Hadron Spectra for Semileptonic Heavy Quark Decay

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Abstract

We calculate the leading perturbative and power corrections to the hadronic invariant mass and energy spectra in semileptonic heavy hadron decays. We apply our results to the B system. Moments of the invariant mass spectrum, which vanish in the parton model, probe gluon bremsstrahlung and nonperturbative effects. Combining our results with recent data on B meson branching ratios, we obtain a lower bound $\bar{\Lambda} > 410\,\mathrm{MeV}$ and an upper bound $m_b^{\mathrm{pole}} < 4.89\,\mathrm{GeV}$. The Brodsky-Lepage-Mackenzie scale setting procedure suggests that higher order perturbative corrections are small for bottom decay, and even tractable for charm decay.

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I. INTRODUCTION

Our understanding of inclusive decays of hadrons containing at least one heavy quark has improved greatly over the last few years. The energy released during semileptonic or radiative decay of heavy hadrons is much larger than the scale $\Lambda_{\rm QCD}$ of the strong interactions, and therefore an operator product expansion (OPE) exists for some observables in these decays, including rates and differential spectra [1,2]. The leading power corrections to the rates and lepton differential spectra for semileptonic decays of heavy hadrons [3–6] have been studied extensively, as have the power corrections to radiative decays [7].

A major result of this analysis is that, except in regions where the expansion becomes singular such as the endpoint of the electron spectrum in semileptonic $b \to u$ decay, the corrections to the parton model are quite small, suppressed by $\mathcal{O}(\Lambda_{\rm QCD}^2/m_b^2)$. While this does mean that the parton model is quite successful, it makes it difficult to test quantitatively the corrections given by the OPE. In particular, if quark/hadron duality does not hold in this energy regime, one would expect to see corrections to the parton model which could not be accounted for by the leading perturbative and $1/m_Q$ corrections. Shifman has recently criticized the related OPE analysis of τ decay on the basis that violations of duality in the Minkowski regime introduce large corrections which are not seen at any finite order in the OPE [8].

In this paper we suggest that hadronic variables, in particular moments of the invariant mass spectrum $d\Gamma/ds_H$ and the hadron energy spectrum $d\Gamma/dE_H$, provide a useful testing ground for the OPE. This is similar to the analogous suggestion, and analysis, for semileptonic τ decays [9,10]. However, unlike the case for τ decays, at tree level the final hadronic state consists of a single quark. Therefore at lowest order in the OPE the final hadronic state has fixed invariant mass $s_H = m_q^2$, and positive moments of $(s_H - m_q^2)$, which are calculable as a double expansion in $\alpha_s(m_b)$ and $1/m_b$, directly probe physics beyond the parton model. Similarly, at leading order in the OPE the maximum hadron energy is $(m_b^2 + m_q^2)/2m_b$ (when the quark q recoils back-to-back with the leptons); the region above this endpoint is populated only by gluon bremsstrahlung and nonperturbative effects.

In this paper we calculate the corrections to the parton model results for these observables, up to $\mathcal{O}(1/m_b^2, \alpha_s/m_b)$. As discussed in Ref. [11], although the leading power corrections to leptonic variables arise at $\mathcal{O}(1/m_b^2)$, for kinematic reasons the leading power corrections to moments of the invariant mass spectrum arise at $\mathcal{O}(1/m_b)$. The $\mathcal{O}(1/m_b^2)$ corrections to the differential hadronic energy spectrum were first examined in Ref. [12]; however, we disagree with the results presented in that work. We also use the results of Ref. [13], in which the one-loop corrections to the hadron energy spectrum were calculated. We combine our results with recent data on B meson branching ratios to obtain a lower bound on the nonperturbative parameter $\bar{\Lambda}$, which is the leading contribution to the difference between heavy quark and heavy meson masses.

Finally, using the BLM prescription [14] to estimate the size of the two-loop perturbative corrections to the moments of the invariant mass spectrum, we demonstrate that these moments appear to have a well-behaved perturbative expansions not only for B decays, but also for D decays, when the results are expressed in terms of physical observables. This suggests that studying hadronic observables in semileptonic decays of charmed hadrons, which are dominated by only two or three resonances, may shed some insight into the
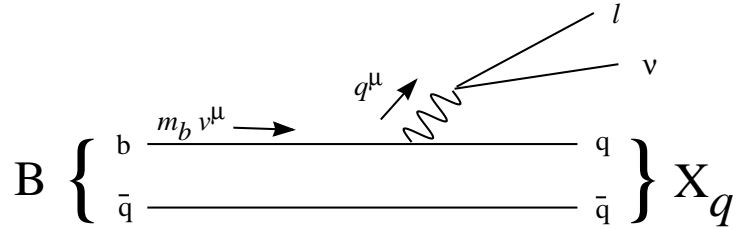
FIG. 1. The kinematics for $B \to X_q \ell \nu$.

applicability of quark/hadron duality at low energies.

II. KINEMATICS

We start by introducing the kinematic variables describing the final state hadrons. For definiteness, we will consider semileptonic B decay, although the analysis extends simply to the decays of charmed hadrons.

The kinematics of the inclusive process $B \to X_q \ell \nu$ is shown in Fig. 1. The four-velocity of the B meson is v^μ, and q^μ is the four-momentum of the lepton pair. The total energy of the leptons in the B rest frame is $v \cdot q$, and their invariant mass is q^2. It is convenient to define dimensionless quantities $\hat{E}_0$ and $\hat{s}_0$,

$$
\begin{aligned}
\hat{E}_0 &= v \cdot (P_b - q)/m_b = 1 - v \cdot \hat{q}\,, \\
\hat{s}_0 &= (P_b - q)^2/m_b^2 = 1 - 2v \cdot \hat{q} + \hat{q}^2\,,
\end{aligned}
\tag{2.1}
$$

where $\hat{q}^\mu = q^\mu/m_b$. At leading order in $1/m_b$, $\hat{E}_0$ and $\hat{s}_0$ are simply the scaled energy and squared invariant mass of the final hadronic state. However, since they are scaled by the b *quark* mass, this identification does not hold at subleading order in $1/m_b$.[1] Instead, they are related to the physical hadronic energy and squared invariant mass,

$$
\begin{aligned}
E_H &= v \cdot (P_B - q) = m_B - v \cdot q\,, \\
s_H &= (P_B - q)^2 = m_B^2 - 2m_B v \cdot q + q^2\,,
\end{aligned}
\tag{2.2}
$$

through

$$
\begin{aligned}
E_H &= \bar{\Lambda} - \frac{\lambda_1 + 3\lambda_2}{2m_B} + \left(m_B - \bar{\Lambda} + \frac{\lambda_1 + 3\lambda_2}{2m_B}\right)\hat{E}_0 + \dots\,, \\
s_H &= m_q^2 + \bar{\Lambda}^2 + (m_B^2 - 2\bar{\Lambda}m_B + \bar{\Lambda}^2 + \lambda_1 + 3\lambda_2)\,(\hat{s}_0 - \hat{m}_q^2) \\
&\quad + (2\bar{\Lambda}m_B - 2\Lambda^2 - \lambda_1 - 3\lambda_2)\hat{E}_0 + \dots\,,
\end{aligned}
\tag{2.3}
$$

where $\hat{m}_q = m_q/m_b$, and the ellipses denote terms higher order in $1/m_b$. The quantities $\bar{\Lambda}$, λ_1 and λ_2 arise in the relationship between the quark and meson masses [15,16],

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[1]This fact was neglected in Ref. [12].

$$m_B = m_b + \bar{\Lambda} - \frac{\lambda_1 + 3\lambda_2}{2m_b} + \dots ,$$

$$m_{B^*} = m_b + \bar{\Lambda} - \frac{\lambda_1 - \lambda_2}{2m_b} + \dots . \tag{2.4}$$

From the measured B–B^* mass splitting, $\lambda_2 \simeq 0.12\,\mathrm{GeV}^2$. We note that in contrast to the lepton spectra, there are $1/m_b$ corrections both to the physical hadronic invariant mass spectrum and to the physical hadronic energy spectrum, although these corrections are absent for the $\hat{s}_0$ and $\hat{E}_0$ spectra [11].

While the complete shape of the $\hat{E}_0$ spectrum may be calculated (away from the parton model endpoint $\hat{E}_0 = \frac{1}{2}(1 + \hat{m}_q^2)$) with the standard OPE analysis, only suitably averaged features, such as moments, of the $\hat{s}_0$ spectrum may be computed reliably. The difference arises because each point of the $\hat{E}_0$ spectrum receives contributions from states of different invariant masses, making the process inclusive, whereas by definition each point of the $\hat{s}_0$ spectrum only receives contributions from states of a single invariant mass. This may be seen explicitly by carrying out the usual OPE analysis for inclusive decays in the variables $\hat{s}_0$ and $\hat{E}_0$, instead of the usual leptonic variables $v \cdot \hat{q}$ and $\hat{q}^2$.

The inclusive B meson decay rate is given by

$$\Gamma(B \to X_q \ell \nu) \sim \int \mathrm{d}\hat{s}_0 \, \mathrm{d}\hat{E}_0 \, \sqrt{\hat{E}_0^2 - \hat{s}_0} L_{\mu\nu}(\hat{s}_0, \hat{E}_0) W^{\mu\nu}(\hat{s}_0, \hat{E}_0) , \tag{2.5}$$

where $L_{\mu\nu}$ is the spin summed lepton tensor $L_{\mu\nu} \propto (q_\mu q_\nu - g_{\mu\nu} q^2)$. Using the optical theorem, the nonperturbative hadronic tensor $W^{\mu\nu}$ is related to the imaginary part of the forward scattering amplitude [1,2],

$$W^{\mu\nu} = \sum_X \langle B| J_h^{\mu\dagger} |X\rangle \langle X| J_h^{\nu} |B\rangle (2\pi)^4 \delta^4(P_B - P_X - q)$$

$$= -2 \,\mathrm{Im} \, \langle B| i \int \mathrm{d}x \, e^{-iq \cdot x} \, T \left[J_h^{\mu\dagger}(x) J_h^{\nu}(0) \right] |B\rangle \tag{2.6}$$

$$\equiv 2 \,\mathrm{Im} \, T^{\mu\nu}$$

where $J_h^\mu = \bar{q} \gamma^\mu \frac{1}{2}(1 - \gamma_5) b$. The time-ordered product $T^{\mu\nu}$ may be written via an operator product expansion as a power series in $\alpha_s(m_b)$ and $1/m_b$.

In the $v \cdot \hat{q}$ plane, for fixed $\hat{q}^2$, the correlator $T^{\mu\nu}$ has the analytic structure shown in Fig. 2a, as discussed in Ref. [2]. There are cuts along the real axis, a physical one (corresponding to B decays) for $v \cdot \hat{q} \leq \frac{1}{2}[1 + \hat{q}^2 - \hat{m}_q^2]$, and an unphysical one (corresponding to scattering processes) for $v \cdot \hat{q} \geq \frac{1}{2}[(2 + \hat{m}_q)^2 - \hat{q}^2 - 1]$. The one-particle pole lies at the right hand end of the physical cut. After an integral over the charged lepton energy, the decay rate is computed by performing an integration over the top of the physical cut, for $\sqrt{\hat{q}^2} \leq v \cdot \hat{q} \leq \frac{1}{2}(1 + \hat{q}^2 - \hat{m}_q^2)$, followed by an integration over $0 \leq \hat{q}^2 \leq (1 - \hat{m}_q)^2$. Note that in the limit $\hat{m}_q \to 0$ and $\hat{q}^2 \to 1$, the physical and unphysical cuts pinch the region of integration. In this corner of the parameter space, the operator product expansion breaks down. Attempts to resum the OPE to all orders in this region have thus far proven inconclusive [4,17,18].

