

Matrix Model Approach to $d > 2$ Non-critical Superstrings

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Abstract

We apply light-cone quantization to a $1+1$ dimensional supersymmetric field theory of large N matrices. We provide some preliminary numerical evidence that when the coupling constant is tuned to a critical value, this model describes a $2+1$ dimensional non-critical superstring.

Matrix models have been remarkably successful [1] in providing us with an understanding of non-critical bosonic string theories in dimensions less than or equal to two. Their extension to dimensions greater than two, however, runs into a number of difficulties, some of them related to the appearance of tachyons in the bosonic string spectrum. Quantization in the infinite momentum frame is a promising approach to these models, where strings emerge as bound states of more elementary “string bits” [2, 3]. Studies of the $c = 2$ matrix model using this technique have indeed revealed the expected tachyon problem [4, 5].

While the bosonic strings are unstable above two dimensions, one expects this instability to be cured by target space supersymmetry. Full space-time supersymmetry requires that the string theory be ten-dimensional (with a possibility of subsequent compactification).¹ It is well known, however, that in $c + 1$ dimensional non-critical string theories the Liouville coordinate enters on a different footing from other dimensions, so that there is no full Poincare invariance. Therefore, one cannot demand a full $c + 1$ dimensional supersymmetry. The most one could require is the c dimensional “space supersymmetry” which was considered in ref. [7]. While this work utilized the NSR approach to the super-Liouville theory, it is interesting to ask whether an equivalent Green-Schwartz formulation exists [8, 9]. An even more ambitious goal is to construct matrix models which describe non-critical superstring theories. In this paper we make a small step in this direction by formulating and numerically studying a two-dimensional supersymmetric matrix field theory which is expected to describe $2 + 1$ dimensional non-critical superstrings.

Our model is a higher-dimensional generalization of the Marinari-Parisi model (the supersymmetric quantum mechanics of a large- N hermitian matrix) [10]. It is not entirely clear whether the continuum limit of the Marinari-Parisi model describes a theory with target space supersymmetry; in fact, the conventional understanding is that this model provides an alternative description of the bosonic $c = 0$ string [10, 11]. There are some signs, however, of remaining target space supersymmetry [12]. Since one-dimensional supersymmetry is quite trivial, we hope that going one dimension higher will reveal a more interesting structure.

¹A successful construction of critical superstrings from string bit models has been found recently [6].

The price we have to pay is that the model is no longer exactly solvable. Luckily, using the techniques of light-cone quantization, we can still set up a scheme for calculating the spectrum of the matrix model in the large- N limit. Our approach is largely an extension of the program of [4, 5], with the advantage that the supersymmetry cures the sickness of the model.

The $c = 2$ matrix model studied in [4] is defined as a two-dimensional field theory with the euclidean action

$$S = \int d^2x \text{Tr} \left(\frac{1}{2} (\partial_\mu \phi)^2 + \frac{1}{2} m^2 \phi^2 - \frac{1}{3\sqrt{N}} \lambda \phi^3 \right) ,$$

where $\phi(x^0, x^1)$ is an $N \times N$ hermitian matrix field. The connection of this model with triangulated random surfaces follows, as usual, after identifying the Feynman graphs with the graphs dual to triangulations. At the leading order in N , we obtain a sum over the planar triangulated random surfaces embedded in two dimensions. If the tadpole graphs are discarded, as they should be because they do not correspond to good triangulations, then the entire perturbative expansion is finite. This is similar to what we find in the matrix models for $c \leq 1$. Therefore, it is sensible to look for a singularity in the sum over the planar graphs as λ approaches some critical value λ_c . For the $c = 1$ model the spectrum of energies becomes continuous at $\lambda = \lambda_c$. This is because we are really dealing with a 1+1 dimensional string theory whose center-of-mass mode satisfies [13]

$$E^2 = p_\phi^2$$

where p_ϕ is the Liouville momentum which possesses continuous spectrum. By analogy one might expect that in the $c = 2$ model the spectrum of $M^2 = 2P_+P_-$ becomes continuous at $\lambda = \lambda_c$. Light-cone quantization of the model revealed, however, that the mass-squared of the lowest state becomes negative for sufficiently large λ and, in fact, tends to $-\infty$ as $\lambda \rightarrow \lambda_c$ [5]. Although it is still possible that a continuum of states is forming around $M^2 = -\infty$, it was difficult to draw definite conclusions from numerical studies of the spectrum. Further evidence for a phase transition at some critical value of the coupling was given [14] based on an approximate identification of the $c = 2$ model with a one-dimensional spin chain. However, the fact remains that the corresponding continuum theory, even if it exists in some sense, is a very sick theory containing a tachyon.

