xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx

Segre types of symmetric two-tensors in n-dimensional spacetimes

J. Santos

Universidade Federal do Rio G. do Norte

Departamento de Física, Cx. Postal 1641

59072-970 Natal - RN, Brazil

Email: janilo@dfte.ufrn.br

M.J. Rebouças and A.F.F. Teixeira

Centro Brasileiro de Pesquisas Físicas

Rua Dr. Xavier Sigaud, 150

22290-180 Rio de Janeiro - RJ, Brazil

Email: reboucas@cat.cbpf.br

July 12, 1995

Abstract

Three propositions about Jordan matrices are proved and applied to algebraically classify the Ricci tensor in n-dimensional Kaluza-Klein-type spacetimes. We show that the possible Segre types are $[1,1\ldots1]$, $[21\ldots1]$, $[31\ldots1]$, $[z\bar{z}1\ldots1]$ and degeneracies thereof. A set of canonical forms for the Segre types is obtained in terms of semi-null bases of vectors.

1 Introduction

The algebraic classification of a symmetric two-tensor (such as the Ricci tensor) defined on a four-dimensional (4-D for short) Lorentzian manifold has been discussed by several authors [1] – [3] and is of interest in, for example, classifying and interpreting matter field distributions [4] – [12], and in the study of limits for non-vacuum spacetimes [13]. It is also important in understanding some purely geometrical features of spacetimes (see, e.g., Plebański, [2] Cormack and Hall [14]), and as a part of the procedure for checking whether apparently different spacetimes are in fact locally the same up to coordinate transformations (the equivalence problem [15] – [19]).

Over the past three decades, particularly after the appearance of supergravity and superstring theories in the 1970's, there has been a resurgence in work on Kaluza-Klein-type theories in higher dimensional settings [20]. This has been basically motivated by the quest for an unification of gravity with the other fundamental interactions. On the other hand, they have been used as a way of finding new solutions of Einstein's equations in four dimensions, without ascribing any physical meaning to the additional components of the metric tensor [21].

In this paper, after stating a proposition concerning orthogonality of null vectors and proving three propositions about Jordan matrices, we apply them to algebraically classify the Ricci tensor in n-dimensional Kaluza-Klein-type Lorentzian spacetimes. The classification is obtained from first principles, without making use of previous results [3, 22], and generalizes a recent article on Segre types in 5-D spacetimes [23]. The Ricci tensor is classified into four Segre types and their degeneracies. Using real semi-null bases we derive a set of canonical forms for the Segre types, extending to n-dimensional Lorentzian spaces the canonical forms obtained for lower dimensional spaces [3], [22] – [24]. Although the Ricci tensor is constantly referred to, the results of the following sections apply to any second order real symmetric tensor on n-dimensional Lorentzian spaces.

2 Mathematical Prerequisites

The algebraic classification of the Ricci tensor in n-dimensional spacetimes can be cast in terms of the eigenvalue problem

$$(R^a{}_b - \lambda\, \delta^a_b)\, V^b \;= 0, \tag{2.1}$$

where λ is a scalar, V^b is a vector and the mixed Ricci tensor $R^a{}_b$ may be thought of as a linear operator $R : T_p(M) \longrightarrow T_p(M)$. Here and in what follows M is a real n-dimensional spacetime manifold locally endowed with a Lorentzian metric of signature $(-++\cdots+)$, $T_p(M)$ denotes the tangent space to M at a point $p \in M$ and latin indices range from 0 to $n-1$, unless otherwise stated. Although the matrix $R^a{}_b$ is real, the eigenvalues λ and the eigenvectors V^b are often complex. A mathematical procedure used to classify matrices in such a case is to reduce them through similarity transformations to canonical forms over the complex field. Among the existing canonical forms the Jordan canonical form (JCF) turns out to be the most appropriate for a classification of $R^a{}_b$.

The mathematical theory concerning JCF of square matrices is well established and can be found in many textbooks on linear algebra [25, 26]. The basic result is that given an n-square matrix A over the complex field, there exist nonsingular matrices X such that

$$X^{-1} A\, X = J \tag{2.2}$$

where J, the JCF of A, is a block diagonal matrix, each block being of the form

$$J_r(\lambda_k) = \begin{pmatrix} \lambda_k & 1 & 0 & \cdots & 0 \\ 0 & \lambda_k & 1 & \cdots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & \lambda_k \end{pmatrix} . \tag{2.3}$$

Here r is the dimension of the block and λ_k is a root of the characteristic equation $\det(A - \lambda I) = 0$. Although X is far from being unique, the JCF is uniquely determined up to the ordering of the blocks along the main diagonal of J. In what follows A of eq. (2.2) is the matrix formed with the mixed components $R^a{}_b$ of the Ricci tensor R.