Mapping from the $v \cdot \hat{q}$ plane to the $\hat{s}_0$ plane at fixed $\hat{E}_0$, one finds two cuts on the positive $\hat{s}_0$ axis, as shown in Fig. 2b. The physical cut, which terminates in the one-particle pole, extends over $\hat{s}_0 \geq \hat{m}_q^2$. The unphysical cut lies away from the pole, at $\hat{s}_0 \geq \hat{m}_q^2 + 4\hat{m}_q + 4\hat{E}_0$.
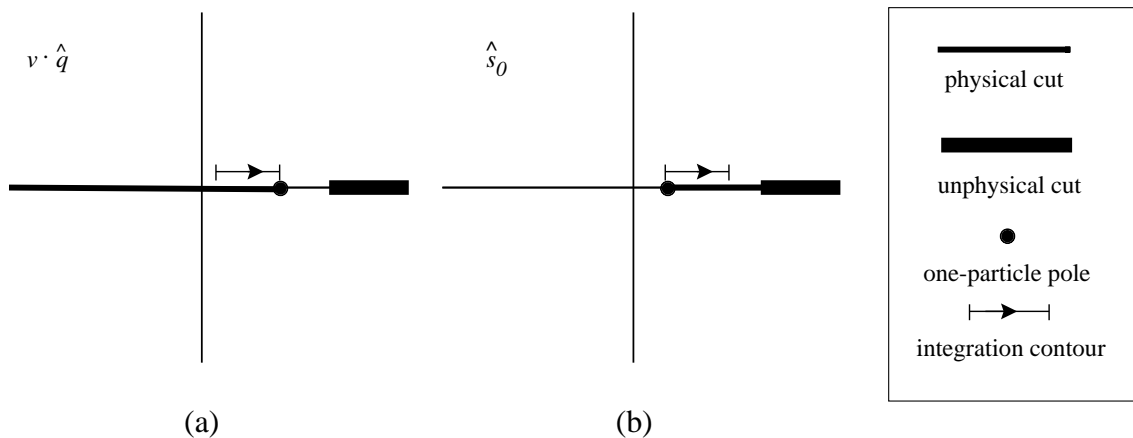
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FIG. 2. The analytic structure of $T^{\mu\nu}$, (a) in the $v \cdot \hat{q}$ plane, and (b) in the $\hat{s}_0$ plane. Both the physical and unphysical cuts are shown, as well as the position of the one-particle pole.

The region of integration in $\hat{s}_0$ is given by $\max(2\hat{E}_0 - 1, \hat{m}_q) \leq \hat{s}_0 \leq \hat{E}_0^2$, to be followed by an integration in $\hat{E}_0$, over $\hat{m}_q \leq \hat{E}_0 \leq 1$.

In these variables, the region of integration only touches the unphysical cut in the limit $\hat{m}_q \to 0$ and $\hat{E}_0 \to 0$, which from Eq. (2.1) is equivalent to the condition for the cuts to pinch in the $v \cdot \hat{q}$ plane. In this singular region, as before, the operator product expansion breaks down. We also note that the integration region covers the one-particle pole at $\hat{s}_0 = \hat{m}_q^2$ only if $\hat{E}_0 \leq \frac{1}{2}(1 + \hat{m}_q^2)$. Indeed, this corresponds to the maximum energy the final quark can take away in the decay process. The cut for $\hat{s}_0 > \hat{m}_q^2$ is populated only by multiparticle final states generated by the radiation of gluons. In perturbation theory, then, the differential spectrum $d\Gamma/d\hat{E}_0$ for $\hat{E}_0 > \frac{1}{2}(1 + \hat{m}_q^2)$ is of order α_s.

As is the case for τ decays, the contour of integration in Eq. (2.5) may be deformed away from the physical region, except at the point the contour crosses the physical cut. However, we note that in contrast to τ decays, the integrand in Eq. (2.5) does not have a double zero where the deformed contour approaches the physical region. It is possible, therefore, that deviations from quark/hadron duality in the Minkowski regime may be more pronounced in semileptonic heavy hadron decay than in τ decay.

III. SPECTRAL MOMENTS

In this section we compute the spectral moments at the parton level. We will treat both the leading power corrections, proportional to λ_1 and λ_2, and the leading perturbative contributions, proportional to $\alpha_s(m_b)$. We take the two types of corrections in turn.

A. Power Corrections

For the computation of the power corrections, it is convenient to decompose the time ordered product $T^{\mu\nu}$ into the form factors

$$T^{\mu\nu}(\hat{s}_0, \hat{E}_0) = -g^{\mu\nu}T_1(\hat{s}_0, \hat{E}_0) + v^\mu v^\nu T_2(\hat{s}_0, \hat{E}_0) + \ldots , \tag{3.1}$$

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where the omitted form factors vanish for massless leptons in the final state. In terms of T_1 and T_2 the differential spectrum is given by

$$\frac{1}{\Gamma_0}\frac{d\Gamma}{d\hat{s}_0 d\hat{E}_0} = -\frac{32}{\pi}\text{Im}\sqrt{\hat{E}_0^2 - \hat{s}_0}\left[3(1 - 2\hat{E}_0 + \hat{s}_0)T_1(\hat{s}_0, \hat{E}_0) + (\hat{E}_0^2 - \hat{s}_0)T_2(\hat{s}_0, \hat{E}_0)\right], \quad (3.2)$$

where

$$\Gamma_0 = \frac{G_F^2 m_b^5 |V_{bq}|^2}{192\pi^3} \quad (3.3)$$

is proportional to the total decay rate.

The leading $1/m_b$ corrections to the hadronic quantities T_1 and T_2 were calculated in Refs. [3,4]. In terms of $\hat{s}_0$ and $\hat{E}_0$, they are given by

$$T_1(\hat{s}_0, \hat{E}_0) = \frac{1}{\hat{s}_0 - \hat{m}_q^2 + i\epsilon}\left[\frac{\hat{E}_0}{2} - \frac{\lambda_1}{12m_b^2} - \frac{\lambda_2}{4m_b^2}\right]$$
$$+ \frac{1}{\left(\hat{s}_0 - \hat{m}_q^2 + i\epsilon\right)^2}\left[\frac{\lambda_1}{6m_b^2}\left(5\hat{E}_0^2 - 3\hat{E}_0 - 2\hat{s}_0\right) + \frac{\lambda_2}{2m_b^2}\left(5\hat{E}_0^2 + \hat{E}_0 - 2\hat{s}_0\right)\right] \quad (3.4)$$
$$+ \frac{1}{\left(\hat{s}_0 - \hat{m}_q^2 + i\epsilon\right)^3}\left[\frac{2\lambda_1}{3m_b^2}\hat{E}_0(\hat{s}_0 - \hat{E}_0^2)\right],$$

$$T_2(\hat{s}_0, \hat{E}_0) = \frac{1}{\hat{s}_0 - \hat{m}_q^2 + i\epsilon}\left[1 - \frac{5\lambda_1}{6m_b^2} - \frac{5\lambda_2}{2m_b^2}\right]$$
$$+ \frac{1}{\left(\hat{s}_0 - \hat{m}_q^2 + i\epsilon\right)^2}\left[\frac{7\lambda_1}{3m_b^2}(\hat{E}_0 - 1) + \frac{\lambda_2}{m_b^2}(5\hat{E}_0 - 3)\right] \quad (3.5)$$
$$+ \frac{1}{\left(\hat{s}_0 - \hat{m}_q^2 + i\epsilon\right)^3}\left[\frac{4\lambda_1}{3m_b^2}(\hat{s}_0 - \hat{E}_0^2)\right].$$

Integrating this expression with respect to $\hat{E}_0$, we find the leading power correction to the invariant mass spectrum. Of course, since there is only a single quark in the final state, this expression is a singular function with support only at $\hat{s}_0 - \hat{m}_q^2$. Only its moments, which we present below, are meaningful. The corrections to the hadronic energy spectrum, obtained by integrating first with respect to $\hat{s}_0$, are more interesting, and are presented in Appendix A.

Because the expansions of T_1 and T_2 in terms of $1/m_b$ contain pole factors $1/(\hat{s}_0 - \hat{m}_q^2)^n$, it is simplest to compute the moments of $(\hat{s}_0 - \hat{m}_q^2)$ rather than those of $\hat{s}_0$. The requisite calculations are straightforward but tedious, and we present only the final results. It is convenient to scale the various contributions to Γ_0 rather than to the full width Γ; the quantities which we will present below are then of the form

$$\mathcal{M}^{(n,m)} = \frac{1}{\Gamma_0}\int(\hat{s}_0 - \hat{m}_q^2)^n \hat{E}_0^m \frac{d\Gamma}{d\hat{s}_0 d\hat{E}_0}d\hat{s}_0 d\hat{E}_0, \quad (3.6)$$

for integers n and m. They are related to the parton level moments by a scaling to the corrected decay rate,

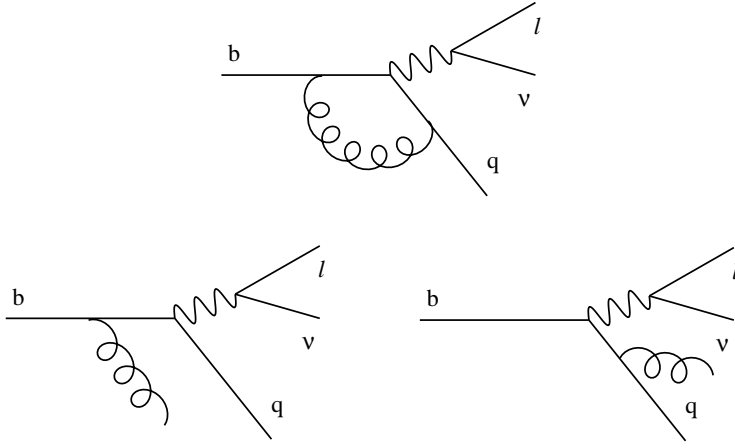$$\langle\hat{E}_0^m(\hat{s}_0 - \hat{m}_q^2)^n\rangle = \frac{\Gamma_0}{\Gamma}\mathcal{M}^{(n,m)}. \quad (3.7)$$

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FIG. 3. The Feynman diagrams which contribute to the moments at order α_s. There are also wave function corrections, which we do not show.

B. Perturbative Corrections

The perturbative corrections to T_1 and T_2 are most conveniently calculated directly from the graphs in Fig. 3. The radiative contributions to $\langle(\hat{s}_0 - \hat{m}_q^2)^n\rangle$ come only from bremsstrahlung graphs and are straightforward to compute for arbitrary $\hat{m}_q$. We find

$$
\frac{1}{\Gamma_0}\frac{d\Gamma}{d\hat{s}_0} = \frac{\alpha_s}{\pi}\frac{1}{\hat{s}_0 - \hat{m}_q^2}\left[\frac{(\hat{s}_0 - 1)}{27\hat{s}_0^2}\left(-9\hat{m}_q^4 - 6\hat{m}_q^6 + \hat{s}_0(18\hat{m}_q^2 + 81\hat{m}_q^4 + 48\hat{m}_q^6)\right.\right. \tag{3.8}
$$
$$
+ \hat{s}_0^2(93 - 316\hat{m}_q^2 + 243\hat{m}_q^4 + 102\hat{m}_q^6) + \hat{s}_0^3(-41 - 478\hat{m}_q^2 + 9\hat{m}_q^4)
$$
$$
\left. + \hat{s}_0^4(-95 - 64\hat{m}_q^2) + 55\hat{s}_0^5\right)
$$
$$
+ \frac{4}{9}\ln\hat{s}_0\left(-3 + 5\hat{m}_q^2 - 18\hat{m}_q^4 - 9\hat{m}_q^6 + \hat{s}_0(-5 + 45\hat{m}_q^2 - 9\hat{m}_q^4 - 3\hat{m}_q^6)\right.
$$
$$
\left.\left. + 9\,\hat{s}_0^2(1 + 2\hat{m}_q^2) + 2\hat{m}_q^2\hat{s}_0^3 - 2\hat{s}_0^4\right)\right],
$$

from which it is easy to extract the moments $\langle(\hat{s}_0 - \hat{m}_q^2)^n\rangle$. Similarly, weighting with extra factors of $\hat{E}_0$ yields the radiative correction to the moments $\mathcal{M}^{(n,m)}$, for $n \geq 1$ and any m.