In this article, we attempt to cure this sickness by considering a 1+1 dimensional large- N hermitian matrix field theory with (1, 1) supersymmetry

$$[P_+, P_-] = [P_\pm, Q_\pm] = \{Q_+, Q_-\} = 0 ,$$

$$Q_+^2 = P_+, \quad Q_-^2 = P_- .$$

From the above algebra it follows that the operators P_+ and P_- are positive definite, since they are squares of hermitian supercharges Q_+ and Q_- . This implies that $M^2 = 2P_+P_-$ is also positive definite. As we tune the parameters, we expect the theory to become effectively 2+1 dimensional due to the appearance of the Liouville mode which makes the spectrum of M^2 continuous. Furthermore, we expect the mass-squared of the lightest state to vanish at this critical point, since the appearance of new massless states is typically a signature

of the phase transition. Indeed, the continuous spectrum with a vanishing mass gap was observed in the $c = 1$ [13] and the Marinari-Parisi models [10]. In this article, we present some evidence that such massless continuum of states does appear in the spectrum of $1 + 1$ dimensional supersymmetric matrix field theory.

The simplest matrix field theory with the above supersymmetry is given by the action

$$S = \frac{1}{4} \int d^2x d^2\theta \text{Tr} \left[\bar{D}\Phi D\Phi + W(\Phi) \right]$$

where Φ is a $(1,1)$ matrix superfield

$$\Phi_{ij} = \phi_{ij} + \bar{\theta}\Psi_{ij} + \bar{\theta}\theta F_{ij},$$

and Ψ_{ij} is a matrix whose elements are two-component spinors

$$\Psi_{ij} = \begin{bmatrix} \Psi_- \\ \Psi_+ \end{bmatrix}_{ij}.$$

In light-cone quantization only Ψ_- turns out to be dynamical. D is the covariant derivative

$$D = \frac{\partial}{\partial\theta} - i\Gamma^\mu\theta\partial_\mu.$$

where Γ^μ are the two dimensional Dirac matrices in the Majorana representation

$$\Gamma^0 = \begin{bmatrix} & -i \\ i & \end{bmatrix}, \quad \Gamma^1 = \begin{bmatrix} & i \\ i & \end{bmatrix}.$$

We consider the simplest superpotential $W(\Phi)$ given by

$$W(\Phi) = \frac{1}{2}\mu\Phi^2 - \frac{\lambda}{3\sqrt{N}}\Phi^3.$$

The Feynman graphs which arise from the perturbative expansion of the path integral generate triangulated random surfaces embedded in the superspace with coordinates $(x^0, x^1, \theta^1, \theta^2)$. We expect that, as λ is tuned to some critical value λ_c , the size of a typical graph diverges so that the world sheet continuum limit may be defined. The dimension two terms in the world sheet action are fixed by the target-space supersymmetry (they may also be inferred from the superspace propagator),

$$S_{\text{world sheet}} = \int d^2\sigma \sqrt{h} h^{\alpha\beta} \left[z_1 \left(\partial_\alpha x^\mu - i\bar{\theta}\Gamma^\mu\partial_\alpha\theta \right) \left(\partial_\beta x_\mu - i\bar{\theta}\Gamma_\mu\partial_\beta\theta \right) + iz_2\partial_\alpha\bar{\theta}\partial_\beta\theta \right], \quad (1)$$

where z_1 and z_2 are normalization constants, and the metric $h_{\alpha\beta}$ is regarded as a dynamical variable. This action resembles the Green-Schwartz action. However, it has no Wess-Zumino term or manifest κ -symmetry. Nevertheless, if we find anything non-trivial from our model, it is a kind of non-critical Green-Schwartz superstring.

