of the temperature, due to their antiperiodic boundary conditions. Their zero modes are missing, and from the point of view of the dimensionally reduced scalar sector, fermions appear as particles with masses of the order of T; they are much heavier than the scalars and decouple. In what follows we discuss in some detail why this does *not* happen.

We proceed along the same lines as in the scalar theory. We analyze the problem of chiral symmetry restoration in a Gross-Neveu model given by the lagrangian $L = \bar{\psi}(i\partial + m + g\sigma)\psi - \frac{1}{2}\sigma^2$. When fermions are integrated out of the model, the Ising symmetry, $\sigma \to -\sigma$, of the effective action becomes manifest. First, we start with the zero-temperature gap equation and corresponding critical exponents. The model can be treated in the large-N limit. To leading order, the fermion self-energy, Σ, comes from the σ-tadpole: $\Sigma = m - g^2 < \bar{\psi}\psi >$. To obtain the scaling properties of the theory, we define the critical coupling as $1 = 4g_c^2 \int_q 1/q^2$. Combining this definition with the gap equation leads to

$$\frac{m}{\Sigma} + (g^2/g_c^2 - 1) = 4g^2 \int_q \frac{\Sigma^2}{q^2(q^2 + \Sigma^2)} \tag{12}$$

Like the scalar example, this form is especially well suited for extracting critical indices since the problem reduces again to the counting of the infra-red divergences on the right hand side [18][14]. The critical indices are defined by $< \bar{\psi}\psi > |_{m=0} \sim t^{\beta_{mag}}, < \bar{\psi}\psi > |_{t=0} \sim m^{1/\delta}, \Sigma|_{m=0} \sim t^{\nu}$, etc.. Here, $t = g^2/g_c^2 - 1$ is the deviation from the critical coupling. Since $\Sigma \sim < \bar{\psi}\psi >$, $\beta_{mag} = \nu$ to leading order. Above four dimensions the integral in eq.(12) is finite in the limit of vanishing Σ and the scaling is mean-field.

Below four dimensions, the $\Sigma \to 0$ limit is singular – the integral scales as Σ^{d-2}. Thus, in the chiral limit $t \sim \Sigma^{d-2}$, and at the critical point, $t = 0$, away from the chiral limit, $m \sim \Sigma^{d-1}$. The resulting exponents are non-gaussian: $\beta_{mag} = 1/(d-2)$ and $\delta = d-1$. The remaining exponents are obtained easily: $\eta = 4-d, \gamma = 1$ [14] and one can check that they obey hyperscaling.

We now consider the Gross-Neveu model at finite-temperature. We choose to stay between two and four dimensions to emphasize how zero-temperature powerlaw scaling changes at finite temperature. The gap equation is now modified to

$$\Sigma = m + 4Tg^2 \sum_n \int_{\vec{q}} \frac{\Sigma}{\omega_n^2 + \vec{q}^2 + \Sigma^2} \tag{13}$$

where $\omega_n = (2n+1)\pi T$. For $g > g_c$ the critical temperature is determined by:

$$1 = 4T_c g^2 \sum_n \int_{\vec{q}} 1/(\omega_{nc}^2 + \vec{q}^2) \tag{14}$$

where $\omega_{nc} = (2n+1)\pi T_c$. Combining the definition of T_c with the finite-temperature gap equation, we can bring it to a form similar to eq.(12)

$$\frac{m}{\Sigma} = (1 - T/T_c) + 4Tg^2 \sum_n \int_{\vec{q}} \frac{\Sigma^2 + \omega_{nc}(\omega_n + \omega_{nc})(T/T_c - 1)}{(\omega_{nc}^2 + \vec{q}^2)(\omega_n^2 + \vec{q}^2 + \Sigma^2)} \tag{15}$$

The extraction of the critical exponents proceeds along the same lines as in the zero-temperature case. One difference relative to eq.(10) becomes apparent immediately: the zero modes are absent here and the integrand in eq.(15) is regular in the $\Sigma \to 0$ limit even below four dimensions. Consequently, the IR divergences are absent from all the integrals and the scaling properties are those of mean-field theory: $\beta_{mag} = \nu = 1/2, \delta = 3$, etc. This is true for any d, below or above four. It appears that in this case, contrary to the scalar example, the effect of making the temporal direction finite $(1/T)$ is to regulate the IR behavior and suppress fluctuations. This is manifest in other thermodynamic quantities as well. For example, to leading order, the scalar susceptibility, $\chi = \partial < \bar{\psi}\psi > /\partial m$, is given by

$$\chi^{-1} = 8g^2 T \sum_n \int_{\vec{q}} \frac{\Sigma^2}{(\omega_n^2 + \vec{q}^2 + \Sigma^2)^2} \tag{16}$$

Once again, because of the absence of the zero mode ($\omega_0 = \pi T$), the integral in eq.(16) is analytic in Σ, and the mean field relation $\chi^{-1} \sim \Sigma^2$ follows. This is equivalent to $\gamma = 2\nu = 1$. The explicit calculation of the momentum dependence of the σ propagator yields $\eta = 0$.

The phase diagram is given by eq.(14) which defines a critical line in the (g, T) plane. For every coupling there exists a critical temperature beyond which the symmetry is restored. Conversely, for a fixed temperature there is a critical coupling, defined by the above expression, corresponding to symmetry restoration. At zero temperature, the symmetry is restored at $g = g_c$. Thus, $(g = g_c, T = 0)$ is the ultra-violet (UV) fixed point. As the coupling moves away from g_c, a higher restoration temperature results. At infinite coupling the end-point, $(g = \infty, T = T_c)$, is the IR fixed point. The critical line connects the UV and IR fixed points dividing the (g, T) plane into two parts. The equation for the critical line can be brought into a compact form by combining the expression for T_c with the definition of the zero-temperature critical coupling. This results in: $(g^2/g_c^2 - 1) \sim T_c^{d-2}(g)$, i.e. $T_c(g) \sim \Sigma(T = 0)$. In this way, for any value of the coupling, the critical temperature remains the same in physical units. This tradeoff of the temperature for the coupling constant can be seen explicitly in the four-fermi model. In particular, since the lattice simulations are performed at a fixed number of temporal links, the critical line is approached by varying g. The critical exponents thus obtained will reflect the singularities as a function of the coupling, not the temperature. However, the tradeoff between the two is always linear so that $g - g_c \sim T - T_c$. To show this, we start with the gap equation on the critical line. We denote the points on the critical line by (g^*, T^*), to distinguish them from the endpoints g_c and T_c. To show that the above statement holds, we compare the gap equation at finite temperature with the expression for the critical line (in the chiral limit).