The one-loop radiative corrections to the hadronic energy spectrum, and hence to the moments $\mathcal{M}^{(0,m)}$, are considerably more difficult to compute. This is because they receive contributions from both virtual graphs and bremsstrahlung graphs, only the sum of which is infrared finite. The complete calculation of the radiative corrections to the differential energy spectrum $d\Gamma/d\hat{E}_0$ was computed by Czarnecki, Jeżabek and Kühn [13].

We present the full perturbative corrections to $\langle\hat{E}_0\rangle$ and $\langle\hat{E}_0^2\rangle$ in Appendix A. In the limit $\hat{m}_q \to 0$ they take the simple form

$$
\mathcal{M}_{\text{pert.}}^{(0,1)}(\hat{m}_q = 0) = \left[\frac{1381}{900} - \frac{7}{30}\pi^2\right]\frac{\alpha_s}{\pi} = -0.768\,\frac{\alpha_s}{\pi}\,,
$$
$$
\mathcal{M}_{\text{pert.}}^{(0,2)}(\hat{m}_q = 0) = \left[\frac{2257}{3600} - \frac{4}{45}\pi^2\right]\frac{\alpha_s}{\pi} = -0.250\,\frac{\alpha_s}{\pi}\,. \tag{3.9}
$$

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C. Corrections to the Moments

We now combine the results of the previous subsections to present the full expressions for the parton-level moments, including the leading perturbative and power corrections.

The first two moments of the hadronic invariant mass spectrum are given by

$$
\begin{aligned}
\mathcal{M}^{(1,0)} = {} & \frac{\alpha_s}{\pi} \left[\frac{91}{450} + \frac{71}{18} \hat{m}_q^2 - \frac{158}{27} \hat{m}_q^4 + \frac{34}{9} \hat{m}_q^6 + \frac{1}{18} \hat{m}_q^8 - \frac{2873}{1350} \hat{m}_q^{10} \right. \\
& \left. + \left(4\hat{m}_q^2 + \frac{40}{9} \hat{m}_q^4 - \frac{8}{3} \hat{m}_q^6 + \frac{20}{3} \hat{m}_q^8 + \frac{56}{45} \hat{m}_q^{10} \right) \ln \hat{m}_q \right] \\
& + \frac{\lambda_1}{2m_b^2} \left[\frac{13}{10} - \frac{27}{2} \hat{m}_q^2 - 16\hat{m}_q^4 + 32\hat{m}_q^6 - \frac{9}{2} \hat{m}_q^8 + \frac{7}{10} \hat{m}_q^{10} - 60\hat{m}_q^4 \ln \hat{m}_q - 12\hat{m}_q^6 \ln \hat{m}_q \right] \\
& + \frac{\lambda_2}{2m_b^2} \left[\frac{3}{2} - \frac{9}{2} \hat{m}_q^2 + 8\hat{m}_q^4 - 24\hat{m}_q^6 + \frac{45}{2} \hat{m}_q^8 - \frac{7}{2} \hat{m}_q^{10} + 12\hat{m}_q^4 \ln \hat{m}_q - 36\hat{m}_q^6 \ln \hat{m}_q \right],
\end{aligned}
\tag{3.10}
$$

$$
\begin{aligned}
\mathcal{M}^{(2,0)} = {} & \frac{\alpha_s}{\pi} \left[\frac{5}{324} - \frac{137}{450} \hat{m}_q^2 - \frac{101}{36} \hat{m}_q^4 + \frac{86}{81} \hat{m}_q^6 - \frac{29}{36} \hat{m}_q^8 + \frac{37}{18} \hat{m}_q^{10} + \frac{6341}{8100} \hat{m}_q^{12} \right. \\
& \left. - \left(\frac{10}{3} \hat{m}_q^4 + \frac{152}{27} \hat{m}_q^6 + \frac{14}{3} \hat{m}_q^8 + 4\hat{m}_q^{10} + \frac{56}{135} \hat{m}_q^{12} \right) \ln \hat{m}_q \right] \\
& + \frac{\lambda_1}{2m_b^2} \left[-\frac{16}{45} + \frac{16}{5} \hat{m}_q^2 - 16\hat{m}_q^4 + 16\hat{m}_q^8 - \frac{16}{5} \hat{m}_q^{10} + \frac{16}{45} \hat{m}_q^{12} - \frac{128}{3} \hat{m}_q^6 \ln \hat{m}_q \right].
\end{aligned}
\tag{3.11}
$$

The first mixed moment is

$$
\begin{aligned}
\mathcal{M}^{(1,1)} = {} & \frac{\alpha_s}{\pi} \left[\frac{9}{100} + \frac{209}{180} \hat{m}_q^2 - \frac{149}{108} \hat{m}_q^4 + \frac{4}{3} \hat{m}_q^6 - \frac{49}{36} \hat{m}_q^8 + \frac{1457}{2700} \hat{m}_q^{10} - \frac{23}{60} \hat{m}_q^{12} \right. \\
& \left. + \left(\frac{7}{5} \hat{m}_q^2 + \frac{10}{9} \hat{m}_q^4 + \frac{4}{3} \hat{m}_q^6 + \frac{2}{3} \hat{m}_q^8 + \frac{23}{45} \hat{m}_q^{10} + \frac{4}{15} \hat{m}_q^{12} \right) \ln \hat{m}_q \right] \\
& + \frac{\lambda_1}{2m_b^2} \left[\frac{23}{90} - \frac{1}{2} \hat{m}_q^2 + 12\hat{m}_q^4 - 16\hat{m}_q^6 + \frac{13}{2} \hat{m}_q^8 - \frac{27}{10} \hat{m}_q^{10} + \frac{4}{9} \hat{m}_q^{12} \right. \\
& \left. + 12\hat{m}_q^4 \ln \hat{m}_q + \frac{20}{3} \hat{m}_q^6 \ln \hat{m}_q \right] \\
& + \frac{\lambda_2}{2m_b^2} \left[\frac{13}{30} + \frac{3}{2} \hat{m}_q^2 - 4\hat{m}_q^4 - \frac{3}{2} \hat{m}_q^8 + \frac{49}{10} \hat{m}_q^{10} - \frac{4}{3} \hat{m}_q^{12} + 12\hat{m}_q^4 \ln \hat{m}_q - 20\hat{m}_q^6 \ln \hat{m}_q \right].
\end{aligned}
\tag{3.12}
$$

The first two moments of the hadron energy spectrum are given by

$$
\begin{aligned}
\mathcal{M}^{(0,1)} = {} & \left[\frac{7}{20} - \frac{5}{4} \hat{m}_q^2 + 8\hat{m}_q^4 - 8\hat{m}_q^6 + \frac{5}{4} \hat{m}_q^8 - \frac{7}{20} \hat{m}_q^{10} + 6\hat{m}_q^4 \ln \hat{m}_q + 6\hat{m}_q^6 \ln \hat{m}_q \right] \\
& + A_1(\hat{m}_q) \frac{\alpha_s}{\pi} \\
& + \frac{\lambda_1}{2m_b^2} \left[1 - 8\hat{m}_q^2 + 8\hat{m}_q^6 - \hat{m}_q^8 - 24\hat{m}_q^4 \ln \hat{m}_q \right] \\
& + \frac{\lambda_2}{2m_b^2} \left[7\hat{m}_q^2 - 20\hat{m}_q^4 + 20\hat{m}_q^8 - 7\hat{m}_q^{10} + 24\hat{m}_q^2 \ln \hat{m}_q - 48\hat{m}_q^4 \ln \hat{m}_q \right],
\end{aligned}
\tag{3.13}
$$

$$
\begin{aligned}
\mathcal{M}^{(0,2)} = {} & \left[\frac{2}{15} - \frac{1}{5} \hat{m}_q^2 - 2\hat{m}_q^4 + 2\hat{m}_q^8 + \frac{1}{5} \hat{m}_q^{10} - \frac{2}{15} \hat{m}_q^{12} - 8\hat{m}_q^6 \ln \hat{m}_q \right] \\
& + A_2(\hat{m}_q) \frac{\alpha_s}{\pi}
\end{aligned}
\tag{3.14}
$$

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$$+ \frac{\lambda_1}{2m_b^2} \left[\frac{43}{90} - \frac{3}{2}\hat{m}_q^2 + 14\hat{m}_q^4 - 16\hat{m}_q^6 + \frac{9}{2}\hat{m}_q^8 - \frac{17}{10}\hat{m}_q^{10} + \frac{2}{9}\hat{m}_q^{12} \right.$$
$$\left. + 12\hat{m}_q^4 \ln \hat{m}_q + \frac{28}{3}\hat{m}_q^6 \ln \hat{m}_q \right]$$
$$+ \frac{\lambda_2}{2m_b^2} \left[\frac{13}{30} - \frac{21}{2}\hat{m}_q^2 + 50\hat{m}_q^4 - 52\hat{m}_q^6 + \frac{21}{2}\hat{m}_q^8 + \frac{49}{10}\hat{m}_q^{10} - \frac{10}{3}\hat{m}_q^{12} \right.$$
$$\left. + 12\hat{m}_q^4 \ln \hat{m}_q + 28\hat{m}_q^6 \ln \hat{m}_q \right],$$

where the functions $A_m(\hat{m}_q)$ are presented in Appendix A.