In terms of component fields, the matrix model action is written

$$S = \int d^2x \text{Tr} \left[\frac{1}{2}(\partial_\mu\phi)^2 + \frac{1}{2}\bar{\Psi}i\not{\partial}\Psi - \frac{1}{2}V^2(\phi) - \frac{1}{2}V'(\phi)\bar{\Psi}\Psi \right] \quad (2)$$

where $V(\phi) = W'(\phi) = \mu\phi - \frac{\lambda}{\sqrt{N}}\phi^2$. As such, it could also be thought of as a matrix analogue of the model considered in [15] with zero central charge. The above Lagrangian contains cubic and quartic interaction terms which depend linearly and quadratically on the coupling constant λ . One could now proceed to enumerate the states and determine the matrix elements of P_- as was done in theories without supersymmetry [4, 5]. Instead, we prefer to determine the matrix elements of supercharges for the enumerated states, as was recently proposed in [16]. This has a number of advantages. Firstly, the supercurrent contains only terms up to cubic order in fields, and depends only linearly on the coupling constant λ . Secondly, the supercharge is a more fundamental dynamical object, being the square root of the Hamiltonian. We find it particularly convenient to consider the combination $I = \sqrt{2}iQ_+Q_-$. This operator is bosonic, commutes simultaneously with P_+ and P_- , and its square is equal to M^2 . I contains more information than M^2 because it fixes the ambiguity of sign when one takes the square root. When we search for evidence for the continuous spectrum, this fact will be rather useful.

The action is invariant under the standard supersymmetry transformation

$$\begin{aligned}\delta\phi_{ij} &= \bar{\epsilon}\Psi_{ij} \\ \delta\Psi_{ij} &= -i\Gamma^\mu\epsilon\partial_\mu\phi_{ij}.\end{aligned}$$

The supercurrent can easily be shown to be

$$J^\mu = \frac{1}{\sqrt{2}}\text{Tr}[(\partial\phi)\Gamma^\mu\Psi + iV(\phi)\Gamma^\mu\Psi].$$

The light-cone components of the supercharges are

$$\begin{aligned}Q_- &= \int dx^- : \text{Tr}[\sqrt{2}(\partial_-\phi)\Psi_-] : \\ Q_+ &= \int dx^- : \text{Tr}[V(\phi)\Psi_-] : .\end{aligned}$$

Expanding in normal modes²

$$\begin{aligned}\phi_{ij}(x^-) &= \frac{1}{\sqrt{2\pi}}\int_0^\infty\frac{dk_-}{\sqrt{2k_-}}\left(a_{ij}(k_-)e^{-ik_-x^-} + a_{ji}^\dagger(k_-)e^{ik_-x^-}\right) \\ \frac{1}{\sqrt{2}}\Psi_{ij}(x^-) &= \frac{1}{\sqrt{2\pi}}\int_0^\infty\frac{dk_-}{\sqrt{2}}\left(b_{ij}(k_-)e^{-ik_-x^-} + b_{ji}^\dagger(k_-)e^{ik_-x^-}\right),\end{aligned}$$

the supercharges become

$$\begin{aligned}iQ_- &= \int_0^\infty dk\sqrt{k}\left(b_{ij}^\dagger(k)a_{ij}(k) - a_{ij}^\dagger(k)b_{ij}(k)\right) \\ \sqrt{2}Q_+ &= \int_0^\infty dk\frac{1}{\sqrt{k}}\left(b_{ij}^\dagger(k)a_{ij}(k) + a_{ij}^\dagger(k)b_{ij}(k)\right) \\ &\quad - \frac{y}{\sqrt{N}}\int dk_1dk_2\left\{\right.\end{aligned}$$

²The factor of $1/\sqrt{2}$ appears due to the unusual normalization of the kinetic term for Ψ in (2).

$$\left. \begin{aligned} & \frac{1}{\sqrt{k_1 k_2}} \left[a_{ij}^\dagger(k_1) a_{jk}^\dagger(k_2) b_{ik}(k_1 + k_2) + b_{ik}^\dagger(k_1 + k_2) a_{ij}(k_1) a_{jk}(k_2) \right] \\ & \frac{1}{\sqrt{k_1(k_1 + k_2)}} \left[a_{ij}^\dagger(k_1) b_{jk}^\dagger(k_2) a_{ik}(k_1 + k_2) + a_{ik}^\dagger(k_1 + k_2) a_{ij}(k_1) b_{jk}(k_2) \right] \\ & \frac{1}{\sqrt{k_2(k_1 + k_2)}} \left[b_{ij}^\dagger(k_1) a_{jk}^\dagger(k_2) a_{ik}(k_1 + k_2) + a_{ik}^\dagger(k_1 + k_2) b_{ij}(k_1) a_{jk}(k_2) \right] \end{aligned} \right\}.$$