We shall first examine the structure of a general Jordan block $J_r(\lambda)$ in J, where $J_r(\lambda)$ begins at row and column s and ends at row and column t $(t = s + r - 1)$. The matricial equation (2.2) can be rewritten as $RX = XJ$. Equating columns s to t on both sides of this matricial equation we have

$$\begin{aligned} R\,\mathbf{X}_s &= \lambda\,\mathbf{X}_s\,, \\ R\,\mathbf{X}_{s+1} &= \lambda\,\mathbf{X}_{s+1} + \mathbf{X}_s\,, \\ &\vdots \\ R\,\mathbf{X}_t &= \lambda\,\mathbf{X}_t + \mathbf{X}_{t-1}\,, \end{aligned} \tag{2.4}$$

where $\mathbf{X}_q$ denotes the vector associated to the q-th column of the matrix X. We shall refer to these relations as a Jordan chain. It should be noted that (a) the column vectors $\mathbf{X}_s, \mathbf{X}_{s+1}, \ldots, \mathbf{X}_t$ are linearly independent, otherwise X would be a singular matrix; (b) the first m vectors in a Jordan chain span an m-dimensional subspace of $T_p(M)$ invariant under $R^a{}_b$, and (c) particularly for $m = 1$ the vector $\mathbf{X}_s$ which starts the Jordan chain defines the sole eigendirection of R associated to that Jordan block. The complete set of vectors $\{\mathbf{X}_a\,;\, a = 1, \ldots, n\}$ are linearly independent and give a Jordan basis, i.e., a basis in which $R^a{}_b$ takes a JCF.

To algebraically classify a second order *symmetric* tensor we shall discuss now three propositions about Jordan blocks $J_r(\lambda)$ $(r > 1)$.

Proposition 1 *The first column vector (eigenvector $\mathbf{X}_s$) of a Jordan block $J_r(\lambda)$ $(r > 1)$ is orthogonal to all vectors of its block, except possibly to the last one ($\mathbf{X}_t$).*

Indeed, from (2.4) a vector $\mathbf{X}_{q+1}$ $(s \leq q < t)$ obeys the equation

$$R\,\mathbf{X}_{q+1} = \lambda \mathbf{X}_{q+1} + \mathbf{X}_q. \tag{2.5}$$

Eq. (2.5) and the first equation (2.4) yield

$$\begin{aligned} R\,\mathbf{X}_{q+1}.\mathbf{X}_s &= \lambda\,\mathbf{X}_{q+1}.\mathbf{X}_s + \mathbf{X}_q.\mathbf{X}_s\,, \\ R\,\mathbf{X}_s.\mathbf{X}_{q+1} &= \lambda\,\mathbf{X}_s.\mathbf{X}_{q+1}\,, \end{aligned} \tag{2.6}$$

where the dot between vectors indicates inner product. As R_{ab} is symmetric, one easily obtains

$$\mathbf{X}_s.\mathbf{X}_q = 0, \quad s \leq q < t\,. \tag{2.7}$$

When $q = s$ eq. (2.7) implies $\mathbf{X}_s.\mathbf{X}_s = 0$, i.e., the eigenvector associated to a Jordan block of dimension $r > 1$ is a null vector.

Proposition 2 *If $\mathbf{X}_p$ and $\mathbf{X}_q$ are two vectors (not eigenvectors) related to a block $J_r(\lambda)$ $(r > 2)$ then*

$$\mathbf{X}_p.\mathbf{X}_q = \mathbf{X}_{p-1}.\mathbf{X}_{q+1}\,, \quad s < p \leq q < t. \tag{2.8}$$

Indeed, from (2.4) one finds that for p and q in the given range, the equations

$$R\,\mathbf{X}_p = \lambda\,\mathbf{X}_p + \mathbf{X}_{p-1}\,, \tag{2.9}$$

$$R\,\mathbf{X}_{q+1} = \lambda\,\mathbf{X}_{q+1} + \mathbf{X}_q \tag{2.10}$$

hold. If one now takes the inner product of (2.9) and of (2.10) by $\mathbf{X}_{q+1}$ and $\mathbf{X}_p$, respectively, and uses the symmetry of R_{ab}, one obtains eqs. (2.8).