Finally, we obtain the leading corrections to the total decay rate by taking the $n = m = 0$ moment. Of course, this result is not new; we present it for completeness and because we will need it to normalize the moments. We find

$$\Gamma(B \to X_q e\bar{\nu}) = \Gamma_0 \left[f_0(\hat{m}_q) + \frac{1}{2m_b^2} f_1(\hat{m}_q, \lambda_1, \lambda_2) + A_0(\hat{m}_q) \frac{\alpha_s}{\pi} \right], \tag{3.15}$$

where

$$f_0(\hat{m}_q) = 1 - 8\hat{m}_q^2 + 8\hat{m}_q^6 - \hat{m}_q^8 - 24\hat{m}_q^4 \ln \hat{m}_q,$$
$$f_1(\hat{m}_q, \lambda_1, \lambda_2) = \lambda_1 \left(1 - 8\hat{m}_q^2 + 8\hat{m}_q^6 - \hat{m}_q^8 - 24\hat{m}_q^4 \ln \hat{m}_q \right) \tag{3.16}$$
$$+ \lambda_2 \left(-9 + 24\hat{m}_q^2 - 72\hat{m}_q^4 + 72\hat{m}_q^6 - 15\hat{m}_q^8 - 72\hat{m}_q^4 \ln \hat{m}_q \right).$$

The power correction $f_1(\hat{m}_q, \lambda_1, \lambda_2)$ was first obtained in Refs. [3,4], and the perturbative correction $A_0(\hat{m}_q)$, which we present in Appendix A, was first found in Ref. [19]. It takes a simple form when $\hat{m}_q \to 0$, for which

$$A_0(\hat{m}_q = 0) = \frac{25}{6} - \frac{2}{3}\pi^2. \tag{3.17}$$

IV. APPLICATION TO B MESON DECAYS

The relations (2.3) allow moments of the physical parameters E_H and s_H to be expressed in terms of the parton-level moments. For the first two moments of s_H we find

$$\langle s_H \rangle = m_q^2 + \bar{\Lambda}^2 + (m_B^2 - 2\bar{\Lambda}m_B + \bar{\Lambda}^2 + \lambda_1 + 3\lambda_2) \langle \hat{s}_0 - \hat{m}_q^2 \rangle$$
$$+ (2\bar{\Lambda}m_B - 2\bar{\Lambda}^2 - \lambda_1 - 3\lambda_2)\langle \hat{E}_0 \rangle,$$
$$\langle s_H^2 \rangle = m_q^4 + 2\bar{\Lambda}^2 m_q^2 + 2m_B^2(m_q^2 + \bar{\Lambda}^2)\langle \hat{s}_0 - \hat{m}_q^2 \rangle$$
$$+ 2m_q^2(2\bar{\Lambda}m_B - 2\bar{\Lambda}^2 - \lambda_1 - 3\lambda_2)\langle \hat{E}_0 \rangle$$
$$+ (m_B^4 - 4\bar{\Lambda}m_B^3 + 6\bar{\Lambda}^2 m_B^2 + 2\lambda_1 m_B^2 + 6\lambda_2 m_B^2)\langle (\hat{s}_0 - \hat{m}_q^2)^2 \rangle \tag{4.1}$$
$$+ 4\bar{\Lambda}^2 m_B^2 \langle \hat{E}_0^2 \rangle + 4\bar{\Lambda}m_B^3 \langle \hat{E}_0(\hat{s}_0 - \hat{m}_q^2) \rangle,$$

where all expressions are valid to relative order $\Lambda_{\text{QCD}}^2/m_b^2$ and to all orders in m_q/m_b. It is straightforward to extend the analysis to higher moments $\langle s_H^n \rangle$. Similarly, the leading moments of E_H are given by

$$\langle E_H \rangle = \bar{\Lambda} - \frac{\lambda_1 + 3\lambda_2}{2m_B} + \left(m_B - \bar{\Lambda} + \frac{\lambda_1 + 3\lambda_2}{2m_B} \right) \langle \hat{E}_0 \rangle \,,$$

$$\langle E_H^2 \rangle = \bar{\Lambda}^2 + (2\bar{\Lambda}m_B - 2\bar{\Lambda}^2 - \lambda_1 - 3\lambda_2)\langle \hat{E}_0 \rangle \tag{4.2}$$
$$+ (m_B^2 - 2\bar{\Lambda}m_B + \bar{\Lambda}^2 + \lambda_1 + 3\lambda_2)\langle \hat{E}_0^2 \rangle \,.$$

On the right hand side of these expressions appear the parton level moments $\langle \hat{E}_0^m \hat{s}_0^n \rangle$, which are obtained from the quantities $\mathcal{M}^{(n,m)}$ by multiplying by the scale factor Γ_0/Γ.

A. Decay to an up quark

For $\hat{m}_q = 0$, such as in the quark decay $b \to u\ell\bar{\nu}$, we find the simple expressions

$$\langle \hat{s}_0 \rangle = \frac{91}{450} \frac{\alpha_s}{\pi} + \frac{13\lambda_1}{20m_B^2} + \frac{3\lambda_2}{4m_B^2} \,,$$

$$\langle \hat{s}_0^2 \rangle = \frac{5}{324} \frac{\alpha_s}{\pi} - \frac{16\lambda_1}{90m_B^2} \,, \tag{4.3}$$

$$\langle \hat{E}_0 \rangle = \frac{7}{20} \left(1 + \frac{137}{630} \frac{\alpha_s}{\pi} + \frac{13\lambda_1}{14m_B^2} + \frac{9\lambda_2}{2m_B^2} \right) \,,$$

$$\langle \hat{E}_0^2 \rangle = \frac{2}{15} \left(1 + \frac{257}{480} \frac{\alpha_s}{\pi} + \frac{31\lambda_1}{24m_B^2} + \frac{49\lambda_2}{8m_B^2} \right) \,,$$

$$\langle \hat{E}_0 \hat{s}_0 \rangle = \frac{9}{100} \frac{\alpha_s}{\pi} + \frac{23\lambda_1}{180m_B^2} + \frac{13\lambda_2}{60m_B^2} \,,$$

accurate up to corrections of order α_s/m_B^2 and $1/m_B^3$. These then yield the physical moments

$$\langle s_H \rangle = m_B^2 \left[\frac{91}{450} \frac{\alpha_s}{\pi} + \frac{7\bar{\Lambda}}{10m_B} \left(1 - \frac{227}{630} \frac{\alpha_s}{\pi} \right) + \frac{3}{10m_B^2} \left(\bar{\Lambda}^2 + \lambda_1 - \lambda_2 \right) \right] \,,$$

$$\langle s_H^2 \rangle = m_B^4 \left[\frac{5}{324} \frac{\alpha_s}{\pi} + \frac{604\bar{\Lambda}}{2025m_B} \frac{\alpha_s}{\pi} + \frac{8}{15m_B^2} \left(\bar{\Lambda}^2 - \frac{\lambda_1}{3} \right) \right] \,, \tag{4.4}$$

$$\langle E_H \rangle = \frac{7}{20} m_B \left[1 + \frac{137}{630} \frac{\alpha_s}{\pi} + \frac{13\bar{\Lambda}}{7m_B} \left(1 - \frac{137}{1170} \frac{\alpha_s}{\pi} \right) + \frac{12\lambda_2}{7m_B^2} \right] \,,$$

$$\langle E_H^2 \rangle = \frac{2}{15} m_B^2 \left[1 + \frac{257}{480} \frac{\alpha_s}{\pi} + \frac{13\bar{\Lambda}}{4m_B} \left(1 + \frac{17}{780} \frac{\alpha_s}{\pi} \right) + \frac{13}{4m_B^2} \left(\bar{\Lambda}^2 - \frac{4}{39}\lambda_1 + \frac{5}{13}\lambda_2 \right) \right] \,.$$

B. Decay to a charm quark

For $b \to c$ decays, we make use of the fact that the charm quark is also heavy to write m_c/m_b as a power series in $1/m_B, 1/m_D$. Let us define the spin-averaged meson masses,

$$\bar{m}_D \equiv \frac{m_D + 3m_{D^*}}{4} = m_c + \bar{\Lambda} - \frac{\lambda_1}{2m_D} + \ldots \simeq 1975 \,\text{MeV} \tag{4.5}$$

$$\bar{m}_B \equiv \frac{m_B + 3m_{B^*}}{4} = m_b + \bar{\Lambda} - \frac{\lambda_1}{2m_B} + \ldots \simeq 5313 \,\text{MeV} \,,$$

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which gives

$$\frac{m_c}{m_b} = \frac{\bar{m}_D}{\bar{m}_B} - \frac{\bar{\Lambda}}{m_B}\left(1 - \frac{\bar{m}_D}{\bar{m}_B}\right) - \frac{\bar{\Lambda}^2}{m_B^2}\left(1 - \frac{\bar{m}_D}{\bar{m}_B}\right) + \frac{\lambda_1}{2m_Bm_D}\left(1 - \frac{\bar{m}_D^2}{\bar{m}_B^2}\right) \tag{4.6}$$

$$= 0.372 - 0.628\frac{\bar{\Lambda}}{m_B} - 0.628\frac{\bar{\Lambda}^2}{m_B^2} + 1.16\frac{\lambda_1}{m_B^2} \, ,$$

accurate up to corrections of order $1/m_Bm_D^2$. This substitution introduces additional $\mathcal{O}(1/m_B, 1/m_Bm_D)$ corrections to the parton level moments.[2] We find

$$\langle \hat{s}_0 - \hat{m}_q^2 \rangle = 0.051\frac{\alpha_s}{\pi} + 0.16\frac{\alpha_s}{\pi}\frac{\bar{\Lambda}}{m_B} + 0.51\frac{\lambda_1}{m_B^2} + 1.14\frac{\lambda_2}{m_B^2} \, ,$$

$$\langle (\hat{s}_0 - \hat{m}_q^2)^2 \rangle = 0.0053\frac{\alpha_s}{\pi} + 0.017\frac{\alpha_s}{\pi}\frac{\bar{\Lambda}}{m_B} - 0.14\frac{\lambda_1}{m_B^2} \, , \tag{4.7}$$

$$\langle \hat{E}_0 \rangle = 0.489\left[1 + 0.043\frac{\alpha_s}{\pi} - 0.78\frac{\bar{\Lambda}}{m_B}\left(1 - 0.12\frac{\alpha_s}{\pi}\right) - 0.44\frac{\bar{\Lambda}^2}{m_B^2} + 1.96\frac{\lambda_1}{m_B^2} + 2.53\frac{\lambda_2}{m_B^2}\right]$$

$$\langle \hat{E}_0^2 \rangle = 0.242\left[1 + 0.099\frac{\alpha_s}{\pi} - 1.50\frac{\bar{\Lambda}}{m_B}\left(1 - 0.12\frac{\alpha_s}{\pi}\right) - 0.19\frac{\bar{\Lambda}^2}{m_B^2} + 3.64\frac{\lambda_1}{m_B^2} + 4.69\frac{\lambda_2}{m_B^2}\right]$$

$$\langle \hat{E}_0(\hat{s}_0 - \hat{m}_q^2) \rangle = 0.030\frac{\alpha_s}{\pi} + 0.077\frac{\alpha_s}{\pi}\frac{\bar{\Lambda}}{m_B} + 0.18\frac{\lambda_1}{m_B^2} + 0.53\frac{\lambda_2}{m_B^2} \, ,$$

and for the total rate,

$$\frac{\Gamma}{\Gamma_0} = 0.369\left(1 - 1.54\frac{\alpha_s}{\pi} + 3.35\frac{\bar{\Lambda}}{m_B}\left(1 - 1.86\frac{\alpha_s}{\pi}\right) + 5.81\frac{\bar{\Lambda}^2}{m_B^2} - 5.69\frac{\lambda_1}{m_B^2} - 7.47\frac{\lambda_2}{m_B^2}\right) . \tag{4.8}$$

The physical moments are then

$$\langle s_H - \bar{m}_D^2 \rangle = m_B^2\left[0.051\frac{\alpha_s}{\pi} + 0.26\frac{\bar{\Lambda}}{m_B}\left(1 + 0.38\frac{\alpha_s}{\pi}\right) + 0.26\frac{1}{m_B^2}\left(\bar{\Lambda}^2 + 3.9\lambda_1 - 1.2\lambda_2\right)\right] \, ,$$

$$\langle (s_H^2 - \bar{m}_D^2)^2 \rangle = m_B^4\left[0.0053\frac{\alpha_s}{\pi} + 0.067\frac{\bar{\Lambda}}{m_B}\frac{\alpha_s}{\pi} + 0.065\frac{1}{m_B^2}\left(\bar{\Lambda}^2 - 2.1\lambda_1\right)\right] \tag{4.9}$$

$$\langle E_H \rangle = 0.489m_B\left[1 + 0.043\frac{\alpha_s}{\pi} + 0.27\frac{\bar{\Lambda}}{m_B}\left(1 + 0.19\frac{\alpha_s}{\pi}\right) + 0.33\frac{1}{m_B^2}\left(\bar{\Lambda}^2 + 4.3\lambda_1 + 2.9\lambda_2\right)\right] \, ,$$

$$\langle E_H^2 \rangle = 0.242m_B^2\left[1 + 0.099\frac{\alpha_s}{\pi} + 0.55\frac{\bar{\Lambda}}{m_B}\left(1 + 0.28\frac{\alpha_s}{\pi}\right) + 0.75\frac{1}{m_B^2}\left(\bar{\Lambda}^2 + 3.5\lambda_1 + 2.2\lambda_2\right)\right] \, ,$$

where the corrections to these expressions are of order α_s/m_B^2 and $1/m_B^3$. Note that in these expansions, there is no hidden dependence on the quark masses; here the coefficients are functions *only* of physical quantities.