$$1 = 4T^* g^{*2} \sum_n \int_{\vec{q}} \frac{1}{\omega_n^{*2} + \vec{q}^2}, \qquad 1 = 4T g^2 \sum_n \int_{\vec{q}} \frac{1}{\omega_n^2 + \vec{q}^2 + \Sigma^2} \tag{17}$$

Since we already have the expression for the scaling in terms of temperature and at fixed coupling, we now fix the temperature to some value $T = T^*$ on the critical line and approach the critical region horizontally. Simple algebra leads to

$$\frac{g^2 - g^{*2}}{g^{*2}} = 4T^* g^2 \sum_n \int_{\vec{q}} \frac{\Sigma^2}{(\omega_n^{*2} + \vec{q}^2)(\omega_n^{*2} + \vec{q}^2 + \Sigma^2)} \tag{18}$$

This results in the scaling $\Sigma^2 \sim (g^2 - g^{*2})$, which, when compared with the scaling at fixed g, gives $|T - T_c| \sim |g^2 - g^{*2}|$. Therefore, the exponents are independent of the direction with which we approach the critical line.

5 Landau Theory and Ginzburg Criterion in Theories with Long Range Interactions

In this section we consider generalizations of the Landau theory and Ginzburg criterion for the case of long-range forces. The particular motivation to analyze this problem in a separate section is the problem of universality that, in the context of chiral transitions, requires some attention. The Gross-Neveu model with interaction term $L = g^2 \sum_i (\bar{\psi}_i \psi_i)^2$ has discrete chiral symmetry irrespective of the number of fermion species which is manifest after introducing an auxiliary field, ϕ and integrating out the fermions. In order to conform with the standard notation, in this section we use ϕ (instead of σ) to denote the scalar field and use σ to denote the range parameter. The effective action is

$$S_{eff} = \frac{1}{2} \int_x \phi^2 - N \mathrm{tr} \ln(i\partial + m + g\phi) \tag{19}$$

Clearly this action has a discrete, Ising-like, symmetry $\phi \to -\phi$. The unexpected feature of this model which is at odds with the expectations based on universality is that the

critical exponents are not Ising. The origin of the failure of the standard universality arguments is usually attributed to the appearence of massless fermions that accompany the chiral transition. They are a source of long range forces which are capable of changing the universality class. In fact, it was shown in [19] that the Gross-Neveu model describes the same physics as a Landau theory with long range forces. However, long range forces alone can not account for the difference between the Gross-Neveu and Ising critical exponents. The relationship between fermionic and scalar σ-models is more intricate and goes beyond naive universality arguments. It turns out that the universality class of the Gross-Neveu model is that of $O(N=-2)$ magnets with with long range forces [19].

Since the effective hamiltonian is not local, we allow for the appearance of a nonintger power in the Landau free energy. To leading order in $1/N$, S_{eff} is given by the saddle point approximation. Thus, keeping only the relevant terms, we obtain from eq.(19)

$$H = \phi(\nabla^2)^{\sigma/2}\phi + \frac{1}{2}t\phi^2 + \lambda\phi^p \tag{20}$$

where we leave p undetermined for the time being. The first and third term in eq.(20) are produced by radiative corrections. The extraction of the critical exponents in this case is straightforward. The result is

$$\alpha = \frac{p-4}{p-2}, \quad \beta_{mag} = \frac{1}{p-2}, \quad \gamma = 1, \quad \delta = p-1, \quad \eta = 2-\sigma, \quad \nu = \frac{1}{\sigma} \tag{21}$$

These are Landau exponents obtained by the mean-field treatment of the Ginzburg-Landau hamiltonian eq.(20). There are two independent external parameters, σ and p, and so hyperscaling is not respected in general. The value of exponent ν follows from the expression for the propagator: $G(k) = 1/(k^\sigma + t)$. Imposing hyperscaling relates p and σ by

$$p = \frac{2d}{d-\sigma} \tag{22}$$

For this value of p, the exponents of eq.(20) read

$$\alpha = 2 - d/\sigma, \quad \beta_{mag} = \frac{d-\sigma}{2\sigma}, \quad \gamma = 1, \quad \delta = \frac{d+\sigma}{d-\sigma}, \quad \eta = 2-\sigma, \quad \nu = \frac{1}{\sigma} \tag{23}$$

For comparison, we rewrite the Gross-Neveu exponents

$$\alpha = \frac{d-4}{d-2}, \quad \beta_{mag} = \frac{1}{d-2}, \quad \gamma = 1, \quad \delta = d-1, \quad \eta = 4-d, \quad \nu = \frac{1}{d-2} \tag{24}$$

Both sets of exponents are obtained from the stationary point solution which neglects the fluctuations in the ϕ field and, thus, represent generalized mean field scaling. It is interesting to compare the two sets of exponents – they coincide for $\sigma = d-2$. Unlike scalar models with long range forces, where σ is an external parameter, in the Gross-Neveu model σ is generated dynamically by the fermions. The restriction $2 < d < 4$ under which the above exponents were obtained, $\sigma = d-2$ translates into $0 < \sigma < 2$. This is exactly the range of values for the σ parameter for which the long range forces are capable of changing the universality class. This is easy to understand by looking at the gradient term in eq.(20). Clearly, for $\sigma < 2$ the dominant contribution at low energy comes from the $(-\nabla^2)^{\sigma/2}$ term; the kinetic term, $(-\nabla^2)$, can be neglected in the IR. The local limit (short range interaction) is recovered for $\sigma = 2$. Note that $\sigma < 2$ introduces anomalous dimensions into the effective theory since the scaling dimension of the scalar field is determined by its gradient term. From eq.(20), the dimension of the ϕ-field is $d_\phi = (d-\sigma)/2$. The absence of $(-\nabla^2)$ in the effective hamiltonian, eq.(20), can be phrased also in terms of the compositeness condition, $Z = 0$. In the context of the Gross-Neveu model this is equivalent to the statement of the existence of the long range forces.

It is a straightforward exercise to verify that the equation of state and other universal quantities like amplitude ratios in the Gross-Neveu model and in Landau theory coincide after the identification $\sigma = d-2$ is made. Massless fermions introduce nonlocal effects into the game and the fermionic model maps onto a σ-model with long-range forces. Thus, the universality class of the Gross-Neveu model is not the standard, short range Ising model,

but a Landau theory with long range forces and a specific value of the range parameter $\sigma = d-2$. Thus, the $N = \infty$ limit of the Gross-Neveu model corresponds to a generalized Landau theory and the exponents of eq.(24) replace the standard mean-field ones.