Proposition 3 *Eigenvectors related to different Jordan blocks are orthogonal provided at least one of the blocks has dimension $r > 1$.*

In fact, let $J_r(\lambda)$ and $J_{r'}(\lambda')$ be blocks of a Jordan matrix, where $J_r(\lambda)$ generates the Jordan chain (2.4) and $J_{r'}(\lambda')$ gives rise to a similar (primed) Jordan chain with r' equations. Then

$$R\,\mathbf{X}_{s'} = \lambda'\mathbf{X}_{s'}. \tag{2.11}$$

From (2.4) and (2.11) one finds

$$R\,\mathbf{X}_{s'}.\mathbf{X}_s = \lambda'\,\mathbf{X}_{s'}.\mathbf{X}_s\,, \tag{2.12}$$

$$R\,\mathbf{X}_s.\mathbf{X}_{s'} = \lambda\,\mathbf{X}_s.\mathbf{X}_{s'}\,, \tag{2.13}$$

$$R\,\mathbf{X}_{s+1}.\mathbf{X}_{s'} = \lambda\,\mathbf{X}_{s+1}.\mathbf{X}_{s'} + \mathbf{X}_s.\mathbf{X}_{s'}\,, \tag{2.14}$$

$$R\,\mathbf{X}_{s'}.\mathbf{X}_{s+1} = \lambda'\,\mathbf{X}_{s'}.\mathbf{X}_{s+1}\,. \tag{2.15}$$

Here again the symmetry of R_{ab} together with (2.12) and (2.13) imply

$$(\lambda' - \lambda)\mathbf{X}_{s'}.\mathbf{X}_s = 0. \tag{2.16}$$

Similarly eqs. (2.14) and (2.15) give

$$(\lambda' - \lambda)\mathbf{X}_{s'}.\mathbf{X}_{s+1} = \mathbf{X}_{s'}.\mathbf{X}_s\,. \tag{2.17}$$

Table 1: Null vectors related to a Jordan block $J_r(\lambda)$.

r	null vectors	orthogonality relations
2	$\mathbf{X}_1$	
3	$\mathbf{X}_1$	$\mathbf{X}_1.\mathbf{X}_2 = 0$
4	$\mathbf{X}_1$, $\mathbf{X}_2$	$\mathbf{X}_1.\mathbf{X}_2 = \mathbf{X}_1.\mathbf{X}_3 = 0$

Finally (2.16) and (2.17) imply $\mathbf{X}_{s'}.\mathbf{X}_s = 0$ regardless of whether $\lambda' = \lambda$ or $\lambda' \neq \lambda$.

To complete the mathematical prerequisites we state a trivial extension of a known result for null vectors in 4-D spacetimes, namely:

Proposition 4 *In an n-D real vector space endowed with a Lorentzian metric two null vectors are orthogonal if and only if they are collinear.*

Employing recurrently proposition 2, then proposition 1, we have found the null vectors related to a Jordan block $J_r(\lambda)$ with $r = 2, 3, 4$. These null vectors and their orthogonality relations with other vectors of the same block are shown in table 1. Although we have included in table 1 the null vectors when $r = 4$, it should be noticed that in the classification of symmetric two-tensors in n-dimensional Lorentzian spaces the Jordan blocks of dimension $r \geq 4$ cannot occur. In the next section we shall discuss this point, and use it together with propositions 3 and 4 to classify $R^a{}_b$.

3 The Classification

In the Jordan classification two square matrices are said to be equivalent if similarity transformations exist such that they can be brought to the same JCF. The JCF of a matrix gives explicitly its eigenvalues and makes apparent the dimensions of the Jordan blocks. However, for many purposes a somehow coarser classification of a matrix is sufficient. In the Segre classification, for example, the value of the roots of the characteristic equation is not relevant — only dimension of the Jordan blocks and degeneracy of eigenvalues matter. The Segre type is a list $[r_1 r_2 \cdots r_m]$ of the dimensions of the Jordan blocks. Equal eigenvalues in distinct blocks are indicated by enclosing the corresponding digits inside round brackets. Thus, for example, in the degenerated Segre type [(42)1] six out of the seven eigenvalues are equal; there are three linearly independent eigenvectors, two of which are associated to the Jordan blocks of dimensions 4 and 2, whereas the last one corresponds to the block of dimension 1.

