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Vacuum functional and fermion condensate in the massive Schwinger model

Christoph Adam
Institut für theoretische Physik, Universität Bern
Sidlerstraße 5, CH-3012 Bern, Switzerland*)

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Abstract

We derive a systematic procedure of computing the vacuum functional and fermion condensate of the massive Schwinger model via a perturbative expansion in the fermion mass. We compute numerical results for the first nontrivial order.

*)permanent address: Institut für theoretische Physik, Universität Wien
Boltzmanngasse 5, 1090 Wien, Austria
email address: adam@pap.univie.ac.at

Introduction

The massless Schwinger model has been studied extensively because, although being exactly soluble, it has quite a rich structure that resembles features of more realistic theories. Among these are: a massive physical state is formed via the chiral anomaly ([1], [2], [15], [12], [14], [8], [7], [6]) (this state is noninteracting in the massless Schwinger model). There are "instanton-like" gauge field configurations present, and therefore the vacuum structure is nontrivial: the true vacuum is a superposition of all instanton sectors, and each possible superposition is labelled by a vacuum angle θ ([2], [17], [3], [4], [9], [7], [6], [5]). However the physics does not depend on the value of θ in the massless case.

The massive Schwinger model is no longer exactly soluble ([10], [11], [13], [8]). But all of the nontrivial features of the massless model persist to hold, at least for small fermion mass. The massive state is now interacting, and its mass acquires corrections due to the fermion mass. Instantons and nontrivial vacuum are present, too, and, in addition, physical quantities now depend on the value of the vacuum angle θ.

Here we will show how some physical quantities may be computed via a perturbative expansion in the fermion mass, and we will arrive at some numerical results for the vacuum functional and fermion condensate. For this we use the wellknown exact path integral solution of the massless Schwinger model as a starting point.

Perturbative computation

The vacuum functional for the massive Schwinger model may be written as a sum of all instanton sectors ($k \ldots$ instanton number)

$$Z(m,\theta) = \sum_{k=-\infty}^{\infty} e^{ik\theta} Z_k(m) \tag{1}$$

where

$$\begin{aligned} Z_k(m) &= N \int D\bar{\Psi} D\Psi D A_k^\mu e^{\int dx \left[\bar{\Psi}(i\partial\!\!\!/ - eA\!\!\!/_k - m)\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right]} \\ &= N \int D\bar{\Psi} D\Psi D\beta_k \sum_{n=0}^{\infty} \frac{m^n}{n!} \prod_{i=1}^{n} \int dx_i \bar{\Psi}(x_i)\Psi(x_i) \cdot \\ &\cdot \exp\{\int dx \Big[\bar{\Psi}(i\partial\!\!\!/ - \epsilon_{\mu\nu}\gamma^\mu\partial^\nu\beta_k)\Psi + \frac{1}{2}\beta_k \Box^2 \beta_k\Big]\} \end{aligned} \tag{2}$$

($A_\mu = \epsilon_{\mu\nu}\partial^\nu\beta$ corresponding to Lorentz gauge) where we expanded the mass term. Therefore the vacuum functional is related to the computation of fermionic VEV of the massless Schwinger model (and in addition some space time integrations thereof). These fermionic VEV of the massless Schwinger model may be computed from the generating functional

$$\begin{aligned} Z_k[\bar{\eta},\eta] &= N \int D\beta \prod_{i_0=0}^{k-1} (\bar{\eta}\Psi^\beta_{i_0})(\bar{\Psi}^\beta_{i_0}\eta) \cdot \\ &\cdot e^{i\int dx dy \bar{\eta}(x) G^\beta(x,y)\eta(y)} e^{\frac{1}{2}\int dx \beta \mathbf{D} \beta} \end{aligned} \tag{3}$$

where the fermions have already been integrated out. $\Psi_{i_0}^{\beta}$ are the fermionic zero modes and G^{β} is the exact fermionic Green's function in the external gauge field β, and $\mathbf{D}$ is the operator of the effective gauge field action, with Green's function $\mathbf{G}$ ($D_\mu(x)(D_0(x))$ is the massive (massless) scalar propagator)

$$\mathbf{G}(x) = \pi(D_\mu(x) - D_0(x))$$
$$D_\mu(x) = -\frac{1}{2\pi}K_0(\mu|x|) \quad , \quad D_0(x) = \frac{1}{4\pi}\ln x^2 \tag{4}$$

($K_0 \ldots$ McDonald function). From this all fermionic VEV may be computed (for details see e.g. [6], [7], [5]).