The Ginzburg criterion that determines the importance of the fluctuations, and therefore the validity of the Landau theory, can be derived for the case of long-range forces by imposing the following requirement on the second cumulant

$$\mathcal{E}_{LG} = \frac{\int_x < \phi(x)\phi(0) >_c}{\xi^d < \phi >^2} \ll 1 \tag{25}$$

The meaning of this requirement is simple: since the mean field replaces the value of the field variable with its expectation value, the connected Green's functions vanish and Landau theory is exact. The Ginzburg criterion, eq.(25), is a requirement that the actual result is not far from the gaussian limit. It is expressed as a dimensionless number that measures a relative error made by the mean-field approximation assuming the physical volume to be determined by the correlation length. It relates the bare parameters of the theory (t and λ in this case). From the Landau-Ginzburg hamiltonian, the expressions for the order parameter and susceptiblity reads

$$< \phi > \sim \left(\frac{|t|}{\lambda}\right)^{1/(p-2)}, \qquad \chi^{-1} \sim |t|, \qquad \xi \sim |t|^{-1/\sigma} \tag{26}$$

The Ginzburg criterion, eq.(25), then becomes

$$\mathcal{E}_{LG} = \frac{|t|^{-1}}{|t|^{-d/\sigma}\left(\frac{|t|}{\lambda}\right)^{2/(p-2)}} = \lambda^{2/(p-2)}|t|^{(d-d_c)/\sigma} \ll 1 \tag{27}$$

where the upper critical dimensionality is defined as $d_c = p\sigma/(p-2)$. In the Gross-Neveu model $d = d_c$ in the leading order calculations. The validity of the Landau theory is then conditioned by the requirement that

$$\lambda^{2/(p-2)} \ll |t|^{(d_c-d)/\sigma} \tag{28}$$

Clearly, for $d < d_c$ the right hand side of eq.(28) shrinks to zero as the critical point is approached and mean field theory fails to describe the critical region. Above d_c, the right hand side diverges leaving the coupling λ unconstrained; the mean field description in this case is adequate. The standard Ginzburg criterion for short range forces is recovered by substituting $\sigma = 2$ and $d_c = 4$ in eq.(28).

Since the exponents satisfy hyperscaling, it follows that the highest power in the expansion of the determinant is determined by the dimensionality of space i.e. $p = d$. Consequently, the upper critical dimensionality is $d_c = p\sigma/(p-2) = d$. Thus, in the large-N limit the mean-field treatment is correct since any d behaves as the critical dimensionality and the Ginzburg criterion is always respected in the sense of eq.(28). Higher order $1/N$ corrections have two effects. They shift the critical coupling and modify the exponents. While the first feature is expected (critical couplings are nonuniversal), the second one might appear as a surprise since different N apparently produce different universality classes which are not related to the symmetry group. This is another feature of the effect of the long range forces.

Unlike local scalar theories, where non-gaussian scaling appears only in a narrow region around the critical point, while mean-field behavior prevails in a wider region away from it, in the Gross-Neveu model non-gaussian behavior holds everywhere, close and away from the critical point, and is exact in the $N \to \infty$ limit. This result of the $1/N$ expansion has been corroborated by the lattice simulations of the model with 12 species [7]. Beyond the $N = \infty$ limit there is a small region in the vicinity of the transition point where the $1/N$ deviations from the Gross-Neveu exponents are observed. The width of that region shrinks as some power of $1/N$ for large-N.

6 Simulations of the N=12 Model

We have done simulations of the Gross-Neveu model in three and four dimensions to test these ideas. Some of the three dimensional simulations have already been reported, but here we will present the data and fits, discuss the results and add new calculations. We

begin with the four dimensional simulations which are new. We chose to simulate the Z_2 four Fermi model because it has been analyzed particularly thoroughly in the $1/N$ expansion and because it is relatively easy to simulate. We simulate the model exactly with the Hybrid Monte Carlo method for N=12. Using a large value for N suppresses fluctuations and allows an accurate study within reasonable computer resources. In addition, since the chiral symmetry is discrete the model can be simulated directly in the chiral limit $m = 0.0$, so particularly accurate determinations of the model's critical indices are possible. The lattice action and further detail concerning the algorithm can be found in ref.7.

–Four Dimensions

We simulated a 8×16^3 lattice as for the O(2) bosonic model, and measured the vacuum expectation value of the auxiliary σ field and its susceptibility χ. The raw data is shown in Table 3. The time step in the algorithm was $dt = 0.10$, the total molecular dynamics 'time' expended at each coupling β ranged from 5,000 to 20,000. These numbers should be compared to typical finite temperature lattice QCD simulations with light quarks where typically the statistics is an order of magnitude less thorough. The error bars in the Table reflect, as usual, the statistical uncertainties in the data accounting for correlations in the usual way. As we discussed for the O(2) model above, we cannot reliably simulate the model too close to the critical point because of tunneling between Z_2 vacua on a finite system. So, although we collected data at $\beta = .5875, .59$, and $.595$, we observed tunneling in these data sets and could not use them for quantitative purposes.

–Order Parameter

We shall estimate the critical indices β_{mag}, γ and δ in exactly the same fashion and using the same notation that we did for the O(2) model on the 8×16^3 lattice. In particular, we fit the σ data over the range of couplings .5700 - .5825 and found an excellent powerlaw fit (confidence level .995) which predicted the parameters $A = 1.6(2)$, $\beta_c = .5910(1)$ and, most importantly, the critical index $\beta_{mag} = .52(6)$. The fit and the data are shown in Fig.5. Our measured value of β_{mag} is nicely consistent with mean field theory ($\beta_{mag} = .50$

), and it is far from the three dimensional Ising value ($\beta_{mag} = .325(2)$) predicted by the dimensional reduction scenerio. We feel that it is interesting and significant that the powerlaw hypothesis fits the data so well. This simplicity is expected from the analytic work we did above in the sense that if the temperature regulates the theory's infra-red fluctuations, then the theory's scaling laws should be particularly accessible. If instead, the 8×16^3 lattice were too small and insufficiently asymmetric to expose the true infra-red behavior of the theory at finite temperature, then simple powerlaw scaling would be quite a surprise. When we simulate the three dimensional Gross-Neveu model at finite temperature we will study several lattice sizes and be able to back up these impressions more quantitatively.

–Susceptibility

Next we fit our susceptibility data with a simple powerlaw as was succesful for the O(2) model and we found an excellent fit (confidence level of .976) ,shown in Fig.6, but the parameters were determined rather crudely ($A = 4.8(4.1)$, $C = -.1(1)$, and $\gamma = .7(3)$ for $\beta_c = .5910$). This value of the critical index γ is certainly consistent with the mean field value of $\gamma = 1.00$ and disagrees with the three dimensional Ising value $\gamma = 1.241(2)$, but the determination is crude and hardly persuasive. As usual, it proves harder to measure susceptibilities than order parameters in simulations and the susceptibilities are subject to larger finite size effects.