In classifying symmetric tensors in a Lorentzian spacetime two refinements to the usual Segre notation are often used. Instead of a digit to denote the dimension of a block with complex eigenvalue a letter is used, and the digit corresponding to a timelike eigenvector is separated from the others by a comma.

We learn from table 1 that the JCF of $R^a{}_b$ cannot have a block with dimension greater than 3. Indeed, the existence of such a block would give rise to at least two null vectors $\mathbf{X}_1$ and $\mathbf{X}_2$ that are orthogonal, hence collinear by proposition 4, which would lead to a singular transformation matrix X. Besides, the JCF of $R^a{}_b$ cannot have more than one block of dimension $r > 1$. This is so because there is one null eigenvector associated to each such block and, by proposition 3, these vectors are orthogonal, hence the matrix X in (2.2) would again be singular. Therefore, in a 5-D spacetime, for example, the JCF with Segre types [5], [41], [32], [221] and their degeneracies are ruled out by the above arguments, in agreement with earlier results on this matter [23]. Similarly, one can find the possible non-degenerated Segre types of $R^a{}_b$ for n-dimensional ($n \geq 3$) spacetimes. In table 2 we present these types, where we have included the complex cases to be discussed below.

When the characteristic equation corresponding to (2.1) has complex roots, one can deal with this case by using an approach borrowed from [3] as follows. Suppose that $\alpha \pm i\beta$ are complex eigenvalues of $R^a{}_b$ corresponding to the eigenvectors $\mathbf{V}_\pm = \mathbf{Y} \pm i\mathbf{Z}$, where α and $\beta \neq 0$ are real and $\mathbf{Y}$, $\mathbf{Z}$ are independent vectors defined on $T_p(M)$. Since R_{ab} is symmetric and the eigenvalues are different, the eigenvectors must be orthogonal and hence equation $\mathbf{Y}.\mathbf{Y} + \mathbf{Z}.\mathbf{Z} = 0$ holds. It follows that either one of the vectors $\mathbf{Y}$ or $\mathbf{Z}$ is timelike and the other spacelike or both are null and, since $\beta \neq 0$, not collinear.

Table 2: Segre types for $R^a{}_b$ in n-dimensional spacetimes.

dimension of spacetime	Segre types (non-degenerated)
3	$[1,11]$, $[21]$, $[3]$, $[z\bar{z}1]$
4	$[1,111]$, $[211]$, $[31]$, $[z\bar{z}11]$
5	$[1,1111]$, $[2111]$, $[311]$, $[z\bar{z}111]$
$\vdots$	$\vdots$
n	$[1,1\ldots1]$, $[21\ldots1]$, $[31\ldots1]$, $[z\bar{z}11\ldots1]$

Whichever is the case, the real and the imaginary part of (2.1) give

$$R^a{}_b Y^b = \alpha Y^a - \beta Z^a \,, \tag{3.1}$$

$$R^a{}_b Z^b = \beta Y^a + \alpha Z^a \,. \tag{3.2}$$

Thus, the vectors $\mathbf{Y}$ and $\mathbf{Z}$ span a timelike two-dimensional subspace of $T_p(M)$ invariant under $R^a{}_b$. Besides, by a procedure similar to that used in 5-D Lorentzian spaces [23] one can show that the $(n-2)$-dimensional space orthogonal to this timelike 2-space is spacelike, is also invariant under $R^a{}_b$ and contains $n-2$ orthogonal eigenvectors of $R^a{}_b$ with real eigenvalues. These eigenvectors, together with $\mathbf{V}_+$ and $\mathbf{V}_-$, form a set of n linearly independent eigenvectors of $R^a{}_b$ at $p \in M$. Therefore, when there exists complex (a conjugate pair of) eigenvalues, $R^a{}_b$ is necessarily diagonalizable over the complex field and possesses $n-2$ real eigenvalues. Its Segre type is $[z\bar{z}1\ldots1]$ or one of its degeneracies.