We have dropped the subscript “minus” on k for brevity, set $\mu = 1$, and introduced the dimensionless coupling parameter $y = \lambda/(2\sqrt{\pi}\mu)$. The theory may be regulated in the infrared by compactifying the x^- direction and imposing periodic boundary conditions on the fields. The momentum k then takes on discrete values n/L , and the integrals above are replaced by sums. For massive particles we may discard the $n = 0$ modes, which give infinite light-cone energy, and restrict to positive integer n . This regularization was also shown to preserve supersymmetry [16]. We introduce discretized oscillators

$$A_{ij}(n) = \frac{1}{\sqrt{L}} a_{ij}(k = n/L), \quad B_{ij}(n) = \frac{1}{\sqrt{L}} b_{ij}(k = n/L),$$

which satisfy commutation relations

$$\begin{aligned} [A_{ij}(n), A_{kl}^\dagger(n')] &= \delta_{nn'} \delta_{ik} \delta_{jl} \\ \{B_{ij}(n), B_{kl}^\dagger(n')\} &= \delta_{nn'} \delta_{ik} \delta_{jl}. \end{aligned} \quad (3)$$

The discretized expressions for the supercharges are

$$\begin{aligned} iQ_- &= \sum_{n=1} \sqrt{n} \left(B_{ij}^\dagger(n) A_{ij}(n) - A_{ij}^\dagger(n) B_{ij}(n) \right) \\ \sqrt{2}Q_+ &= \sum_{n=1} \frac{1}{\sqrt{n}} \left(B_{ij}^\dagger(n) A_{ij}(n) + A_{ij}^\dagger(n) B_{ij}(n) \right) \\ &\quad - y \sum_{n_1=1} \sum_{n_2=1} \left\{ \begin{aligned} & \frac{1}{\sqrt{n_1 n_2}} \left[A_{ij}^\dagger(n_1) A_{jk}^\dagger(n_2) B_{ik}(n_1 + n_2) + B_{ik}^\dagger(n_1 + n_2) A_{ij}(n_1) A_{jk}(n_2) \right] \\ & \frac{1}{\sqrt{n_1(n_1 + n_2)}} \left[A_{ij}^\dagger(n_1) B_{jk}^\dagger(n_2) A_{ik}(n_1 + n_2) + A_{ik}^\dagger(n_1 + n_2) A_{ij}(n_1) B_{jk}(n_2) \right] \\ & \frac{1}{\sqrt{n_2(n_1 + n_2)}} \left[B_{ij}^\dagger(n_1) A_{jk}^\dagger(n_2) A_{ik}(n_1 + n_2) + A_{ik}^\dagger(n_1 + n_2) B_{ij}(n_1) A_{jk}(n_2) \right] \end{aligned} \right\}. \end{aligned}$$

The Hilbert space of this model can be constructed by acting on the Fock vacuum with mode creation operators A_{ij}^\dagger and B_{ij}^\dagger . As in ref. [4, 5], we restrict our attention to singlet states,³ i.e. states of the form

$$|\Phi\rangle \sim \text{Tr} \left[A^\dagger(n_1) A^\dagger(n_2) B^\dagger(n_3) \cdots A^\dagger(n_i) \right] |0\rangle.$$

³The non-singlet states do not have an obvious stringy interpretation. As discussed in ref. [17] they may be eliminated by adding some amount of the gauge interaction.

The light-cone momentum of such a state is given by $P_- = K/L$, where $K = \sum_i n_i$, and we restrict ourselves to states of fixed P_- . The integer K plays the role of the cut-off and, remarkably, for a finite K the total number of states is finite [18]. The continuum limit is recovered as L and K are sent to infinity. For the sake of illustration, we display the set of states with $K = 3$. There are five bosonic states