–δ Exponent

Now we turn to our determination of the index δ. We measured the order parameter σ at the fermion masses .002, .004, .006, .008, and .010 at the critical coupling $\beta_c = .5910$. The data is collected in Table 4. Powerlaw fits to the m-dependence, using the same notation as for the O(2) model of Sec.3, produced a very good fit (confidence level of .80), and predicted the parameters $A = .969(9)$, and the critical index $\delta = 3.23(3)$. The value of δ obtained here is close to the mean field value of 3.00, and we interpret the discrepancy between the two numbers as a finite size effect. It is well-known that estimates of δ coming from finite lattice simulations produce numbers which are systematically higher

than the actual value and the correct value is obtained only in the thermodynamic limit. Nonetheless, the important point is that the δ exponent obtained here is far from that of the O(2) model simulated also on a 8×16^3 lattice (recall from the previous section that $\delta = 4.7(2)$ in that case). The data and the fits for the Z_2 Gross-Neveu model are shown in Fig.7 and 8. In Fig.9 we combine Fig.4 and 8, showing $ln(\sigma)$ vs. $ln(m)$ for the two cases. In both cases the powerlaw ansatz holds well but the predicted values of δ are very different - the O(2) model satisfies dimensional reduction while the four Fermi model does not. Naturally it would be good to repeat these sorts of simulations on larger lattices and do a thorough finite size scaling analysis of both models.

We have argued elsewhere that a particularly visual way of looking for approximate mean field scaling is to plot the theory's 'transverse susceptibility', m/σ, against the square of the order parameter at criticality. In mean field theory such a plot would be a straight line passing through the origin. In Fig.10 we show such a plot for the data in Table 3. The plot has slight curvature because our measured critical indices differ slightly from mean field theory. To the accuracy of our data and its fit, the dashed curve passes through the origin.

–Three Dimensions

The conceptual issues discussed in this paper can be challenged more persuasively by considering the three dimensional Z_2 Gross-Neveu model at finite temperature and examining whether its critical behavior is that of the two dimensional Ising model, as predicted by the dimensional reduction scenerio, or mean field theory, as predicted by the large-N solution of the model. In three dimensions we can simulate larger lattices with greater statistics so the model's critical indices can be determined with greater accuracy and confidence. Finite size scaling studies are more feasible and the width of the critical region at nonzero temperature can be examined numerically. And lastly, the critical indices and universal amplitude ratios of the two dimensional Ising model are *far* from mean field theory, so the two scenerios can be distinguished numerically with greater confidence than in four dimensions.

Several studies have been made of the Z_2 Gross-Neveu model in three dimensions and the predictions of the $1/N$ expansion have been reproduced by the same simulation methods as used here. The zero temperature, large-N critical indices of the three dimensional Gross-Neveu model are recorded in Table 5. As emphasized in the text and is well-known to workers in the field, the fixed point of the three dimensional Gross-Neveu model is *different* from that of the Z_2 ϕ^4 model. The physical origin of the difference lies in the fact that the fermionic Gross-Neveu model cannot be expressed as a *local* bosonic model with the same symmetries. Therefore, it can and does determine a 'new' universality class, which at large-N has the critical indices listed in the table. The relation between the two fixed points has been studied in detail within the context of Yukawa models. We also list in the table the mean field exponents expected to describe the finite temperature transitions in Gross-Neveu models in any dimension, as well as the two dimensional Ising model indices which should apply to the finite temperature transition of a bosonic model in three dimensions which satisfies dimensional reduction. As we discussed in a recent letter on this subject [5], the mean field and the two dimensional Ising exponents are distinctly different. In particular the two dimensional Ising model has $\delta = 15$ which is five times larger than the mean field prediction!

Notice from the table that the critical indices β_{mag}, δ, and γ do not change monotonically as we pass from the $d = 3$ Gross-Neveu model to the $T \neq 0$ Gross-Neveu model to the two dimensional Ising model : while β_{mag} decreases from 1 to 1/2 to 1/8 and δ increases from 2 to 3 to 15, the index γ varies from 1 to 1 to 7/4. So, if we confirm the mean field exponents in our finite temperature simulations, it will not be easy to interpret the results as evidence that 'our lattices are too small and not sufficiently aymmetric to measure the true finite temperature indices'. In addition, we will do simulations on several lattice sizes to check that we are in the scaling region of the finite temperature transition of interest. The lattices themselves will have many more sites per dimension than typical four dimensional simulations.

6.1 6×30^2 Lattice

We begin with our simulations on 6×30^2 lattices. This lattice size was chosen because its asymmetry ratio is large (30/6) and because our past studies of the zero temperature model indicated that the lattice is just barely large enough to obtain critical indices and powerlaw scaling laws with good accuracy [7]. Lattices which are smaller than this in the 'temperature' direction would produce results badly distorted by discreteness. We present the data in Table 6. The notation is the same as previous tables. The statistics for each data point are far better than the four dimensional simulations - more than 100,000 sweeps were done at each parameter setting in each of our three dimensional simulations, unless stated otherwise. The notation for our fits will follow the same conventions as we used in the four dimensional study presented above. First, consider the order parameter and its critical index β_{mag}. A simple powerlaw fit over the range of couplings .70 through .76 works well and predicts the parameters $A = 1.33(2)$ and $\beta_{mag} = .52(2)$ in fine agreement with mean field theory. The data and the fit are shown in Fig.11. (The figure also includes the data and fit from a simulation on a much larger lattice 12×36^2 which will be discussed below.) The critical coupling on the 6×30^2 lattice is determined to be .792(1). Next we considered the susceptibility data and fit $1/\chi$ as discussed earlier in Sec.3. The range of couplings .68 through .75 produced an adequate powerlaw fit with the parameters $A = 12.9(1.3)$, $C = -.095(65)$, and the critical index $\gamma = .99(7)$. The data and its fit are shown in Fig.12 along with the 12×36^2 data. As usual, susceptibility data and fits are considerably less accurate than order parameter studies. Nonetheless, the results are in fine agreement with mean field theory and contrast sharply with the two dimensional Ising value of $\gamma = 7/4$. Finally we consider the model at criticality and estimate the index δ. The data is given in Table 7 and powerlaw fits over the mass range of .00375 - .015 produce the parameters $A = .89(1)$ and $\delta = 3.40(4)$. The data and its fit are plotted in Fig.13 along with the 12×36^2 results. As expected the measured δ is slightly higher than the 'expected' mean field result due to finite size effects. Nonetheless, since the two dimensional Ising value is $\delta = 15$, the numerical distinction between the two scenerios is brilliantly clear. It is significant that simple powerlaw fits were adequate in this

application. This result gives more credence to the fact that the simulation is detecting the true finite temperature scaling law of the model and is not seriously distorted by a possible sluggish cross-over between the three dimensional Gross-Neveu model and the two dimensional Ising model.