4 A Set of Canonical Forms

An often used approach in physics to establish a canonical form for a tensor is to align the basis vectors along the preferred directions intrinsically defined by the tensor. As far as the Ricci tensor R is concerned, the existence of null eigenvectors suggests to choose a semi-null basis $\mathcal{B}$ for $T_p(M)$, consisting of 2 null vectors and $n-2$ spacelike vectors,

$$\mathcal{B} = \{\mathbf{l}, \mathbf{m}, \mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(n-2)}\}\,, \tag{4.1}$$

such that the only non-vanishing inner products are

$$\mathbf{l}.\mathbf{m} = \mathbf{x}^{(1)}.\mathbf{x}^{(1)} = \mathbf{x}^{(2)}.\mathbf{x}^{(2)} = \ldots = \mathbf{x}^{(n-2)}.\mathbf{x}^{(n-2)} = 1 \tag{4.2}$$

The most general decomposition of R_{ab} in terms of the basis $\mathcal{B}$ is

$$\begin{aligned} R_{ab} \;=\; & 2\rho_1 l_{(a}m_{b)} + \rho_2 l_a l_b + \\ & \rho_3 x_a^{(1)} x_b^{(1)} + \rho_4 x_a^{(2)} x_b^{(2)} + \cdots + \rho_n x_a^{(n-2)} x_b^{(n-2)} + \rho_{n+1} m_a m_b + \\ & 2\rho_{n+2} l_{(a} x_{b)}^{(1)} + \cdots + 2\rho_{n(n+1)/2} x_{(a}^{(n-3)} x_{b)}^{(n-2)} \,, \end{aligned} \tag{4.3}$$

where the coefficients $\rho_1, \ldots, \rho_{n(n+1)/2}$ are real scalars.

We shall now show that semi-null bases $\mathcal{B}$ can always be chosen so that R_{ab} takes one of the following canonical forms at $p \in M$:

Segre type	**Canonical form**
$[1,1\ldots1]$	$R_{ab} = 2\rho_1 l_{(a}m_{b)} + \rho_2(l_a l_b + m_a m_b) + \rho_3 x_a^{(1)} x_b^{(1)} + \rho_4 x_a^{(2)} x_b^{(2)} + \cdots + \rho_n x_a^{(n-2)} x_b^{(n-2)}\,,$ (4.4)
$[21\ldots1]$	$R_{ab} = 2\rho_1 l_{(a}m_{b)} \pm l_a l_b + \rho_3 x_a^{(1)} x_b^{(1)} + \rho_4 x_a^{(2)} x_b^{(2)} + \cdots + \rho_n x_a^{(n-2)} x_b^{(n-2)}\,,$ (4.5)
$[31\ldots1]$	$R_{ab} = 2\rho_1 l_{(a}m_{b)} + 2 l_{(a}x_{b)} + \rho_1 x_a^{(1)} x_b^{(1)} + \rho_4 x_a^{(2)} x_b^{(2)} + \cdots + \rho_n x_a^{(n-2)} x_b^{(n-2)}\,,$ (4.6)
$[z\bar{z}11\ldots1]$	$R_{ab} = 2\rho_1 l_{(a}m_{b)} + \rho_2(l_a l_b - m_a m_b) + \rho_3 x_a^{(1)} x_b^{(1)} + \rho_4 x_a^{(2)} x_b^{(2)} + \cdots + \rho_n x_a^{(n-2)} x_b^{(n-2)}\,,$ (4.7)

where the coefficients $\rho_1, \ldots, \rho_n$ are real scalars and $\rho_2 \neq 0$ in (4.7). Clearly these coefficients are related to the eigenvalues of $R^a{}_b$.

Using the non-vanishing inner products of the basis $\mathcal{B}$, it is not difficult to show that each of the above expressions for R_{ab} leads to the corresponding Segre type indicated on the left. However, to show the reciprocal is not as simple as that and we shall examine case by case.

Segre type [1,1...1]. For this Segre type one writes down a n-dimensional general real symmetric matrix for the metric tensor g_{ab} and imposes the condition