----

[2]In the rest of this section, we will treat m_B/m_D as $\mathcal{O}(1)$. Thus, by $\mathcal{O}(1/m_B)$ we denote corrections both of order $1/m_B$ and $1/m_D$.

We can also expand our results about the small velocity (SV) limit [20], $\Lambda_{\text{QCD}} \ll m_b - m_c \ll m_c < m_b$. In this limit, only the D and D^* states are produced. Expanding in powers of $1 - \hat{m}_c$ and $\Lambda_{\text{QCD}}/(m_b - m_c)$, we find

$$\langle s_H - \bar{m}_D^2 \rangle = m_B^2 (1 - \hat{m}_c)^3 \left[\frac{4}{21} \frac{\alpha_s}{\pi} + \frac{4\bar{\Lambda}}{m_b - m_c} + \frac{\lambda_2 - 2\lambda_1}{(m_b - m_c)^2} + \dots \right] + \mathcal{O}(1 - \hat{m}_c)^4. \quad (4.10)$$

Note that in the SV limit, as expected, the average invariant mass of the final hadron is $\bar{m}_D$, and therefore the D and D^* are produced in the ratio 1:3. Furthermore, corrections to the average invariant mass due to production of excited states are suppressed by $(1 - \hat{m}_c)^3$.

Finally, all of these results may be applied to the inclusive decays of the Λ_b, with the obvious replacements $\bar{\Lambda} \to \bar{\Lambda}_\Lambda, \lambda_1 \to \lambda_{1\Lambda}, \ \lambda_2 \to 0$, where

$$m_{\Lambda_b} = m_b + \bar{\Lambda}_\Lambda - \frac{\lambda_{1\Lambda}}{2m_b} + \dots . \quad (4.11)$$

We also note that, in order to avoid introducing factors of $\bar{\Lambda}$ and λ_1 from the meson sector into the expansion, in Eq. (4.5) the spin-averaged meson masses should be replaced by baryon masses. Since the uncertainty in m_{Λ_b} is ± 50 MeV [21], this introduces large uncertainties into the moments of Λ_b spectra, when written in terms of physical masses.

V. A LOWER BOUND ON $\bar{\Lambda}$

Although the invariant mass spectrum for $B \to X_c e\bar{\nu}$ has not been measured, we may use the recent OPAL measurement [22] of the branching ratio to the narrow P wave charmed mesons, the $D_1(2420)$ and $D_2^*(2460)$, to place a lower limit on $\bar{\Lambda}$. In Ref. [22], the branching ratio to these states was estimated to be $34 \pm 7\%$. From Ref. [23], we take the ratio of D to D^* production in $B \to X_c \ell\nu$, for which several experimental measurements have been combined consistently:

$$\frac{\Gamma(B \to D^* e\bar{\nu}_e)}{\Gamma(B \to De\bar{\nu}_e) + \Gamma(B \to D^* e\bar{\nu}_e)} = 0.65 \pm 0.06 . \quad (5.1)$$

We estimate the minimum value for the first moment of the invariant mass spectrum by taking the 1σ limits of these experimental results. Hence, we take a 27% branching fraction to the P wave states, and assume that the rest of the branching fraction is saturated by the D and D^* in the ratio 0.41:0.59. The minimum value for the first moment $\langle s_H - \bar{m}_D^2 \rangle$ is then

$$\langle s_H - \bar{m}_D^2 \rangle_{\text{min.}} \simeq 0.27 \left[(2.450\,\text{GeV})^2 - (1.975\,\text{GeV})^2 \right] + 0.43 \left[(2.010\,\text{GeV})^2 - (1.975\,\text{GeV})^2 \right]$$
$$+ 0.30 \left[(1.869\,\text{GeV})^2 - (1.975\,\text{GeV})^2 \right] \quad (5.2)$$
$$= 0.51\,\text{GeV}^2 .$$

For the second moment, we will be conservative and neglect the small (and positive) contribution of the ground state doublet. We find

$$\langle (s_H - \bar{m}_D^2)^2 \rangle_{\text{min.}} \simeq 0.27 \times \left[(2.450\,\text{GeV})^2 - (1.975\,\text{GeV})^2 \right]^2 = 1.2\,\text{GeV}^4. \quad (5.3)$$
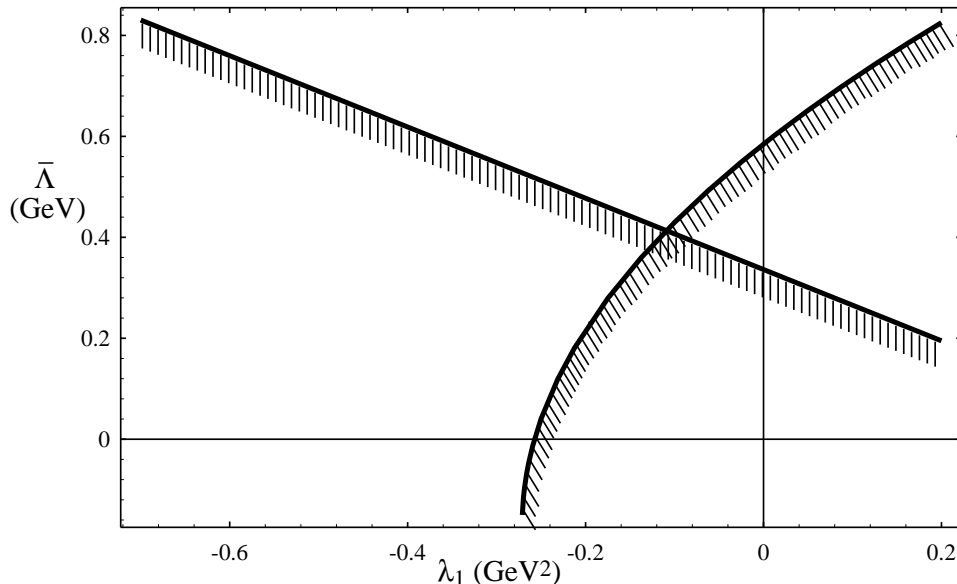
FIG. 4. Correlated one-loop limits on $\bar{\Lambda}$ and λ_1. The shaded region is ruled out by our analysis of the first two moments of $(s_H - \bar{m}_D^2)$.

Solving Eq. (4.9) for the first moment, we find

$$\bar{\Lambda} > \left[0.41 - 1.41 \frac{\alpha_s}{\pi} - 0.07 \left(\frac{\lambda_1}{0.1 \, \text{GeV}^2} \right) \right] \text{GeV}. \tag{5.4}$$

The nonperturbative parameters $\bar{\Lambda}$ and λ_1 are well-defined only at a given order in pertur-bation theory [24]. Our limits apply to these quantities defined at one loop. We will use the coupling constant $\alpha_s(m_b) = 0.2$ in what follows. Since λ_1 is closely related to minus the kinetic energy of the b quark in the B meson, it is expected to be negative. Under this assumption, we obtain the lower bound

$$\bar{\Lambda} > 340 \, \text{MeV}. \tag{5.5}$$

This limit corresponds to an upper bound on the b quark pole mass of $m_b^{\text{pole}} < 4.97 \, \text{GeV}$. In Ref. [25], the stringent inequality $\lambda_1 \leq -3\lambda_2 \approx -0.35 \, \text{GeV}^2$ was proposed; in such a case we would find the more restrictive bound

$$\bar{\Lambda} > 570 \, \text{GeV}, \tag{5.6}$$

corresponding to the upper limit $m_b^{\text{pole}} < 4.71 \, \text{GeV}$.

If we also use the bound (5.3) on the second moment, we may relax the assumptions on λ_1 and obtain correlated limits on $\bar{\Lambda}$ and λ_1. These are plotted in Fig. 4. By this method, we obtain the lower bound

$$\bar{\Lambda} > 410 \, \text{MeV}, \tag{5.7}$$

independent of λ_1. Where the bound on $\bar{\Lambda}$ is saturated, $\lambda_1 = -0.11 \, \text{GeV}^2$. Our result implies the upper limit $m_b^{\text{pole}} < 4.89 \, \text{GeV}$, *without* any assumption on λ_1 being made.

This approach complements the recent proposal [26] that $\bar{\Lambda}$ and λ_1 be extracted from moments of the photon energy spectrum in the rare process $B \to X_s \gamma$.

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VI. HIGHER LOOPS

In order to apply our results consistently, it is important to know the scale at which to evaluate $\alpha_s(\mu)$ in the radiative corrections. It has been shown recently [27] that the naïve choice $\mu = m_b$ significantly underestimates the size of the two-loop effects. In particular, the prescription of Brodsky, Lepage and Mackenzie (BLM) [14] suggests that the relevant scale for the radiative corrections in $B \to X_u e\bar{\nu}$ decay is $\mu \sim 0.07\,m_b$, when expressed in terms of the b quark pole mass, indicating that two-loop effects are substantial.

It has also been stressed, however, that the BLM prescription may give a misleadingly low scale when relating unphysical quantities [28]. In particular, $\bar{\Lambda}$ is related to the pole mass of the heavy quark, which is not an observable, and in fact suffers from an inherent ambiguity in its definition [24]. In this section, we show that although the BLM prescription indicates that radiative corrections to the first two moments of the invariant mass spectrum for semileptonic $b \to u$ decay are uncontrolled when expressed in terms of the HQET parameter $\bar{\Lambda}$, they are well behaved when expressed in terms of physical quantities.