It is useful to rewrite the fermionic scalar bilinears in terms of chiral bilinears like

$$S(x) := \bar{\Psi}(x)\Psi(x) = \bar{\Psi}(x)P_+\Psi(x) + \bar{\Psi}(x)P_-\Psi(x) =: S_+(x) + S_-(x), \tag{5}$$

because for a VEV of products of $S_\pm$ only a definite instanton sector contributes. E.g. the fermion condensate is

$$\langle S(x)\rangle = \langle S_+(x)\rangle^{k=1} + \langle S_-(x)\rangle^{k=-1} = \frac{1}{\pi}e^{2\mathbf{G}(0)} = \frac{e^\gamma}{2\pi}\mu =: \Sigma \tag{6}$$

where γ is the Euler constant and μ the Schwinger mass $\mu^2 = \frac{e^2}{\pi}$.

For higher VEV of n_+S_+ and n_-S_- one finds

$$\langle S_{H_1}(x_1)\cdots S_{H_n}(x_n)\rangle = \Big(\frac{\Sigma}{2}\Big)^n \exp\Big[\sum_{i<j}(-)^{\sigma_i\sigma_j}4\pi D_\mu(x_i - x_j)\Big] \tag{7}$$

where $\sigma_i = \pm 1$ for $H_i = \pm$. The sole contribution to this VEV stems from the instanton sector $k = \sum_i \sigma_i = n_+ - n_-$ (see e.g. [6], [5], [16]).

Now we just have to insert this result into the perturbation expansion (2). For a first, rough approximation we may use the fact that the massive scalar propagator $D_\mu(x)$ vanishes exponentially for large argument. Therefore, when integrating over space time and expanding the exponential, all contributions from D_μ^l will be ignored for the moment, supposing that the space time volume V is sufficiently large. In this case the integrations in (2) just produce factors of V. Further, when inserting (7) in (2) we have to sum over all possible distributions of $n_+ = n - n_-$ pluses and n_- minuses on n scalar densities S. This results in a factor $\binom{n}{n_-}$. Therefore we find for the n-th order term

$$\langle\prod_{i=1}^n \int dx_i S(x_i)\rangle \sim \Big(\frac{\Sigma}{2}\Big)^n V^n \sum_{n_-=0}^n \binom{n}{n_-} e^{i(n-2n_-)\theta} = (\Sigma V\cos\theta)^n \tag{8}$$

and for the normalized vacuum functional

$$\frac{Z(m,\theta)}{Z(0,0)} \sim \exp(m\Sigma V\cos\theta) \tag{9}$$

(we ignored terms like m^nV^{n-1} in this approximation compared to $m^{n-1}V^{n-1}$, therefore (9) is the first order result in m). This result is wellknown, and its consequences for the vacuum structure and spectrum of the Dirac operator are discussed in great detail in [18].

To obtain higher order results, we rewrite the exponential in (7) like e.g.

$$\exp\Big(\sum(-)^{\sigma_i\sigma_j}4\pi D_\mu(x_i-x_j)\Big)=\exp\Big(+4\pi D_\mu(x_1-x_2)\Big)\cdot\exp\Big(-4\pi D_\mu(x_1-x_3)\Big)\cdot\ldots$$
$$=\Big(1+E(x_1-x_2)\Big)\cdot\Big(1+F(x_1-x_3)\Big)\cdot\ldots \tag{10}$$

where

$$E(x)=e^{4\pi D_\mu(x)}-1 \quad , \quad F(x)=e^{-4\pi D_\mu(x)}-1$$
$$E\equiv\int d^2xE(x) \quad , \quad F\equiv\int d^2xF(x). \tag{11}$$

Both functions $E(x), F(x)$ vanish exponientially for large argument.

Using this notation and inserting (7) into the perturbation expansion (2) we obtain, order by order:

n=1:

$$\frac{m}{1!}\frac{\Sigma}{2}\int dx(e^{i\theta}+e^{-i\theta})=\frac{m}{1!}\frac{\Sigma}{2}V(e^{i\theta}+e^{-i\theta}) \tag{12}$$

n=2:

$$\frac{m^2}{2!}\Big(\frac{\Sigma}{2}\Big)^2\int dx_1dx_2\Big[e^{2i\theta}e^{4\pi D_\mu(x_1-x_2)}+2e^{-4\pi D_\mu(x_1-x_2)}+e^{-2i\theta}e^{4\pi D_\mu(x_1-x_2)}\Big]$$
$$=\frac{m^2}{2!}\Big(\frac{\Sigma}{2}\Big)^2\Big[V^2(e^{2i\theta}+2+e^{-2i\theta})+V(Ee^{2i\theta}+2F+Ee^{-2i\theta})\Big] \tag{13}$$

n=3:

$$\ldots=\frac{m^3}{3!}\Big(\frac{\Sigma}{2}\Big)^3\Big[V^3(e^{3i\theta}+3e^{i\theta}+3e^{-i\theta}+e^{-3i\theta})+$$
$$V^2\Big(3Ee^{3i\theta}+3(E+2F)e^{i\theta}+3(E+2F)e^{-i\theta}+3Ee^{-3i\theta}\Big)+$$
$$V\Big((3E^2+E\times E\times E)e^{3i\theta}+3(2EF+F^2+E\times F\times F)e^{i\theta}+$$
$$3(2EF+F^2+E\times F\times F)e^{-i\theta}+(3E^2+E\times E\times E)e^{-3i\theta}\Big)\Big] \tag{14}$$

n=4:

$$\ldots=\frac{m^4}{4!}\Big(\frac{\Sigma}{2}\Big)^4\Big[V^4(e^{4i\theta}+4e^{2i\theta}+6+4e^{-2i\theta}+e^{-4i\theta})+$$
$$V^3\Big(6Ee^{4i\theta}+12(E+F)e^{2i\theta}+12(E+2F)+12(E+F)e^{-2i\theta}+6Ee^{-4i\theta}\Big)+$$
$$V^2\Big((15E^2+4E\times E\times E)e^{4i\theta}+4(3E^2+9EF+3F^2+E\times E\times E+3E\times F\times F)e^{2i\theta}+$$
$$+6(E^2+8EF+F^2+4E\times F\times F)+4(3E^2+9EF+3F^2+E\times E\times E+3E\times F\times F)e^{-2i\theta}+$$
$$+(15E^2+4E\times E\times E)e^{-4i\theta}\Big)+\ldots\Big] \tag{15}$$

$$\ldots$$

where e.g.

$$E\times E\times E\equiv\int dy_1dy_2E(y_1)E(y_1+y_2)E(y_2) \tag{16}$$

and we displayed the result up to the accuracy we need. Observe that the result is not obtained by just expanding polynomials like $(1+E(x_i))^l$, because e.g. a third power in $E(x_i)$ may contribute to $V^{n-3}E^3$ or to $V^{n-2}E \times E \times E$.

In a next step we rearrange the terms (12) – (15) in rising powers of V:

$$\frac{V}{1!}\Big[m\frac{\Sigma}{2}(e^{i\theta}+e^{-i\theta})+\frac{m^2}{2}\Big(\frac{\Sigma}{2}\Big)^2(Ee^{2i\theta}+2F+Ee^{-2i\theta})+$$

$$\frac{m^3}{6}\Big(\frac{\Sigma}{2}\Big)^3\Big((3E^2+E\times E\times E)(e^{3i\theta}+e^{-3i\theta})+3(2EF+F^2+E\times F\times F)(e^{i\theta}+e^{-i\theta})\Big)+\ldots\Big]+$$

$$\frac{V^2}{2!}\Big[m^2\Big(\frac{\Sigma}{2}\Big)^2(e^{2i\theta}+2+e^{-2i\theta})+m^3\Big(\frac{\Sigma}{2}\Big)^3\Big(E(e^{3i\theta}+e^{-3i\theta})+(E+2F)(e^{i\theta}+e^{-i\theta})\Big)+$$

$$\frac{m^4}{12}\Big(\frac{\Sigma}{2}\Big)^4\Big((15E^2+4E\times E\times E)(e^{4i\theta}+e^{-4i\theta})+$$

$$4(3E^2+9EF+3F^2+E\times E\times E+3E\times F\times F)(e^{2i\theta}+e^{-2i\theta})+$$

$$6(E^2+8EF+F^2+4E\times F\times F)\Big)+\ldots\Big]+$$

$$\frac{V^3}{3!}\Big[m^3\Big(\frac{\Sigma}{2}\Big)^3(e^{3i\theta}+3e^{i\theta}+3e^{-i\theta}+e^{-3i\theta})+$$

$$\frac{m^4}{2}\Big(\frac{\Sigma}{2}\Big)^4\Big(3E(e^{4i\theta}+e^{-4i\theta})+6(E+F)(e^{2i\theta}+e^{-2i\theta})+(E+2F)\Big)+\ldots\Big]+$$

$$\frac{V^4}{4!}\Big[m^4\Big(\frac{\Sigma}{2}\Big)^4(e^{4i\theta}+4e^{2i\theta}+6+4e^{-2i\theta}+e^{-4i\theta})+\ldots\Big]+\ldots$$

$$=:\frac{V}{1!}\alpha+\frac{V^2}{2!}\alpha^2+\frac{V^3}{3!}\alpha^3+\ldots \tag{17}$$