$$g_{ac}\,R^c{}_b = g_{bc}\,R^c{}_a \tag{4.8}$$

to account for the symmetry of R_{ab}. In a basis where $R^a{}_b$ is diagonal (4.8) implies that g_{ab} is also diagonal. As det $(g_{ab}) < 0$, then all $g_{ii} \neq 0$. Each basis vector is an eigenvector of $R^a{}_b$ and has norm g_{ii}, so they are either timelike or spacelike and orthogonal to each other. Actually, owing to the Lorentzian signature we have one timelike and $n-1$ spacelike vectors, which suitably normalized give a n-dimensional Minkowski basis $\bar{\mathcal{B}} = \{\mathbf{t}, \mathbf{w}, \mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(n-2)}\}$ where $\mathbf{t}$ is the timelike eigenvector. If one now writes down the expression for R_{ab} in terms of this Minkowski basis, and then introduces the null vectors $\mathbf{l} = \frac{1}{\sqrt{2}}(\mathbf{w} + \mathbf{t})$ and $\mathbf{m} = \frac{1}{\sqrt{2}}(\mathbf{w} - \mathbf{t})$ to form a semi-null basis $\mathcal{B} = \{\mathbf{l}, \mathbf{m}, \mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(n-2)}\}$, one finally obtains R_{ab} in the canonical form (4.4). The eigenvalues are $\rho_1 - \rho_2$, $\rho_1 + \rho_2$, ρ_3, $\rho_4, \ldots, \rho_n$ and the corresponding eigenvectors are $\mathbf{l} - \mathbf{m}$, $\mathbf{l} + \mathbf{m}$, $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$, $\ldots, \mathbf{x}^{(n-2)}$.

Segre type [21...1]. From table 1 one learns that for this case the first vector of the Jordan basis, namely the eigenvector $\mathbf{X}_1$, is a null vector, and using proposition 3 one finds that the eigenvectors $\mathbf{X}_3, \mathbf{X}_4, \ldots, \mathbf{X}_n$ are spacelike, mutually orthogonal and orthogonal to $\mathbf{X}_1$. We are then naturally led to choose $\mathbf{l}$ along the direction of $\mathbf{X}_1$, and $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n-2)}$ along the directions of $\mathbf{X}_3, \ldots, \mathbf{X}_n$, respectively. To complete the semi-null basis we choose $\mathbf{m}$ along the direction uniquely defined by $\mathbf{l}, \mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n-2)}$. Imposing that these last vectors are eigenvectors of R one finds that R_{ab} in (4.3) simplifies to

$$\begin{aligned} R_{ab} &= 2\rho_1 l_{(a} m_{b)} + \rho_2 l_a l_b + \rho_3 x_a^{(1)} x_b^{(1)} + \rho_4 x_a^{(2)} x_b^{(2)} \\ &\quad + \cdots + \rho_n x_a^{(n-2)} x_b^{(n-2)}\,, \end{aligned} \tag{4.9}$$

where the condition $\rho_2 \neq 0$ must be imposed otherwise $\mathbf{m}$ would be a n-th linearly independent eigenvector. We finally make the transformation $\mathbf{l} \to \mathbf{l}\,|\rho_2|^{-1/2}$, $\mathbf{m} \to \mathbf{m}\,|\rho_2|^{1/2}$, to bring R_{ab} from the general form (4.3) into the form (4.5) with ρ_1, ρ_3, $\ldots, \rho_n$ the eigenvalues associated to the eigenvectors $\mathbf{l}$, $\mathbf{x}^{(1)}$, $\ldots, \mathbf{x}^{(n-2)}$.

Segre type $[\mathbf{31\ldots 1}]$. For this Segre type one learns from table 1 that the Jordan basis contains a null vector $\mathbf{X}_1$, orthogonal to the spacelike vector $\mathbf{X}_2$. Moreover, employing proposition 3 one also finds that $\mathbf{X}_4, \mathbf{X}_5, \ldots, \mathbf{X}_n$ are spacelike eigenvectors orthogonal to $\mathbf{X}_1$, and a further inspection of the various Jordan chains shows that they are also mutually orthogonal as well as orthogonal to $\mathbf{X}_2$ and $\mathbf{X}_3$. Nevertheless the vectorial character of $\mathbf{X}_3$ remains open. So, to form a semi-null basis for this case a new basis $\{\tilde{\mathbf{X}}_a; a = 1, \cdots, n\}$ ought to be found. It can be shown that the following basis transformation leaves invariant the matrix $J^a{}_b$:

$$
\begin{aligned}
\tilde{\mathbf{X}}_1 &= a\mathbf{X}_1 \,, \\
\tilde{\mathbf{X}}_2 &= a\mathbf{X}_2 + b\mathbf{X}_1 \,, \\
\tilde{\mathbf{X}}_3 &= a\mathbf{X}_3 + b\mathbf{X}_2 + c\mathbf{X}_1 \,, \\
\tilde{\mathbf{X}}_i &= d_i\mathbf{X}_i \qquad \text{(no sum)} \,,
\end{aligned}
\tag{4.10}
$$