The portion of the two loop correction to Eq. (4.4) which is proportional to the QCD evolution parameter β_0 may be determined from the one loop correction, calculated with a massive gluon in the final state, using the techniques of Ref. [29]. Some of the details of the computation are given in Appendix B; we find, for $\hat{m}_q = 0$,

$$\frac{1}{m_B^2}\langle s_H \rangle = \frac{91}{450}\frac{\alpha_s(m_b)}{\pi} + \left(\frac{53}{180}\pi^2 - \frac{276043}{108000}\right)\beta_0\left(\frac{\alpha_s(m_b)}{\pi}\right)^2 + \frac{7}{10}\frac{\bar{\Lambda}}{m_B} + \ldots \qquad (6.1)$$

$$\simeq 0.20\frac{\alpha_s(m_b)}{\pi} + 3.15\left(\frac{\alpha_s(m_b)}{\pi}\right)^2 + \frac{7}{10}\frac{\bar{\Lambda}}{m_B} + \ldots$$

$$\simeq 0.013 + 0.013 + \frac{7}{10}\frac{\bar{\Lambda}}{m_B} + \ldots ,$$

where $\beta_0 = 11 - 2n_f/3$ and in the last line we have taken $\alpha_s(m_b) \simeq 0.2$. Clearly the perturbation expansion is poorly controlled. In the BLM scale-setting prescription, the scale μ_{BLM} of the coupling is chosen such that the two-loop contribution proportional to β_0 is absorbed into the one-loop correction. The poor convergence of the series is reflected in the low BLM scale for this process:

$$\mu_{\mathrm{BLM}} = m_b \exp\left[-2\left(\frac{53}{180}\pi^2 - \frac{276043}{108000}\right)\bigg/\frac{91}{450}\right] \simeq 0.03 m_b \simeq 140\,\mathrm{MeV} . \qquad (6.2)$$

However, our expression for $\langle s_H \rangle$ is given in terms of the unphysical parameter $\bar{\Lambda}$. While this is perfectly acceptable as an intermediate step, since we are ultimately interested only in relations between observable quantities, it has the effect of making the perturbative expansion appear ill-behaved. Instead, let us define the "decay mass" of the b quark, m_b^Γ, via the charmless semileptonic partial width of the B meson,

$$\Gamma(B \to X_u e\bar{\nu}_e) \equiv \frac{G_F^2|V_{ub}|^2}{192\pi^3}(m_b^\Gamma)^5. \qquad (6.3)$$

The decay mass m_b^Γ is a physical observable and is therefore well-defined. It is related to the pole mass via the expansion

$$m_b^\Gamma = m_b^{\text{pole}} \left[1 + \left(\frac{5}{6} - \frac{2}{15}\pi^2 \right) \frac{\alpha_s(m_b)}{\pi} - (0.596\beta_0 + c) \left(\frac{\alpha_s(m_b)}{\pi} \right)^2 + \ldots + \frac{\lambda_1 - 9\lambda_2}{10m_b^2} + \ldots \right].$$
(6.4)

The two-loop term proportional to β_0, which one expects to dominate the two loop result, was calculated in Ref. [27]. The constant c has not been computed. Since m_b^{pole} is not well-defined due to renormalon effects, the perturbation series in Eq. (6.4) has a renormalon ambiguity at $\mathcal{O}(1/m_b)$.

Defining a physical version of the parameter $\bar{\Lambda}$,[3]

$$\bar{\Lambda}_b^\Gamma \equiv m_B - m_b^\Gamma,$$
(6.5)

we have

$$\frac{\bar{\Lambda}}{m_B} = \frac{\bar{\Lambda}_b^\Gamma}{m_B} + \left(\frac{5}{6} - \frac{2}{15}\pi^2 \right) \frac{\alpha_s(m_b)}{\pi} - (0.596\beta_0 + c) \left(\frac{\alpha_s(m_b)}{\pi} \right)^2 + \mathcal{O}(\alpha_s^3, 1/m_b^2),$$
(6.6)

and Eq. (6.1) becomes

$$\frac{1}{m_B^2}\langle \hat{s}_H \rangle \simeq (0.202 - 0.337)\frac{\alpha_s(m_b)}{\pi} + (3.151 - 3.752) \left(\frac{\alpha_s(m_b)}{\pi} \right)^2 + \frac{7}{10}\frac{\bar{\Lambda}_b^\Gamma}{m_B}$$

$$\simeq -0.135\frac{\alpha_s(m_b)}{\pi} - 0.601 \left(\frac{\alpha_s(m_b)}{\pi} \right)^2 + \frac{7}{10}\frac{\bar{\Lambda}_b^\Gamma}{m_B}$$

$$\simeq -0.0086 - 0.0024 + \frac{7}{10}\frac{\bar{\Lambda}_b^\Gamma}{m_B}.$$
(6.7)

The perturbation expansion clearly has improved dramatically. The corresponding BLM scale is now

$$\mu_{\text{BLM}} = m_b \exp\left[-(2/9)0.601/0.135\right] \simeq 0.37\, m_b,$$
(6.8)

which is significantly greater than before.

It is interesting to note that the cancellation we observe in Eq. (6.7) persists at higher orders in the bubble sum. Using the techniques of Ref. [30] we can calculate the n loop bubble graph, from which we may extract the coefficient of $\beta_0^n \alpha_s^{n+1}$ in the perturbative expansion for $\langle \hat{s}_0 \rangle$. Although there is no reason to believe that this is the dominant contribution at this order, since there is no $\beta_0 \to \infty$ limit of QCD in which the quark and gluon bubble graphs dominate, it does give one class of contributions to the n loop graphs which displays a factorial divergence at large orders in perturbation theory.

Using the techniques of Ref. [30], the perturbation series in Eq. (6.1) continues as

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[3]Note that unlike $\bar{\Lambda}$, $m_Q - m_q^\Gamma$ is not universal for heavy quarks, and differs in the b and c systems. Since it explicitly violates heavy quark symmetry, it is not useful to reformulate HQET in terms of this more physical quantity.

$$\frac{1}{m_B^2}\langle\hat{s}_H\rangle = 0.202\frac{\alpha_s(m_b)}{\pi} + 3.151\left(\frac{\alpha_s(m_b)}{\pi}\right)^2 + 51.91\left(\frac{\alpha_s(m_b)}{\pi}\right)^3 \tag{6.9}$$

$$+940.52\left(\frac{\alpha_s(m_b)}{\pi}\right)^4 + 19347.5\left(\frac{\alpha_s(m_b)}{\pi}\right)^5 + \frac{7}{10}\frac{\bar{\Lambda}}{m_B} + \ldots,$$

and using the results of Ref. [31] for the higher order relation between m_b and m_b^{Γ}, Eq. (6.7) continues as

$$\frac{1}{m_B^2}\langle\hat{s}_H\rangle = (0.202 - 0.337)\frac{\alpha_s(m_b)}{\pi} + (3.151 - 3.752)\left(\frac{\alpha_s(m_b)}{\pi}\right)^2$$

$$+(51.91 - 50.37)\left(\frac{\alpha_s(m_b)}{\pi}\right)^3 + (940.52 - 782.42)\left(\frac{\alpha_s(m_b)}{\pi}\right)^4$$

$$+(19347.5 - 14424.2)\left(\frac{\alpha_s(m_b)}{\pi}\right)^5 + \frac{7}{10}\frac{\bar{\Lambda}_b^{\Gamma}}{m_B} + \ldots$$

$$= -0.135\frac{\alpha_s(m_b)}{\pi} - 0.601\left(\frac{\alpha_s(m_b)}{\pi}\right)^2 + 1.56\left(\frac{\alpha_s(m_b)}{\pi}\right)^3 \tag{6.10}$$

$$+148.1\left(\frac{\alpha_s(m_b)}{\pi}\right)^4 + 4923.\left(\frac{\alpha_s(m_b)}{\pi}\right)^5 + \frac{7}{10}\frac{\bar{\Lambda}_b^{\Gamma}}{m_B} + \ldots$$

$$\simeq -0.0086 - 0.0024 + 0.0004 + 0.0026 + 0.0051 + \frac{7}{10}\frac{\bar{\Lambda}_b^{\Gamma}}{m_B} + \ldots$$

Note that even at higher orders there is significant cancellation between the two series. The remaining bad behaviour presumably reflects the presence of unphysical parameters (such as $\bar{\Lambda}^2$ and λ_1) at higher orders in the operator product expansion. Assuming the series is asymptotic, the size of the smallest term in the expansion gives a measure of the uncertainty in the sum of the series.

We find a similar cancellation for the second moment of s_H. However, the cancellation occurs between terms of different orders in β_0, providing a simple example of a large effect (in this case, a significant cancellation) which is missed by the "naïve nonabelianization" of Refs. [30,31].[4] For $\hat{m}_q = 0$ and to order $1/m_b$, we find

$$\frac{1}{m_B^4}\langle s_H^2\rangle = \langle\hat{s}_0^2\rangle + 4\frac{\bar{\Lambda}}{m_B}\left(\langle\hat{E}_0\hat{s}_0\rangle - \langle\hat{s}_0^2\rangle\right) + \ldots. \tag{6.11}$$

Since $\langle\hat{E}_0\hat{s}_0\rangle$ and $\langle\hat{s}_0^2\rangle$ are both order α_s, there is no $\beta_0\alpha_s^2$ term introduced by expressing $\langle s_H^2\rangle$ in terms of $\bar{\Lambda}_b^{\Gamma}$. However, naïvely counting of powers of β_0 does not work here, because $\langle\hat{s}_0^2\rangle \ll \langle\hat{E}_0\hat{s}_0\rangle, \langle\hat{s}_0\rangle$. Instead, the $\mathcal{O}(\beta_0\alpha_s^2)$ correction to $\langle\hat{s}_0^2\rangle$ is the same order as the $\mathcal{O}(\alpha_s^2)$ term introduced by expressing $\langle s_H^2\rangle$ in terms of $\bar{\Lambda}_b^{\Gamma}$. Using the $\beta_0\alpha_s^2$ term as an estimate of the full two loop correction to $\langle\hat{s}_0^2\rangle$ alone, we find, using the same technique as before,

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[4]We also note that, since the cancellation occurs between different powers of β_0, it has nothing to do with the cancellation of renormalons in $\langle s_H^2\rangle$, at least in the bubble approximation.

$$\langle s_0^2 \rangle = \frac{5}{324}\frac{\alpha_s(m_b)}{\pi} + \left(\frac{277}{648}\pi^2 - \frac{25511}{6075}\right)\beta_0\left(\frac{\alpha_s(m_b)}{\pi}\right)^2 + \ldots , \qquad (6.12)$$

and so

$$\frac{1}{m_B^4}\langle s_H^2 \rangle = 0.015\frac{\alpha_s(m_b)}{\pi} + 0.0196\beta_0\left(\frac{\alpha_s(m_b)}{\pi}\right)^2 + 0.298\left(\frac{\bar{\Lambda}_b^\Gamma}{m_B} - 0.48\frac{\alpha_s(m_b)}{\pi}\right)\frac{\alpha_s(m_b)}{\pi} + \ldots$$

$$= 0.015\frac{\alpha_s(m_b)}{\pi} + 0.033\left(\frac{\alpha_s(m_b)}{\pi}\right)^2 + 0.298\frac{\bar{\Lambda}_b^\Gamma}{m_B}\frac{\alpha_s(m_b)}{\pi} + \ldots$$

$$\simeq 9.5 \times 10^{-4} + 1.3 \times 10^{-4} + 0.298\frac{\bar{\Lambda}_b^\Gamma}{m_B}\frac{\alpha_s(m_b)}{\pi} + \ldots . \qquad (6.13)$$

VII. SUMMARY AND CONCLUSIONS

We have used the operator product expansion and the heavy quark limit to compute the hadronic energy and invariant mass spectra in semileptonic heavy meson decays. Our expressions are complete up to order α_s in perturbation theory, and up to order α_s/m_b and $1/m_b^2$ in the heavy quark expansion. The effects of finite final state quark masses have been taken into account, so it is possible to apply our results to the important decay $b \to c\,\ell\nu$.

Our analysis provides a test of the applicability of the OPE to these decays, and of the crucial underlying concept of global duality. Only appropriately weighted integrals of the theoretical spectra may be compared meaningfully with experiment, and we focus on the leading moments. As an initial application, we used the recent measurement of the B branching fraction to excited D mesons to put bounds on the nonperturbative parameters $\bar{\Lambda}$ and λ_1. We found $\bar{\Lambda} > 410\,\mathrm{MeV}$, which led to a constraint on the b quark pole mass, $m_b^{\mathrm{pole}} < 4.89\,\mathrm{GeV}$. More stringent tests will have to await the availability of more precise data. The success or failure of our predictions will determine the confidence with which one will trust these theoretical techniques in the extraction of CKM matrix elements from semileptonic bottom and charm decays.