$$\begin{aligned}
|1\rangle_b &= \frac{1}{N^{3/2}\sqrt{3}} \text{Tr} [A^\dagger(1)A^\dagger(1)A^\dagger(1)] |0\rangle \\
|2\rangle_b &= \frac{1}{N^{3/2}} \text{Tr} [A^\dagger(1)B^\dagger(1)B^\dagger(1)] |0\rangle \\
|3\rangle_b &= \frac{1}{N} \text{Tr} [A^\dagger(2)A^\dagger(1)] |0\rangle \\
|4\rangle_b &= \frac{1}{N} \text{Tr} [B^\dagger(2)B^\dagger(1)] |0\rangle \\
|5\rangle_b &= \frac{1}{N^{1/2}} \text{Tr} [A^\dagger(3)] |0\rangle
\end{aligned}$$

and five fermionic states

$$\begin{aligned}
|1\rangle_f &= \frac{1}{N^{3/2}} \text{Tr} [A^\dagger(1)A^\dagger(1)B^\dagger(1)] |0\rangle \\
|2\rangle_f &= \frac{1}{N^{3/2}\sqrt{3}} \text{Tr} [B^\dagger(1)B^\dagger(1)B^\dagger(1)] |0\rangle \\
|3\rangle_f &= \frac{1}{N} \text{Tr} [A^\dagger(2)B^\dagger(1)] |0\rangle \\
|4\rangle_f &= \frac{1}{N} \text{Tr} [B^\dagger(2)A^\dagger(1)] |0\rangle \\
|5\rangle_f &= \frac{1}{N^{1/2}} \text{Tr} [B^\dagger(3)] |0\rangle.
\end{aligned}$$

(4)

For higher values of K all states may be generated systematically by considering all partitions of K into positive integers, assigning one of the two oscillators, A^\dagger or B^\dagger , to each integer, and eliminating null and redundant states. Once the basis of states for a given K has been found, it is straightforward to determine the matrix elements of I .

As in the bosonic case, the mode operators create or annihilate “string bits,” except now there are two species of such bits. The stringy states we are interested in may be thought of as bound states of these string bits. Various terms in the supercharges implement joining and splitting of neighboring bits or an interchange of a bosonic and a fermionic bits.

In what follows, we describe the result of our analysis for the spectrum of $I = \sqrt{2}iQ_+Q_-$. We constructed the exact matrix representation of I symbolically for finite K and then evaluated its spectrum numerically. The goal is to examine the convergence of the spectrum in the large K limit in order to recover the continuum theory. Unfortunately, the dimension of matrix I grows rapidly with K . Our computer resources have allowed us to perform diagonalizations for $K = 3, 4, 5, 6, 7, 8, 9$, and 10, where the dimensions of the matrix I are 5, 10, 25, 62, 157, 410, 1097, and 2954, respectively. The number of states for each K is greater than that found in [16] by one, since our model is not gauged and we include the single string bit state.

In figure 1 we plot the spectrum of I as a function of y for K ranging from 5 to 10 (we show only the eigenvalues whose magnitude does not exceed 3). Note that, for each