where a, b, c, d_i $(i = 4, \cdots, n)$ are n arbitrary real constants with $a \neq 0$ and $d_i \neq 0$. To endow the new basis $\{\tilde{\mathbf{X}}_a\}$ with the orthonormality relations of a semi-null basis (4.2) we simply have to impose the n constraints

$$\tilde{\mathbf{X}}_2.\tilde{\mathbf{X}}_2 = \tilde{\mathbf{X}}_4.\tilde{\mathbf{X}}_4 = \cdots = \tilde{\mathbf{X}}_n.\tilde{\mathbf{X}}_n = 1 \,, \tag{4.11}$$

$$\tilde{\mathbf{X}}_2.\tilde{\mathbf{X}}_3 = \tilde{\mathbf{X}}_3.\tilde{\mathbf{X}}_3 = 0 \,. \tag{4.12}$$

The constraints define the values of a, b, c, d_i in (4.10), namely

$$a = (\mathbf{X}_2.\mathbf{X}_2)^{-1/2} \,, \qquad b = -\frac{1}{2}a^3\mathbf{X}_2.\mathbf{X}_3 \,, \tag{4.13}$$

$$c = \frac{3}{2}b^2a^{-1} - \frac{1}{2}a^3\mathbf{X}_3.\mathbf{X}_3, \qquad d_i = (\mathbf{X}_i.\mathbf{X}_i)^{-1/2} \,. \tag{4.14}$$

We finally choose $\mathbf{l} = \tilde{\mathbf{X}}_1$, $\mathbf{x}^{(1)} = \tilde{\mathbf{X}}_2$, $\mathbf{m} = \tilde{\mathbf{X}}_3$, $\mathbf{x}^{(i-2)} = \tilde{\mathbf{X}}_i$ $(i = 4, \cdots, n)$ and write R_{ab} as (4.6) with $\rho_1, \rho_4, \rho_5, \cdots, \rho_n$ the eigenvalues associated to the eigenvectors $\mathbf{l}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(n-2)}$, respectively.

Segre type $[\mathbf{z\,\bar{z}1\ldots 1}]$. For this type all vectors of a Jordan basis are eigenvectors of $R^a{}_b$. Two of them, namely $\mathbf{V}_\pm = \mathbf{Y} \pm i\,\mathbf{Z}$, are complex conjugate with eigenvalues $\alpha \pm i\,\beta$, respectively (see section 2). The other basis vectors $\mathbf{X}_3, \cdots, \mathbf{X}_n$ are spacelike eigenvectors mutually orthogonal and orthogonal to the timelike 2-D space spanned by

$\mathbf{Y}$ and $\mathbf{Z}$. Through a linear combination of $\mathbf{Y}$ and $\mathbf{Z}$ two null vectors $\mathbf{l}$ and $\mathbf{m}$ can be constructed satisfying $\mathbf{l}.\mathbf{m} = 1$ and

$$
\begin{aligned}
R^a{}_b\, l^b &= \alpha l^a - \beta m^a \,, && (4.15)\\
R^a{}_b\, m^b &= \beta l^a + \alpha m^a \,. && (4.16)
\end{aligned}
$$

One can then form a semi-null basis $\{\mathbf{l}, \mathbf{m}, \mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(n-2)}\}$, where each $\mathbf{x}^{(i)}$ is the spacelike eigenvector $\mathbf{X}_i$ suitably normalized. In terms of this basis, and taking into account (4.15) and (4.16), the Ricci tensor reduces from the general form (4.3) to the form (4.7), where $\rho_1 = \alpha$ and $\rho_2 = \beta$. The eigenvalues are $\rho_1 + i\rho_2$, $\rho_1 - i\rho_2$, $\rho_3, \cdots, \rho_n$ and the corresponding eigenvectors are $\mathbf{l} + i\mathbf{m}\,, \mathbf{l} - i\mathbf{m}$, $\mathbf{x}^{(1)}, \cdots, \mathbf{x}^{(n-2)}$.

5 Acknowledgments

We take this occasion to express our warmest thanks to our friend Graham Hall for many motivating and valuable discussions during his visit to CBPF (Brazilian Center for Physics Research).