We also investigated the behaviour of the perturbation series at higher order in α_s, to gain insight into the trustworthiness of the lowest order calculation and the choice of renormalization scale μ. We found that when written in terms of the unphysical quantity $\bar{\Lambda}$, the perturbation series for $\langle s_H \rangle$ seems to be quite badly behaved, with a BLM scale μ_{BLM} too low to be meaningful. However, when we define a more physical "decay mass" m_b^Γ, and through it a physical $\bar{\Lambda}_b^\Gamma$, the perturbation series improves dramatically. The cancellations which we find persist to higher order in α_s, at least when one includes the leading powers of β_0. The perturbation series for $\langle s_H^2 \rangle$ also is improved by writing it in terms of physical quantities, although the required cancellations occur between terms of different order in β_0, indicating the failure of "naïve nonabelianization" in the computation of this quantity.

We have focused on the application to B decays, however, the BLM analysis suggests that the perturbative corrections are under control for D decays as well. The extension of our results to charm is straightforward.

APPENDIX A: THE PARTON LEVEL HADRONIC ENERGY SPECTRUM

In this appendix we discuss the corrections to the parton level hadronic energy spectrum, $\mathrm{d}\Gamma_0/\mathrm{d}\hat{E}_0$. Both the perturbative and the power corrections are somewhat unwieldy; we present them here for completeness.

The power correction may be computed by integrating the doubly differential spectrum (3.2) over $\hat{s}_0$. The integral will be nonzero only if $\hat{E}_0 \leq \frac{1}{2}(1 + \hat{m}_q^2)$, because, as discussed in Section II, only in this case does the integration region overlap with the one-particle pole at $\hat{s}_0 = \hat{m}_q^2$. This is a reflection of the fact that the maximum energy a single quark can carry away from the decay is $\frac{1}{2}(1 + \hat{m}_q^2)$. In the presence of additional strongly interacting particles such as gluons, the total hadronic energy $\hat{E}_0$ can exceed $\frac{1}{2}(1 + \hat{m}_q^2)$.

However, the initial motion of the b quark inside the B meson can produce fluctuations of the maximum allowed final quark energy above $\frac{1}{2}(1 + \hat{m}_q^2)$. These fluctuations appear in the differential rate as singular functions $\delta(\hat{E}_0 - \frac{1}{2}(1 + \hat{m}_q^2))$ and $\delta'(\hat{E}_0 - \frac{1}{2}(1 + \hat{m}_q^2))$, which are resummed into a smooth function extending beyond the parton model endpoint. For a more detailed discussion of this subject see Refs. [4,17,18].

Including the leading power corrections, then, the expression for the hadronic energy spectrum is given by[5]

$$\frac{1}{\Gamma_0}\frac{\mathrm{d}\Gamma}{\mathrm{d}\hat{E}_0} = 16\sqrt{\hat{E}_0^2 - \hat{m}_q^2}\left[3\hat{E}_0 - 4\hat{E}_0^2 - 2\hat{m}_q^2 + 3\hat{E}_0\hat{m}_q^2\right]$$

$$+ \frac{16}{\sqrt{\hat{E}_0^2 - \hat{m}_q^2}}\left[\frac{\lambda_1}{2m_b^2}\left(-6\hat{E}_0^2 + 12\hat{E}_0^3 + \frac{20}{3}\hat{E}_0^4 + 3\hat{m}_q^2 - 6\hat{E}_0\hat{m}_q^2 - \frac{52}{3}\hat{E}_0^2\hat{m}_q^2 + \frac{23}{3}\hat{m}_q^4\right)\right.$$

$$\left.+ \frac{\lambda_2}{2m_b^2}\left(-3\hat{E}_0 - 6\hat{E}_0^2 + 36\hat{E}_0^3 + 20\hat{E}_0^4 + 3\hat{m}_q^2 - 21\hat{E}_0\hat{m}_q^2 - 52\hat{E}_0^2\hat{m}_q^2 + 23\hat{m}_q^4\right)\right]$$

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[5]We do not agree with the expression presented in Ref. [12].

$$+(1-\hat{m}_q^2)^3\left[\frac{\lambda_1}{3m_b^2}(5-\hat{m}_q^2)-\frac{\lambda_2}{m_b^2}(1-5\hat{m}_q^2)\right]\delta\left(\hat{E}_0-\frac{1}{2}(1+\hat{m}_q^2)\right)$$

$$+\frac{\lambda_1}{6m_b^2}(1-\hat{m}_q^2)^5\,\delta'\left(\hat{E}_0-\frac{1}{2}(1+\hat{m}_q^2)\right)+\mathcal{O}(\alpha_s,1/m_b^3). \tag{A1}$$

Integrating this expression with respect to $\hat{E}_0$, we find the power corrections (3.13) to the moments $\langle\hat{E}_0\rangle$ and $\langle\hat{E}_0^2\rangle$.

The expression for the perturbative correction to the hadronic energy spectrum is even more cumbersome. For the complete spectrum at finite $\hat{m}_q$, we refer the reader to Ref. [13]. As an illustration we present here the perturbative corrections at $\hat{m}_q=0$, separately for $\hat{E}_0<\frac{1}{2}$ and $\hat{E}_0>\frac{1}{2}$:

$$\frac{1}{\Gamma_0}\frac{d\Gamma}{d\hat{E}_0}\bigg|_{0<\hat{E}_0<\frac{1}{2}}^{\text{pert.}}=\frac{\alpha_s}{\pi}E_0^2\bigg[36-\frac{32}{3}\pi^2-\frac{496}{9}\hat{E}_0+\frac{128}{9}\pi^2\hat{E}_0+\frac{52}{3}\hat{E}_0^2-\frac{112}{45}\hat{E}_0^3+\frac{64}{135}\hat{E}_0^4$$

$$-24\ln(2\hat{E}_0)+\frac{64}{3}\hat{E}_0\ln(2\hat{E}_0)+16\ln(2\hat{E}_0)\ln(1-2\hat{E}_0) \tag{A2}$$

$$-\frac{64}{3}\hat{E}_0\ln(2\hat{E}_0)\ln(1-2\hat{E}_0)+16\text{Li}_2(2\hat{E}_0)-\frac{64}{3}\hat{E}_0\text{Li}_2(2\hat{E}_0)\bigg],$$

$$\frac{1}{\Gamma_0}\frac{d\Gamma}{d\hat{E}_0}\bigg|_{\frac{1}{2}<\hat{E}_0<1}^{\text{pert.}}=\frac{\alpha_s}{\pi}\bigg[\frac{208}{45}+\frac{1058}{45}\hat{E}_0-\frac{646}{9}\hat{E}_0^2+\frac{1592}{27}\hat{E}_0^3-\frac{52}{3}\hat{E}_0^4+\frac{112}{45}\hat{E}_0^5-\frac{64}{135}\hat{E}_0^6$$

$$+\frac{5}{9}\ln(2\hat{E}_0-1)+\frac{8}{3}\hat{E}_0\ln(2\hat{E}_0-1)+\frac{16}{3}\hat{E}_0^2\ln(2\hat{E}_0-1) \tag{A3}$$

$$-\frac{64}{9}\hat{E}_0^3\ln(2\hat{E}_0-1)+8\hat{E}_0^2\ln^2(2\hat{E}_0-1)-\frac{32}{3}\hat{E}_0^3\ln^2(2\hat{E}_0-1)$$

$$-16\hat{E}_0^2\ln(2\hat{E}_0)\ln(2\hat{E}_0-1)+\frac{64}{3}\hat{E}_0^3\ln(2\hat{E}_0)\ln(2\hat{E}_0-1)$$

$$-16\hat{E}_0^2\text{Li}_2\left(\frac{1}{2\hat{E}_0}\right)+\frac{64}{3}\hat{E}_0^3\text{Li}_2\left(\frac{1}{2\hat{E}_0}\right)+16\hat{E}_0^2\text{Li}_2\left(\frac{2\hat{E}_0-1}{2\hat{E}_0}\right)$$

$$-\frac{64}{3}\hat{E}_0^3\text{Li}_2\left(\frac{2\hat{E}_0-1}{2\hat{E}_0}\right)\bigg].$$

This spectrum is shown in Fig. 5. The logarithmic divergence as $\hat{E}_0\to\frac{1}{2}$ from above is integrable. The region $\hat{E}_0>\frac{1}{2}$ receives contributions only from brehmsstrahlung graphs. Note that the spectrum falls extremely rapidly with increasing $\hat{E}_0$.

The radiative corrections $A_m(\hat{m}_q)$ to the moments $\langle\hat{E}_0^m\rangle$ may be obtained by integrating the full expressions found in Ref. [13]. We find

$$A_0(\hat{m}_q)=\frac{25}{6}-\frac{2}{3}\pi^2-\frac{478}{9}\hat{m}_q^2+\frac{64}{3}\pi^2(1+\hat{m}_q^2)\hat{m}_q^3-\frac{32}{3}\pi^2\hat{m}_q^4+\frac{478}{9}\hat{m}_q^6$$

$$-\left(\frac{25}{6}+\frac{2}{3}\pi^2\right)\hat{m}_q^8-\frac{2}{3}(36+\hat{m}_q^4)\hat{m}_q^4\ln^2\hat{m}_q^2$$

$$+\left(-\frac{40}{3}\hat{m}_q^2+\frac{256}{3}(1+\hat{m}_q^2)\ln(1+\hat{m}_q)\hat{m}_q^3-60\hat{m}_q^4+\frac{8}{9}\hat{m}_q^6-\frac{34}{9}\hat{m}_q^8\right)\ln\hat{m}_q^2$$

$$+\left(-\frac{34}{9}+\frac{128}{9}\hat{m}_q^2-\frac{128}{9}\hat{m}_q^6+\frac{34}{9}\hat{m}_q^8\right)\ln(1-\hat{m}_q^2) \tag{A4}$$

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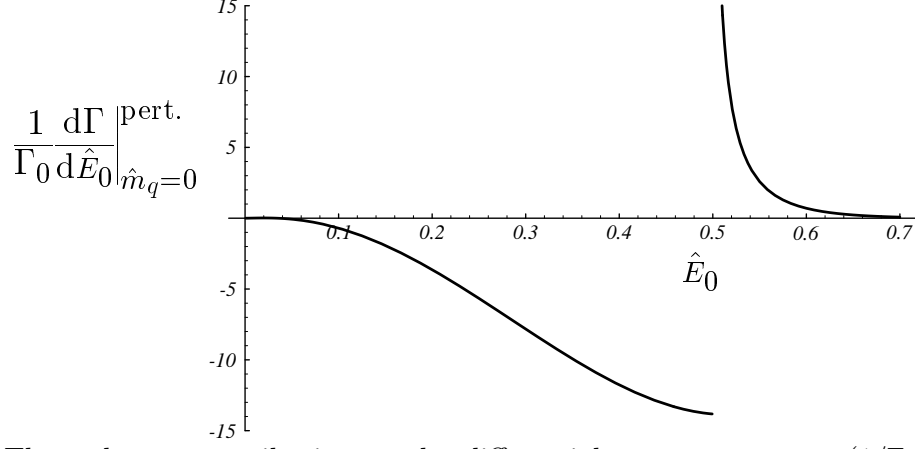FIG. 5. The order α_s contribution to the differential energy spectrum $(1/\Gamma_0)\mathrm{d}\Gamma/\mathrm{d}\hat{E}_0$, for $\hat{m}_q = 0$, in units of α_s/π. In the region $\hat{E}_0 > \frac{1}{2}$, this is the leading nonzero contribution. The logarithmic divergence at $\hat{E}_0 = \frac{1}{2}$ is integrable.