We find that the coefficient of $\frac{V^n}{n!}$ is the n-th power of some specific function α where α does not depend on V any more. This feature could have been expected, because the fermion condensate should not depend on the volume V. With this function α,

$$\alpha(m,\theta)=m\frac{\Sigma}{2}2\cos\theta+\frac{m^2}{2!}\Big(\frac{\Sigma}{2}\Big)^2(2E\cos 2\theta+2F)+$$

$$\frac{m^3}{3!}\Big(\frac{\Sigma}{2}\Big)^3\Big((3E^2+E\times E\times E)2\cos 3\theta+3(2EF+F^2+E\times E\times E)2\cos\theta\Big)+\ldots \tag{18}$$

the normalized vacuum functional may be written like

$$\frac{Z(m,\theta)}{Z(0,0)}=e^{V\alpha(m,\theta)}. \tag{19}$$

From this it is very easy to compute the fermion condensate

$$\langle\bar\Psi\Psi\rangle(m,\theta)\equiv\frac{1}{V}\frac{\partial}{\partial m}\ln Z(m,\theta)=\frac{\partial}{\partial m}\alpha(m,\theta) \tag{20}$$

$$\langle\bar\Psi\Psi\rangle(m,\theta)=\Sigma\cos\theta+\frac{m}{2}\Sigma^2(E\cos 2\theta+F)+$$

$$\frac{m^2}{8}\Sigma^3\Big((3E^2 + E\times E\times E)\cos 3\theta + 3(2EF + F^2 + E\times F\times F)\cos\theta\Big) + \dots \tag{21}$$

For a numerical evaluation of order m we have to compute the coefficients E and F. First, both E and F are proportional to $\frac{1}{\mu^2}$. For $\mu = 1$, E is

$$\begin{aligned} E = \int d^2x E(x) &= \int d^2x(e^{-2K_0(|x|)} - 1) \\ &= 2\pi\int_0^\infty drr(e^{-2K_0(r)} - 1) = -8.9139 \end{aligned} \tag{22}$$

$E(x)$ is well behaving ($E(0) = -1$), so the numerical integration is straight forward. $F(x)$ is singular at $x = 0$, $F(x) \sim \frac{1}{x^2}$ for $x \to 0$, but this singularity can easily be understood and removed in a unique way. Indeed, this singularity is just the free fermion singularity, as can be seen by rewriting $F(x)$ like

$$\Sigma^2(F(x) + 1) = G_0^2(x)e^{4\pi(\mathbf{G}(0)-\mathbf{G}(x))} \overset{|x|\to 0}{\longrightarrow} G_0^2(x) = \frac{1}{4\pi^2x^2}. \tag{23}$$

This singularity may be isolated by a partial integration:

$$\begin{aligned} F &= \int d^2x(e^{2K_0(|x|)} - 1) = 2\pi\int_0^\infty \frac{dr}{r}(e^{2K_0(r)+2\ln r} - r^2) \\ &= 2\pi\Big[\ln r(e^{2K_0(r)+2\ln r} - r^2)\Big]_{\epsilon\to 0}^\infty \\ &+ 2\pi\int_0^\infty dr 2\ln r((K_1(r) - \frac{1}{r})e^{2K_0(r)+2\ln r} + r) \end{aligned} \tag{24}$$

($K_0^{'} = -K_1$). Observe that the first term precisely leads to the free field singularity at the lower boundary (and vanishes at the upper boundary). So the second term is the unique and finite result we are looking for. The numerical integration gives

$$F = 9.7384 \tag{25}$$

With these results we find for the fermion condensate

$$\langle\bar\Psi\Psi\rangle(m,\theta) = \Sigma\cos\theta + \frac{m}{2}\frac{\Sigma^2}{\mu^2}(-8.9139\cos 2\theta + 9.7384) + o(m^2) \tag{26}$$

or, using (6),

$$\begin{aligned} \frac{1}{\mu}\langle\bar\Psi\Psi\rangle(m,\theta) &= \frac{e^\gamma}{2\pi}\cos\theta + \frac{m}{\mu}\frac{e^{2\gamma}}{8\pi^2}(E\cos 2\theta + F) + o(\frac{m^2}{\mu^2}) \\ &= 0.2835\cos\theta + \frac{m}{\mu}(0.7825 - 0.7163\cos 2\theta) + o(\frac{m^2}{\mu^2}). \end{aligned} \tag{27}$$

This is our final result. Observe that the correction is minimal for $\theta = 0$.

Summary

We have derived a systematic procedure of computing the vacuum functional and fermion condensate of the massive Schwinger model in a mass perturbation theory, order by order. For the first nontrivial order we even displayed numerical results. For higher orders the numerical calculations are more involved (because of the occurrence of linked functions in the integrations, like $E \times E \times E$), but in principle they could be performed, e.g. in order to compare the result to the results of other approaches.

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