References

[1] R. V. Churchill, *Trans. Amer. Math. Soc.* **34**, 784 (1932).

[2] J. Plebański, *Acta Phys. Pol.* **26**, 963 (1964).

[3] G. S. Hall, *Diff. Geom.* **12**, 53 (1984). This reference contains an extensive bibliography on the classification of the Ricci tensor on 4-D Lorentzian spacetime manifolds.

[4] G. S. Hall, *Arab. J. Sci. Eng.* **9**, 87 (1984).

[5] G. S. Hall and D. A. Negm, *Int. J. Theor. Phys.* **25**, 405 (1986).

[6] M. J. Rebouças, J. E. Åman and A. F. F. Teixeira, *J. Math. Phys.* **27**, 1370 (1986).

[7] M. J. Rebouças and J. E. Åman, *J. Math. Phys.* **28**, 888 (1987).

[8] M. O. Calvão, M. J. Rebouças, A. F. F. Teixeira and W. M. Silva Jr., *J. Math. Phys.* **29**, 1127 (1988).

[9] J. J. Ferrando, J.A. Morales and M. Portilla, *Gen. Rel. Grav.* **22**, 1021 (1990).

[10] M. J. Rebouças and A. F. F. Teixeira, *J. Math. Phys.* **32**, 1861 (1991).

[11] M. J. Rebouças and A. F. F. Teixeira, *J. Math. Phys.* **33**, 2855 (1992).

[12] J. Santos, M. J. Rebouças and A. F. F. Teixeira, *J. Math. Phys.* **34**, 186 (1993).

[13] G. S. Hall, M. A. H. MacCallum, F. M. Paiva and M. J. Rebouças, *Limits in the Segre classification of the energy-momentum tensor in general relativity* (in preparation).

[14] W. J. Cormack and G. S. Hall, *J. Phys. A* **12**, 55 (1979).

[15] A. Karlhede, *Gen. Rel. Grav.* **12**, 693 (1980).

[16] See M. A. H. MacCallum and J. E. F. Skea, "SHEEP: A Computer Algebra System for General Relativity", in *Algebraic Computing in General Relativity, Lecture Notes from the First Brazilian School on Computer Algebra*, Vol. II, edited by M. J. Rebouças and W. L. Roque. Oxford U. P., Oxford (1994) and refereces therein.

[17] M. A. H. MacCallum, "Classifying Metrics in Theory and Practice", in *Unified Field Theory in More Than 4 Dimensions, Including Exact Solutions*, edited by V. de Sabbata and E. Schmutzer. World Scientific, Singapore (1983).

[18] M. A. H. MacCallum, "Algebraic Computing in General Relativity", in *Classical General Relativity*, edited by W. B. Bonnor, J. N. Islam and M. A. H. MacCallum. Cambridge U. P., Cambridge (1984).

[19] M. A. H. MacCallum, "Computer-aided Classification of Exact Solutions in General Relativity", in *General Relativity and Gravitational Physics (9th Italian Conference)*, edited by R. Cianci, R. de Ritis, M. Francaviglia, G. Marmo, C. Rubano and P. Scudellaro. World Scientific Publishing Co., Singapore (1991).

[20] For a recent review of Kaluza-Klein theories see, for example, T. Matos and J.A. Nieto, *Revista Mexicana de Física* **39**, 81 (1993), and references therein.

[21] See, for example, R. J. Gleiser and M. C. Diaz, *Phys. Rev. D* **37**, 3761 (1988) and references therein quoted on this subject.

[22] G. S. Hall, T. Morgan and Z. Perjés, *Gen. Rel. Grav.* **19**, 1137 (1987).

[23] J. Santos, M. J. Rebouças and A. F. F. Teixeira, *Classification of second order symmetric tensors in 5-dimensional Kaluza-Klein-type theories*, to appear in J. Math. Phys. (1995).

[24] J. Santos, M. J. Rebouças and A. F. F. Teixeira, in *Gravitation: the Spacetime Structure*, proceedings of the 8th Latin American Symposium on Relativity and Gravitation, edited by P. S. Letelier and W. A. Rodrigues, Jr. World Scientific Publishing Co., p. 251 – 258 (1994).

[25] G.E. Shilov, *Linear Algebra* (Dover Publ. Inc., New York, 1977).

[26] I.N. Herstein and D.J. Winter, *Matrix Theory and Linear Algebra* (Macmillan Publ. Co., New York, 1988).