$$+ \left(\frac{8}{3} - \frac{128}{3}(1 + \hat{m}_q^2)\hat{m}_q^2 + 80\hat{m}_q^4 + \frac{8}{3}\hat{m}_q^8\right)\ln \hat{m}_q^2 \ln(1 - \hat{m}_q^2)$$

$$+ \left(4 + \frac{128}{3}(1 + \hat{m}_q^2)\hat{m}_q^3 + 64\hat{m}_q^4 + 4\hat{m}_q^8\right)\mathrm{Li}_2(\hat{m}_q^2)$$

$$- \frac{512}{3}\hat{m}_q^3(1 + \hat{m}_q^2)\mathrm{Li}_2(\hat{m}_q)\,,$$

$$
\begin{aligned}
A_1(\hat{m}_q) =\ & \frac{1381}{900} - \frac{7}{30}\pi^2 - \left(\frac{3133}{900} + \frac{5}{18}\pi^2\right)\hat{m}_q^2 + \left(\frac{99329}{1350} + \frac{16}{3}\pi^2\right)\hat{m}_q^4 - \frac{1408}{45}\pi^2\hat{m}_q^5 \\
& - \left(\frac{100729}{1350} - \frac{16}{3}\pi^2\right)\hat{m}_q^6 + \left(\frac{4933}{900} - \frac{5}{18}\pi^2\right)\hat{m}_q^8 - \left(\frac{6743}{2700} + \frac{7}{30}\pi^2\right)\hat{m}_q^{10} \\
& + \left(6\hat{m}_q^4 + \frac{34}{3}\hat{m}_q^6 - \frac{5}{18}\hat{m}_q^8 - \frac{7}{30}\hat{m}_q^{10}\right)\ln^2 \hat{m}_q^2 \\
& + \left(-\frac{47}{30}\hat{m}_q^2 + \frac{1651}{45}\hat{m}_q^4 - \frac{5632}{45}\ln(1 + \hat{m}_q)\hat{m}_q^5 + \frac{1391}{45}\hat{m}_q^6 + \frac{121}{135}\hat{m}_q^8 - \frac{409}{450}\hat{m}_q^{10}\right)\ln \hat{m}_q^2 \\
& + \left(-\frac{61}{50} + \frac{97}{54}\hat{m}_q^2 - 4\hat{m}_q^4 + 4\hat{m}_q^6 - \frac{97}{54}\hat{m}_q^8 + \frac{61}{50}\hat{m}_q^{10}\right)\ln(1 - \hat{m}_q^2) \\
& + \left(\frac{14}{15} + \frac{10}{9}\hat{m}_q^2 - \frac{100}{3}\hat{m}_q^4 + \frac{2816}{45}\hat{m}_q^5 - \frac{100}{3}\hat{m}_q^6 + \frac{10}{9}\hat{m}_q^8 + \frac{14}{15}\hat{m}_q^{10}\right)\ln \hat{m}_q^2 \ln(1 - \hat{m}_q^2) \\
& + \left(\frac{7}{5} + \frac{5}{3}\hat{m}_q^2 - 32\hat{m}_q^4 - \frac{2816}{45}\hat{m}_q^5 - 32\hat{m}_q^6 + \frac{5}{3}\hat{m}_q^8 + \frac{7}{5}\hat{m}_q^{10}\right)\mathrm{Li}_2(\hat{m}_q^2) \\
& + \frac{11264}{45}\hat{m}_q^5\mathrm{Li}_2(\hat{m}_q)\,,
\end{aligned}
\tag{A5}
$$

and

$$
\begin{aligned}
A_2(\hat{m}_q) =\ & \frac{2257}{3600} - \frac{4}{45}\pi^2 + \left(\frac{2929}{5400} - \frac{1}{5}\pi^2\right)\hat{m}_q^2 - \frac{324727}{10800}\hat{m}_q^4 + \frac{64}{5}\pi^2(1 + \hat{m}_q^2)\hat{m}_q^5 \\
& + \left(\frac{173}{162} - \frac{208}{27}\pi^2\right)\hat{m}_q^6 + \frac{304877}{10800}\hat{m}_q^8 + \left(\frac{1297}{1800} - \frac{1}{5}\pi^2\right)\hat{m}_q^{10}
\end{aligned}
$$

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$$-\left(\frac{36283}{32400}+\frac{4}{45}\pi^2\right)\hat{m}_q^{12}-\left(\frac{116}{9}\hat{m}_q^6+\frac{1}{5}\hat{m}_q^{10}+\frac{4}{45}\hat{m}_q^{12}\right)\ln^2\hat{m}_q^2 \tag{A6}$$

$$+\left(-\frac{2}{45}\hat{m}_q^2-\frac{131}{20}\hat{m}_q^4+\frac{256}{5}(1+\hat{m}_q^2)\ln(1+\hat{m}_q)\hat{m}_q^5-\frac{5467}{135}\hat{m}_q^6-\frac{829}{180}\hat{m}_q^8\right.$$

$$\left.+\frac{23}{450}\hat{m}_q^{10}-\frac{173}{675}\hat{m}_q^{12}\right)\ln\hat{m}_q^2$$

$$+\left(-\frac{298}{675}+\frac{1}{25}\hat{m}_q^2+2\hat{m}_q^4-2\hat{m}_q^8-\frac{1}{25}\hat{m}_q^{10}+\frac{298}{675}\hat{m}_q^{12}\right)\ln(1-\hat{m}_q^2)$$

$$+\left(\frac{16}{45}+\frac{4}{5}\hat{m}_q^2-\frac{128}{5}(1+\hat{m}_q^2)\hat{m}_q^5+\frac{440}{9}\hat{m}_q^6+\frac{4}{5}\hat{m}_q^{10}+\frac{16}{45}\hat{m}_q^{12}\right)\ln\hat{m}_q^2\ln(1-\hat{m}_q^2)$$

$$+\left(\frac{8}{15}+\frac{6}{5}\hat{m}_q^2+\frac{128}{5}(1+\hat{m}_q^2)\hat{m}_q^5+\frac{416}{9}\hat{m}_q^6+\frac{6}{5}\hat{m}_q^{10}+\frac{8}{15}\hat{m}_q^{12}\right)\mathrm{Li}_2(\hat{m}_q^2)$$

$$-\frac{512}{5}(1+\hat{m}_q^2)\hat{m}_q^5\,\mathrm{Li}_2(\hat{m}_q)\,.$$

The correction $A_0(\hat{m}_q)$ to the total rate is equivalent to the result presented in Ref. [19].

APPENDIX B: BUBBLE GRAPHS

The n-loop bubble graph contribution to moments of $\hat{s}_0$ may be calculated from the one loop graph evaluated with a finite gluon mass [29,30]. In this appendix, we briefly outline this calculation using the methods of Ref. [30]. Only the bremmstrahlung graphs in Fig. 3 contribute to the moments of $\hat{s}_0$ for $n \geq 1$. We consider the expansion

$$\frac{d\Gamma}{d\hat{s}_0}=\sum_{j=0}^{\infty}d_j(\hat{s}_0)\beta_0^j\left(\frac{\alpha_s}{\pi}\right)^{j+1}+\dots\,, \tag{B1}$$

where $\beta_0 = 11 - 2n_f/3$ and the ellipses denote terms which have fewer powers of $\beta_0\alpha_s$ and hence are not obtainable from the bubble graphs. Note that these are not suppressed terms in any limit of QCD, although they may be numerically small. The nth moment of $\hat{s}_0$ then has the expansion

$$\mathcal{M}^{(n,0)}=\sum_{j=0}^{\infty}m_j^{(n)}\beta_0^j\left(\frac{\alpha_s}{\pi}\right)^{j+1}+\dots\,, \tag{B2}$$

where

$$m_j^{(n)}=\int_0^1 d\hat{s}_0\,\hat{s}_0^n\,d_j(\hat{s}_0)\,. \tag{B3}$$

Define $d_0(\hat{s}_0,\hat{\lambda}^2)$ and $m_0^{(n)}(\hat{\lambda}^2)$ to be the one-loop corrections calculated with a finite gluon mass λ, and $\hat{\lambda}\equiv\lambda/m_b$. Then

$$m_0^{(n)}(\hat{\lambda}^2)=\int_{\hat{\lambda}^2}^1 d\hat{s}_0^n\,d_0(\hat{s}_0,\hat{\lambda}^2)\,, \tag{B4}$$

and we have [30]

$$d_j(\hat{s}_0) = -\frac{1}{4^j}\frac{\mathrm{d}^j}{\mathrm{d}u^j}\bigg|_{u=0}\frac{\sin(\pi u)}{\pi u}\int_0^{\hat{s}_0}\left(\hat{\lambda}^2 e^C\right)^{-u}\frac{\mathrm{d}}{\mathrm{d}\hat{\lambda}^2}d_0(\hat{s}_0,\hat{\lambda}^2)\,, \tag{B5}$$

where C is a scheme-dependent constant. In the V scheme [14], $C = 0$, while in the $\overline{\mathrm{MS}}$ scheme, $C = -5/3$. Eqn. (B5) may be written in the form

$$d_j(\hat{s}_0) = \sum_{k=0}^{j} c_k\, J_k(\hat{s}_0)\,, \tag{B6}$$

where J_k is defined by

$$J_k(\hat{s}_0) \equiv \int_0^{\hat{s}_0}\mathrm{d}\hat{\lambda}^2\,\ln^k(\hat{\lambda}^2)\frac{\mathrm{d}}{\mathrm{d}\hat{\lambda}^2}d_0(\hat{s}_0,\hat{\lambda}^2)\,. \tag{B7}$$

.

  Taking moments of both sides of Eq. (B6), we have

$$\begin{aligned}
m_j^{(n)} &= \sum_{k=0}^{j} c_k \int_0^1 \mathrm{d}\hat{s}_0\,\hat{s}_0^n \int_0^{\hat{s}_0}\mathrm{d}\hat{\lambda}^2\,\ln^k(\hat{\lambda}^2)\frac{\mathrm{d}}{\mathrm{d}\hat{\lambda}^2}d_0(\hat{s}_0,\hat{\lambda}^2)\\
&= \sum_{k=0}^{j} c_k \int_0^1 \mathrm{d}\hat{\lambda}^2 \int_{\hat{\lambda}^2}^1 \mathrm{d}\hat{s}_0\,\hat{s}_0^n\,\ln^k(\hat{\lambda}^2)\frac{\mathrm{d}}{\mathrm{d}\hat{\lambda}^2}d_0(\hat{s}_0,\hat{\lambda}^2)\\
&= \sum_{k=0}^{j} c_k \int_0^1 \mathrm{d}\hat{\lambda}^2\,\ln^k(\hat{\lambda}^2)\frac{\mathrm{d}}{\mathrm{d}\hat{\lambda}^2}m_0^{(n)}(\hat{\lambda}^2)\,,
\end{aligned} \tag{B8}$$

where we have used the fact that $d_0(\hat{\lambda}^2,\hat{\lambda}^2) = 0$ to move the $\hat{s}_0$ integral to the right of the $\hat{\lambda}^2$ derivative.

  It is straightforward to derive analytic expressions for the moments $m_0^{(n)}(\hat{\lambda}^2)$ from the graphs in Fig. 3; however the resulting formulas are lengthy and we will not reproduce them here. For $j = 1$ the integrals in Eq. (B8) may be performed analytically, giving the $\mathcal{O}(\alpha^2\beta_0)$ correction to $\mathcal{M}^{(n,0)}$, while for $j > 1$ we performed the integrals numerically to obtain the contribution from higher loops in the bubble sum.

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