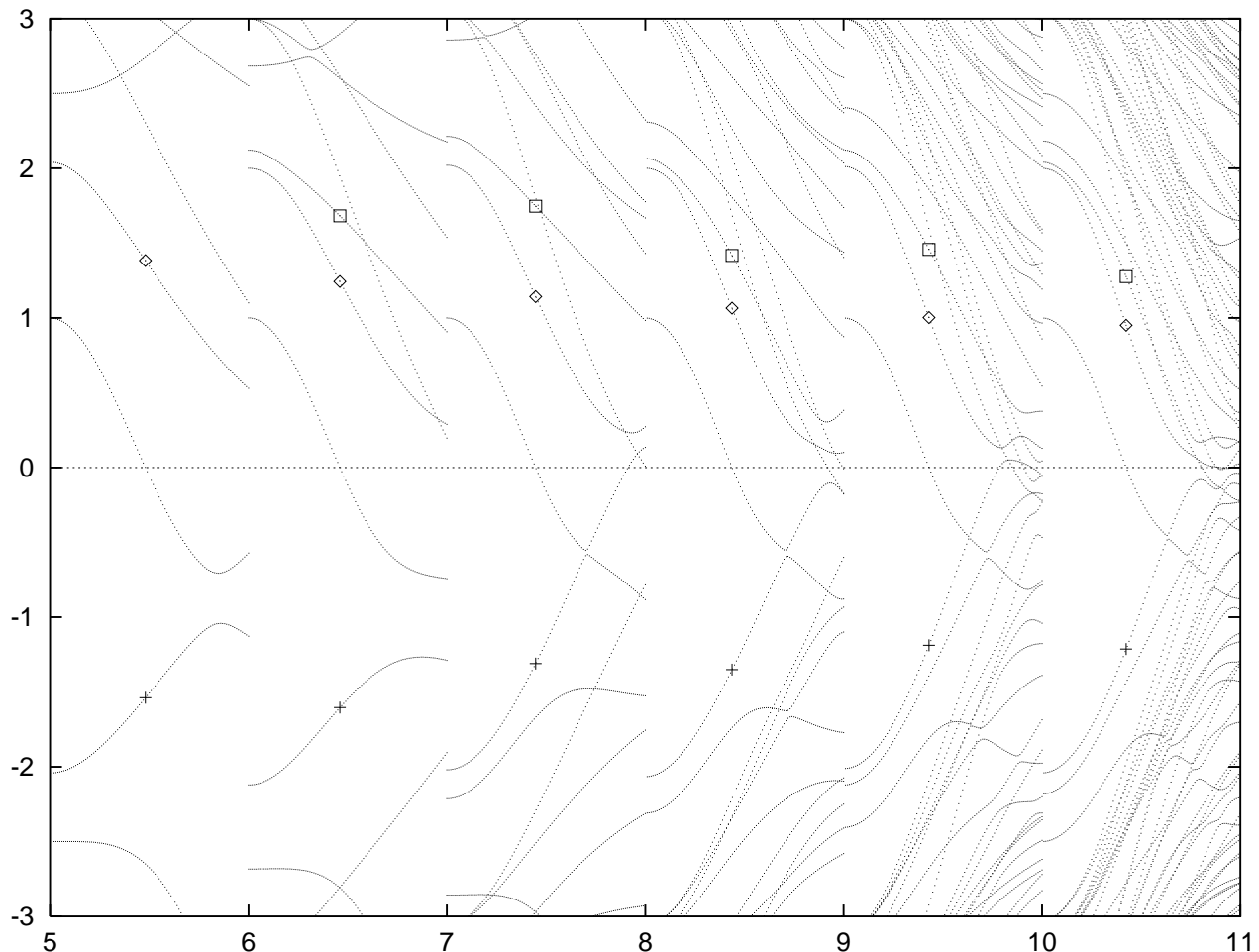


Figure 1: Low lying spectrum of I for $0 < y < 1$ for K from 5 to 10

K , there is a special value of y where the first zero eigenvalue appears. It is plausible that this phenomenon persists to arbitrarily large values of K , and that this zero crossing signifies a phase transition. Indeed, we found the locations of these zero crossings apparently converging like a power with increasing K . We have calculated these locations and tried a least squares fit to the functional form $a + b/K^c$. This gives an estimate of $y_c = 0.37$ for the zero crossing of the first level in the large K limit.

We are also interested in the behavior of other states with low mass-squared. If a continuum of states is forming as we conjectured, the n -th level for any finite n should develop a zero eigenvalue at $y = y_c$ in the limit $K \rightarrow \infty$. This behavior is not at all obvious from figure 1. To examine the situation more closely, we found the eigenvalues of other low lying states for the value of y which corresponds to the zero crossing of lowest state and studied their dependence on K . These points are marked on figure 1 by tick marks on the spectrum. We demonstrate the K dependence of these quantities in figure 2. Figure 2 is plotted on a log-log scale in anticipation of the power law dependence. The linearity of the graph for the second level appears to confirm this. The third and fourth level also appear to follow