6.2 12×36^2 and 12×72^2 Lattices

Now we turn to the 12×36^2 simulations and extensive results on a relatively huge 12×72^2 lattice. The 12×36^2 data is given in Tables 8 and 9, in the same format as above. We simulated these larger lattices to gather further evidence that we are measuring properties of the finite temperature behavior of the model. The larger 'temperature' extent of the lattice should remove some of the distortion in the 6×30^2 lattice due to discreteness. The asymmetry ratio of 36/12 should be sufficient to resolve the zero temperature, bulk trasition reviewed in Table 5 from the true finite temperature transition. If the 6×30^2 results were just indicative of a sluggish cross-over rather than a true finite temperature transition, the the 12×36^2 results should be distinctly different. We shall find no evidence for this view. Instead, the 12×36^2 results will be in good agreement with the 6×30^2 results and mean field theory.

–Order Parameter and β_{mag}

In Fig.11 we show the data and its fit for the order parameter as the coupling varies. The fit is done over the range of couplings .80 through .85. Its confidence level is .78 and its parameters are $A = 1.13(4)$, and $\beta_{mag} = .58(3)$ for $\beta_c = .881$. The critical index β_{mag} is slightly above the 6×30^2 and the mean field prediction of 1/2. This small discrepancy is probably due to the smaller asymmetry ratio used here. The result is far from the two dimensional Ising value of $1/8 = .125$. Next we fit the susceptibility data over the coupling range .82 through .85 and determine the parameters $A = 8.5(2.1)$ and $\gamma = .9(1)$ with a confidence level of .17. The critical index γ is in good agreement with the 6×30^2 simulation and with mean field theory. The data and the fit have already been plotted in Fig.12. Even though the uncertainties are larger than we would have liked, the results are clearly distinct from the Ising value of 7/4. Finally, we simulated the model at the

critical coupling and determined the index δ. A powerlaw fit for the range of bare fermion masses .0009 through .0100 produced the parameters $A = .91(3)$ and $\delta = 2.90(6)$ with confidence level .13. The data and the fit have already been shown in Fig.13. If we fit the data over the mass range .0009 through .0025, a slightly better fit results (confidence level .22) with the parameters $A = .94(9)$ and $\delta = 2.8(2)$. In either case the results are in good agreement with mean field theory and support the analytic large-N results.

Finally, we checked the 12×36^2 results with simulations on a 12×72^2 lattice. The data are given in Table 10. The results for the order parameter σ are within a standard deviation of the 12×36^2 results and they support the view that the lattices studied here are large enough and asymmetric enough to detect the true finite temperature properties of the theory. Note that $\sigma = .1020(10)$ at $\beta = .86$. Since σ is also the dynamical fermion mass at large-N, and since the lowest scalar mass at large-N is $m_\sigma = 2\sigma$, the largest correlation length in this data set is approximately 5 lattice spacings. Vacuum tunneling made it impossible for us to simulate the model closer to the critical point than this. However, in the *symmetric* phase we certainly simulated larger correlation lengths. As discussed briefly in the Introduction, in the 2-d Ising model as well as in mean field theory, the correlation function and the susceptibility in the symmetric phase are closely related to these same quantities in the broken phase. In the 2-d Ising model, the correlation function amplitude ratio is universal and is 2, and the susceptibility amplitude ratio is also universal and is 37.69. In mean field theory these universal amplitude ratios are $2^{1/2}$ and 2, respectively. We will see below the 12×72^2 data predicts that the critical coupling is very near .87, so if our data is described by the 2-d Ising model then the correlation length at $\beta = .88$ would be approximately 10 while if mean field theory applies it would be approximately 7 there. In either case these are substantial correlation lengths, so it would not be foolish to expect the data to exhibit the true critical behavior of the finite temperature transition.

Since the 12×72^2 data for the order parameter agrees with our 12×36^2 data, we should extract the same index β_{mag} as before. The 12×72^2 data is, however, slightly

more accurate and closer to the critical point. Powerlaw fits to the σ vs. β data predict $\beta_{mag} = .53(2)$, $A = 1.00(3)$ and a critical coupling $\beta_c = .875$ with a confidence level of .53. The agreement with mean field theory is almost perfect and the 2-d Ising model prediction of $\beta_{mag} = 1/8$ is easily ruled out.

–Susceptibility and γ

Measurements of the theory's susceptibility suffer from larger statistical errors although typically 100,000 sweeps of the algorithm were run at each parameter setting near the critical coupling. Powerlaw fits to the susceptibility in the broken phase predicted a critical index of $\gamma = .72(14)$, $A = 6.0(1.7)$ and a critical coupling $\beta_c = .865$ with a confidence level of .56. The result for γ is slightly more than one standard deviation below the prediction $\gamma = 1.00$ of mean field theory and is far from the 2-d Ising model's $\gamma = 1.75$. The susceptibility data in the symmetric phase is slightly better because of the absence of tunneling and the larger correlation lengths. Here the fits predict $\gamma = 1.17(15)$, $A = 7.38(3.23)$ and a critical coupling of $\beta_c = .870$ with a confidence level of .92.

–Universal Amplitude Ratios

Perhaps our most persuasive evidence for mean field behavior follows from the universal susceptibility ratio. Mean field theory and the 2-d Ising model differ by a factor approaching 20 for this ratio so even crude data within the theory's scaling region will distinguish the two alternatives. In Fig.14 we show the inverse susceptibility and the mean field fit assuming $\gamma = 1.0$. The fit contains only two free parameters – the critical coupling (which must be near .87 to agree with other fits) and the amplitude (in the broken phase). The resulting figure looks fairly compelling and should be compared to Fig.15 where the same procedure has been tested assuming the applicability of the 2-d Ising model. That figure shows that the amplitude ratio of 37.69 characteristic of the 2-d Ising model utteringly fails.

7 Simulations of the N=4 Model

Now we turn to extensive simulations of the model with one-third the number of fermion species. Our motivation for setting N to four is to investigate whether mean field predictions only apply to the large-N limit of the model. The following secenerio has been suggested in Ref.17 based on experience with large-N scalar models: Since $1/N$ is a dimensionless quantity, it might determine the true width of the scaling region of the model. At infinite N mean field behavior could apply in a region of width of order unity around β_c. However, at finite N a region of width O(1/N) might emerge around the critical point inside of which dimensional reduction applies and the true scaling behavior is that of the 2-d Ising model. For a finite extent outside the 'true' scaling region, mean field behavior would apply as predicted by the large-N gap equation. Outside that region the correlation length would be sufficiently small that the zero temperature critical point would control the theory's scaling. As discussed above, we have no theoretical reasons to suspect that this scenerio applies to fermionic models and the physical picture developed in this article does not hint at any subtle N dependence. Nonetheless, in the absence of a rigorous solution to the model at finite N, we must address this question computationally. Luckily, the computer algorithm works well for any N. Of course, at small N we expect more serious fluctuations and vacuum tunneling so the computer simulations will have their usual limitations. We shall try to master these practical problems with long simulations on relatively large lattices, but some additional, more sophisticated studies would also be welcome.

Extensive simulations of the $N = 4$ model were run on 6×36^2 and 12×72^2 lattices. In Table 11 we collect the 6×36^2 data for σ and its susceptibility for couplings β ranging from .60 to .80. To achieve useful accuracy 250,000 sweeps of the algorithm were run at β values near the transition (from .64 through .72) while 100,000 sweeps were accumulated elsewhere. We could not collect useful data at β values .66, .67 or .68 because of vacuum tunneling (Our 12×72^2 simulations will improve on this limitation.). The error bars in the table account for correlations in the raw data sets. Even with longer runs near the

critical point, the error bars increase near criticality in Table 11.

– $< \bar{\psi}\psi >$

First consider the critical index β_{mag}. In Fig.16 we plot σ^2 vs. β. The curve is well-approximated by a straight line, especially near the critical point, as expected if mean field theory applies. A powerlaw fit predicts $\beta_{mag} = .4(1)$, $A = 1.31(9)$ and critical coupling $\beta_c = .67$ with a confidence level of .39.

–*Susceptibility*

Next consider the susceptibility results. In Fig.17 we plot the inverse susceptibility vs. coupling. We are particularly interested in determining the critical index γ here and in seeing whether the data is compatible with the universal amplitude ratio of 2 expected of mean field theory. The dashed lines in the figure show the fit which incorporates $\gamma = 1.0$. The fit used the data in the broken phase close to the critical point, at $\beta = .64$ and .65, to determine the susceptibility amplitude and critical coupling, and it then predicted the susceptibility in the symmetric phase. We see from the figure that the fit reproduces the susceptibility quite well. Again, the 12×72^2 simulation will yield more useful data near the critical point, so the fit will be even more compelling.

The extensive susceptibility data in the symmetric phase allowed us to test the mean field prediction of $\gamma = 1.0$ directly. Powerlaw fits to the data gave $\gamma = 1.2(2)$, $A = 1.6(8)$, and $\beta = .670$ with a confidence level of .57. The 2-d Ising model value of 1.75 for γ is not compatible with the data. To emphasize this point we show in Fig.18 the 2-d Ising model fit to the susceptibility data incorporating $\gamma_{Ising} = 1.75$ and the universal amplitude ratio 37.69. As before we fit to the data near the critical point in the broken phase and then 'predicted' the data in the broken phase. We see that the 'prediction', the dashed line in the figure, does not resemble the data. The only limitation of this fit is the fact that we do not have much data in the scaling region on the broken coupling side of the transition. On this size lattice we are simply unable to avoid considerable vacuum tunneling.

Finally, consider the 12×72^2 data for the $N = 4$ model. The data is shown in Table 12. As usual, the error bars reflect correlations in the data set. The runs were as long

as 250,000 sweeps near the critical point. Note that the order parameter is .164(5) at $\beta = .745$. We estimate, therefore, that the scalar correlation length is roughly 3.05 at this coupling. We could not simulate the model in the broken phase closer to the critical point because of vacuum tunneling, so we took extensive measurements in the symmetric phase to produce data with larger correlation lengths. (We shall see that within mean field fits, the correlation length at $\beta = .76$ is roughly 7.7.). There were signs of vacuum tunneling in the $\beta = .745$ data sets, and this accounts for the relatively large error bars for the order parameter and its susceptibility there. Luckily, the tunneling events were rare enough in the data that they could be isolated and discarded.

$-< \bar{\psi}\psi >$

First consider the dependence of the order parameter on the coupling. We show σ^2 vs. β in Fig.19, and the linearity of the fit clearly supports mean field theory. Powerlaw fits to the data for β ranging from .70 through .745 predict $\beta_{mag} = .51(4)$, in almost perfect aggreement with the mean field scenerio. The remaining parameters of the fit are $A = 1.22(6)$, $\beta_c = .7635$, and the confidence level of the fit is .67.

–Susceptibility

Next consider the susceptibility data. The error bars on the susceptibility data are larger than we would like, so fits to determine the index γ are not very informative. Powerlaw fits to the data in the broken phase give $\gamma = .8(4)$ and fits in the symmetric phase give $\gamma = 1.4(2)$. Comparisons of the amplitude ratios to the 2-d Ising and mean field prediction are, however, much cleaner. In Fig.20 we show the inverse of the susceptibility vs. coupling. The dashed line gives the mean field prediction for the entire curve based on just two parameters, β_c and the amplitude in the broken phase. The curve is in general agreement with the data for a very reasonable $\beta_c = .764$ which is nicely compatible with the order parameter data. In Fig.21 we show the results of the same exercise using the fitting form of the 2-d Ising model. We see that if we fit the susceptibility data in the broken phase near β_c, then we fail qualitatively to describe the data in the symmetric phase. One might try to salvage the 2-d Ising scenerio by moving the estimate of β_c to

a smaller value in order to accommodate the susceptibility data in the symmetric phase at the expense of a poorer 'fit' in the broken phase. In Fig.22 we show the case with $\beta_c = .758$. The description of the data remains unsuccessful.

–δ Exponent

Finally, we turn to a determination of the critical index δ. This is a useful universal quantity to calculate because is differs by a factor of five in the two scenerios. The data for the order parameter at criticality (we estimated $\beta_c = .760$ for these runs) vs. the bare fermion mass is shown in Table 13. Note that our lowest σ value in the Table is .0856 which corresponds to a correlation length of 5.8. A plot of $\ln(1/\sigma)$ vs. $\ln(1/m)$ should have a slope of $1/\delta$. The data is plotted this way in Fig. 23. The darker dashed line is the mean field prediction of $\delta = 3.0$ and the lighter line is the 2-d Ising prediction of $\delta = 15$. The data with the largest correlation lengths are compatible with $\delta = 3.0$ while none of the data points resemble the 2-d Ising scenerio.

Of course, one can always imagine doing more simulations on larger lattices with greater statistics and resolution, complete with a thorough finite size scaling analysis. Perhaps the results presented here will inspire such activity. Certainly our simulations strongly favor the physical scenerio presented above where fermions are essential at the finite temperature transition and dimensional reduction does not occur.

8 Conclusions

We have studied the universality class of the chiral restoration transition in the Z_2 Gross-Neveu model at finite temperature using large-N and lattice simulation. Our findings strongly support mean field scaling instead of that of a dimensionally reduced Ising model. As we mentioned in the introduction, we found our initial findings, reported in [5], perplexing and, to some extent, circumstantial, and so we continued and refined our study in order to close possible loopholes in our past results. In particular, returning to the three points from the Introduction, we addressed the question of the narrowness of the scaling region and the possible failure of the $1/N$ expansion in this context. The leading

order calculations do not hint at any departure from mean field behavior. However, it is possible that non-Gaussian scaling is resurrected beyond the leading order. If that were the case, there should be a scaling region whose width scales as some power of $1/N$ in which the deviations from mean field scaling are visible. Due to numerical limitations, it is conceivable that the $N = 12$ simulations are not sufficiently good to enter that region. For that purpose we simulated the $N = 4$ theory. We observed no change in our conclusions. Our lattices were sufficiently large to see any such departures if they were there. After all, in past simulations the correct physics was recovered when the 2d Gross-Neveu model was studied on lattices of similar extent [20].

Our task of excluding the dimensional reduction scenario for the 3d Gross-Neveu model was relatively easy. This was so because the exponents δ and β_{mag} differ by factors of 5 and 4, respectively, in the mean field and 2d Ising models. Furthermore, the susceptibility amplitude ratios differ by almost a factor of 20 in these two cases and there is virtually no chance they can be confused with each other even on modest lattice sizes studied with modest statistics.

In Table 5 we record the critical indices of the 3d large-N Gross-Neveu model, mean field indices and those of the 2d Ising model. If the mean field scaling we observed is just an effect of a sluggish crossover from $T = 0$ to 2d Ising scaling, then all the exponents should experience a uniform approach to the Ising values. From the table it is clear that there is no such uniformity. The 'crossover' interpretation of our simulation results is hard to reconcile with this observation.

To eliminate any doubts about observing dimensional reduction in lattice simulations, we simulated the scalar $O(2)$ model where the standard scenario seems inevitable. We undertook this in order to get some ideas about the lattice sizes and correlation lengths needed for the dimensional reduction scenerio to manifest itself. We found scaling consistent with the dimensional reduction on lattices of the modest size, considerably smaller than we used for the Gross-Neveu model.

Therefore, we conclude that the finite temperature scaling of the finite temperature

Gross-Neveu is not that of a dimensionally reduced model.

As far as QCD is concerned, the moral of the Gross-Neveu exercise is twofold. First, from the physics point of view, four-fermi models are closer in spirit to QCD than the σ-models since they both realize mesons as quark composites. Therefore, they share this apparently crucial aspect. It is conceivable, but by no means inevitable that the QCD transition is in the same universality class as the Nambu-Jona-Lasinio model [21]. At least, there is no apriori argument against it. From the numerical point of view, the Gross-Neveu exercise is illustrative of the degree of difficulty involved in addressing questions like the universality class of the theory. This is especially relevant in light of the fact that the question of what symmetry gets restored [22] at finite temperature lattice QCD requires computer power capable of distinguishing the exponents of the $O(2)$ and $O(4)$ models, which are within a few percent of each other [1]. We have had some success in studying this same question in a different, simpler context: our task was to distinguish mean field scaling laws from those of the 2d Ising model where differences of a factor of 20 occur. Nevertheless, this proved to be a nontrivial task and there might still be some scepticism remaining about our findings. In this light the question about the universality class of QCD remains one that will be open for quite a while. Small lattice simulations of the finite temperature transition in QCD cannot distinguish between mean field, $O(2)$, or $O(4)$ behavior at this time [1]. Perhaps some of the observations and methods of analysis reported in this paper will help decide this important question. Certainly, before one studies the dynamics of the chiral transition and heavy ion phenomenology, one needs to determine the universality class of the static transition.

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Table 1: Four-dimensional scalar $O(2)$ model on a 8×16^3 lattice

β	σ	χ
0.19	0.646(1)	0.695(5)
0.185	0.622(1)	0.835(3)
0.18	0.595(1)	1.023(4)
0.175	0.561(1)	1.302(3)
0.17	0.421(1)	1.761(7)
0.165	0.469(1)	2.569(13)
0.1625	0.437(1)	3.221(7)
0.16	0.400(1)	4.313(44)
0.1575	0.353(1)	6.334(31)
0.155	0.292(1)	10.884(52)
0.1525	0.199(1)	31.76(62)

0

Table 2: Order parameter at criticality for the four-dimensional scalar $O(2)$ model on a 8×16^3 lattice

m	σ
0.002	0.219(2)
0.003	0.235(1)
0.004	0.249(1)
0.005	0.260(1)
0.006	0.271(1)
0.007	0.279(1)
0.008	0.287(1)
0.009	0.295(1)
0.01	0.302(1)
0.015	0.332(1)
0.02	0.354(1)
0.03	0.390(1)
0.04	0.417(1)
0.05	0.440(1)

Table 3: Data compilation for the four-dimensional Z_2 four-Fermi model on a 8×16^3 lattice

β	σ	χ
0.56	0.2660(3)	2.80(15)
0.565	0.2420(4)	3.28(15)
0.57	0.2159(3)	4.18(12)
0.5725	0.2020(3)	4.73(9)
0.575	0.1870(4)	5.48(15)
0.5775	0.1711(5)	6.71(21)
0.58	0.1534(7)	8.80(39)
0.5825	0.1336(9)	12.48(79)

Table 4: Order parameter at criticality 2 for the four-dimensional Z_2 four-Fermi model on a 8×16^3 lattice

m	σ
0.002	0.1415(5)
0.004	0.1757(4)
0.006	0.1994(3)
0.008	0.2177(2)
0.010	0.2332(2)

Table 5: Critical indices for the models of interest

	3d Gross-Neveu		2d Ising
	$T = 0$	$T \neq 0$	
β	1	1/2	1/8
δ	2	3	15
γ	1	1	7/4
ν	1	1/2	1
η	1	0	1/4

Table 6: Data compilation for the order parameter, σ and its susceptibility, χ, versus the coupling, β in the Z_2 four-Fermi model on a 6×30^2 lattice

β	σ	χ
0.64	0.5116(2)	0.545(3)
0.66	0.4701(2)	0.624(6)
0.68	0.4276(2)	0.737(3)
0.69	0.4061(3)	0.799(5)
0.70	0.3836(2)	0.897(3)
0.71	0.3608(4)	1.023(5)
0.72	0.3368(4)	1.14(2)
0.73	0.3119(4)	1.405(5)
0.74	0.2847(4)	1.68(2)
0.75	0.2557(8)	2.26(5)
0.76	0.2199(5)	3.60(10)
0.77	0.1734(13)	8.47(1.00)

Table 7: Data compilation for σ vs. m, the bare fermion mass, at the critical point on a 6×30^2 lattice for the Z_2 four-Fermi model

m	σ
0.00125	0.1080(13)
0.002	0.1358(11)
0.0025	0.1489(7)
0.00375	0.1719(9)
0.005	0.1885(5)
0.0075	0.2131(5)
0.01	0.2317(3)
0.015	0.2595(4)
0.02	0.2811(1)
0.03	0.3136(1)
0.04	0.3373(1)

Table 8: Same as Table 6 on a 12×36^2 lattice

β	σ	χ
0.70	0.4299(8)	0.729(7)
0.72	0.3943(2)	0.736(9)
0.74	0.3591(2)	0.791(9)
0.76	0.3233(4)	0.934(20)
0.78	0.2871(5)	1.145(20)
0.79	0.2693(4)	1.220(20)
0.80	0.2499(4)	1.279(15)
0.81	0.2304(3)	1.535(15)
0.82	0.2108(8)	1.784(10)
0.83	0.1884(3)	2.160(20)
0.84	0.1645(5)	2.920(20)
0.85	0.1384(5)	4.29(10)

Table 9: Same as Table 7 on a 12×36^2 lattice

m	σ
0.0009	0.0798(11)
0.00125	0.0951(58)
0.0018	0.1051(19)
0.0025	0.1148(5)
0.005	0.1475(9)
0.010	0.1834(15)

Table 10: 12×72^2 lattice, $N = 12$

β	σ	χ	sweeps ($\times 1000$)
0.80	0.2499(1)	1.29(2)	40
0.81	0.2304(3)	1.54(5)	40
0.82	0.2101(4)	1.72(5)	20
0.83	0.1884(3)	2.16(9)	20
0.84	0.1645(4)	2.82(12)	50
0.85	0.1370(5)	4.54(14)	50
0.86	0.1020(10)	12.0(1.0)	50
0.88		24.40(84)	80
0.89		11.91(5)	70
0.90		7.83(47)	70
0.91		5.51(28)	80
0.92		4.35(27)	80
0.93		3.41(13)	50

Table 11: 6×36^2 lattice, $N = 4$

β	σ	χ	sweeps ($\times 1000$)
0.60	0.4869(8)	3.51(12)	90
0.61	0.4575(6)	4.00(16)	90
0.62	0.4271(9)	4.86(16)	90
0.63	0.3933(9)	6.34(21)	90
0.64	0.3514(17)	11.8(1.8)	250
0.65	0.3029(21)	17.6(1.3)	250
0.69		59.7(2.9)	250
0.70		39.5(1.6)	250
0.71		27.5(1.3)	250
0.72		20.62(79)	250
0.73		17.70(94)	90
0.74		14.77(77)	90
0.75		11.75(54)	90
0.76		9.87(52)	90
0.77		8.60(28)	90
0.78		7.62(30)	90
0.79		6.90(27)	80
0.80		6.13(22)	80

Table 12: 12×72^2 lattice, $N = 4$

β	σ	χ	sweeps ($\times 1000$)
0.70	0.3061(2)	4.45(7)	60
0.71	0.2813(7)	6.48(15)	120
0.72	0.2534(3)	7.32(39)	180
0.73	0.2238(11)	9.96(49)	120
0.74	0.1866(15)	17.3(1.2)	140
0.745	0.164(5)	27.9(5.3)	100
0.77		70.5(9.1)	140
0.78		42.2(1.7)	140
0.79		25.1(2.3)	100
0.80		20.2(1.4)	120
0.81		18.1(1.3)	220
0.82		13.01(86)	120
0.83		10.29(29)	100
0.84		9.42(57)	100
0.85		8.02(75)	60

Table 13: 12×72^2 lattice, $N = 4$ at Criticality

m	σ	χ	sweeps ($\times 1000$)
0.00025	0.0856(78)	99.2(9.0)	240
0.0005	0.1077(24)	43.5(9.9)	200
0.001	0.1363(13)	21.5(1.3)	320
0.002	0.1637(17)	13.9(7)	200
0.003	0.1820(8)	8.16(43)	200
0.004	0.1946(9)	7.63(30)	200
0.005	0.2058(2)	6.48(43)	80

Figure Captions

1. Order parameter vs. coupling for $O(2)$ model on an 8×16^3 lattice. The dashed line is the fit discussed in the text.

2. Same as Fig. 1 except for the inverse susceptibility

3. Order parameter vs. mass for $O(2)$ model on an 8×16^3 lattice at the critical coupling. The dashed line is the fit discussed in the text.

4. log-log plot of the data in Fig. 3.

5. Same as Fig.1 except for the Nambu Jona-Lasinio model.

6. Same as Fig. 5 except for the inverse susceptibility.

7. Same as Fig.3 except for NJL model.

8. log-log plot of the data in Fig. 7

9. Figs. 4 and 8 combined to illustrate the different slopes (critical indices δ).

10. Tranverse susceptibility (m/σ) vs σ^2 for the NJL model at the critical coupling on a 8×16^3 lattice. A straight line passing through the origin would agree exactly with mean field theory.

11. Order parameter squared vs. coupling for the NJL model on 6×30^2 (lower curve) and 12×36^2 lattices.

12. Same as Fig. 11 except for the inverse susceptibility.

13. Log-log plot of the order parameter vs. mass at the critical point for both the 6×30^2 lattice (lower set of data points and their fit discussed in the text) and the 12×36^2 lattice. The dashed-dotted curve is the 2-d Ising model prediction.

14. Inverse susceptibility χ vs. coupling for the NJL model on a 12×72^2 lattice. The dashed curve is the mean field fit discussed in the text.

15. $\chi^{4/7}$ vs. coupling for the data plotted in Fig. 14. The dashed line is the prediction of the dimensional reduction scenerio.

16. Order parameter squared vs. coupling for the NJL model on 6×36^2 lattice with four flavors. The dashed line is the fit discussed in the text.

17. Same as Fig. 16 except for the inverse susceptibility.

18. Same as Fig. 15 for the four flavor model on a 6×36^2 lattice.

19. Same as Fig. 16 except on a 12×72^2 lattice.

20. Same as Fig. 14 except for the four flavor model.

21. Same as Fig. 15 except for the four flavor model.

22. Same as Fig. 21 except the fit tries a critical couling of .758.

23. Log-log plot of the order parameter vs. mass at criticality. The dashed line is given by mean field theory, while the dotted line is given by the 2-d Ising model.