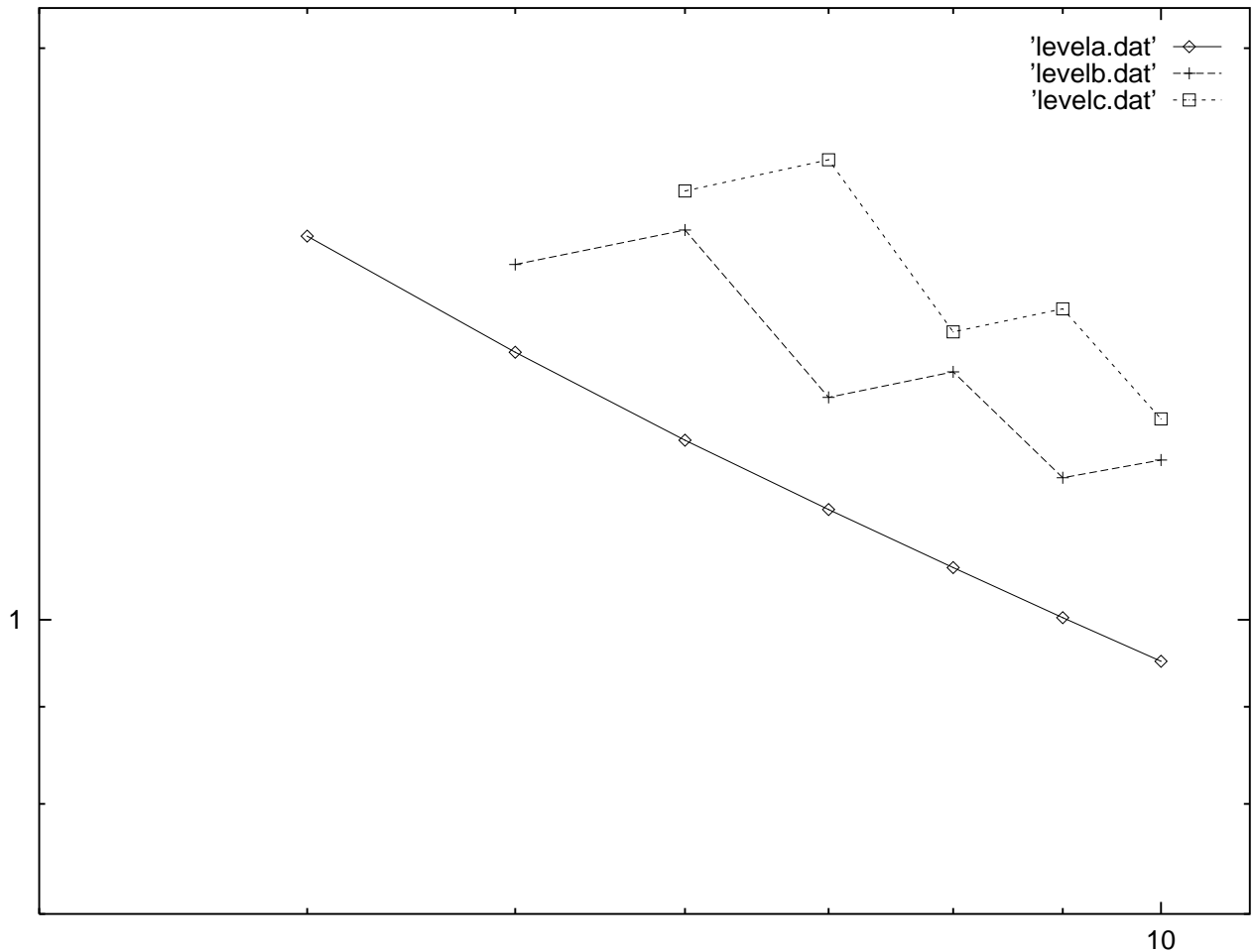


Figure 2: Spectrum 2nd, 3rd, and 4th levels at the zero of 1st level v.s. K

this trend superposed on some oscillations. The oscillation is likely due to the difference between the regularizations with odd and even K . For high enough K this oscillation should be suppressed and these low lying states will hopefully exhibit a power law decay similar to the second level. Alternatively we could extrapolate for even and odd K separately, but we do not have enough data for that.

As a final check, we concentrated on the two lowest states and attempted to extrapolate their masses to infinite K for each y . Again, the spectrum appeared to be converging like a power. Using best fits to the form $a + b/K^c$, we determined our best guesses for the large K limit of the two lowest masses. The result of this calculation is illustrated in figure 3 where we plot the dependence on y of the two lowest masses (we show the results for fixed K as well as the extrapolations to infinite K). We are encouraged to find that the extrapolated masses for the two lowest states come very close to vanishing simultaneously at y near 0.37, which is also the value of y_c that we found by extrapolating the locations of the zero crossings.

At the moment our results are far from being thoroughly convincing, since our extrapolations are no more than the best guesses we can offer based on the available computational data. With the computer resources available to us we were unable to extend the calculation to $K > 10$. Bigger computers, or improved algorithms, should certainly permit the extension

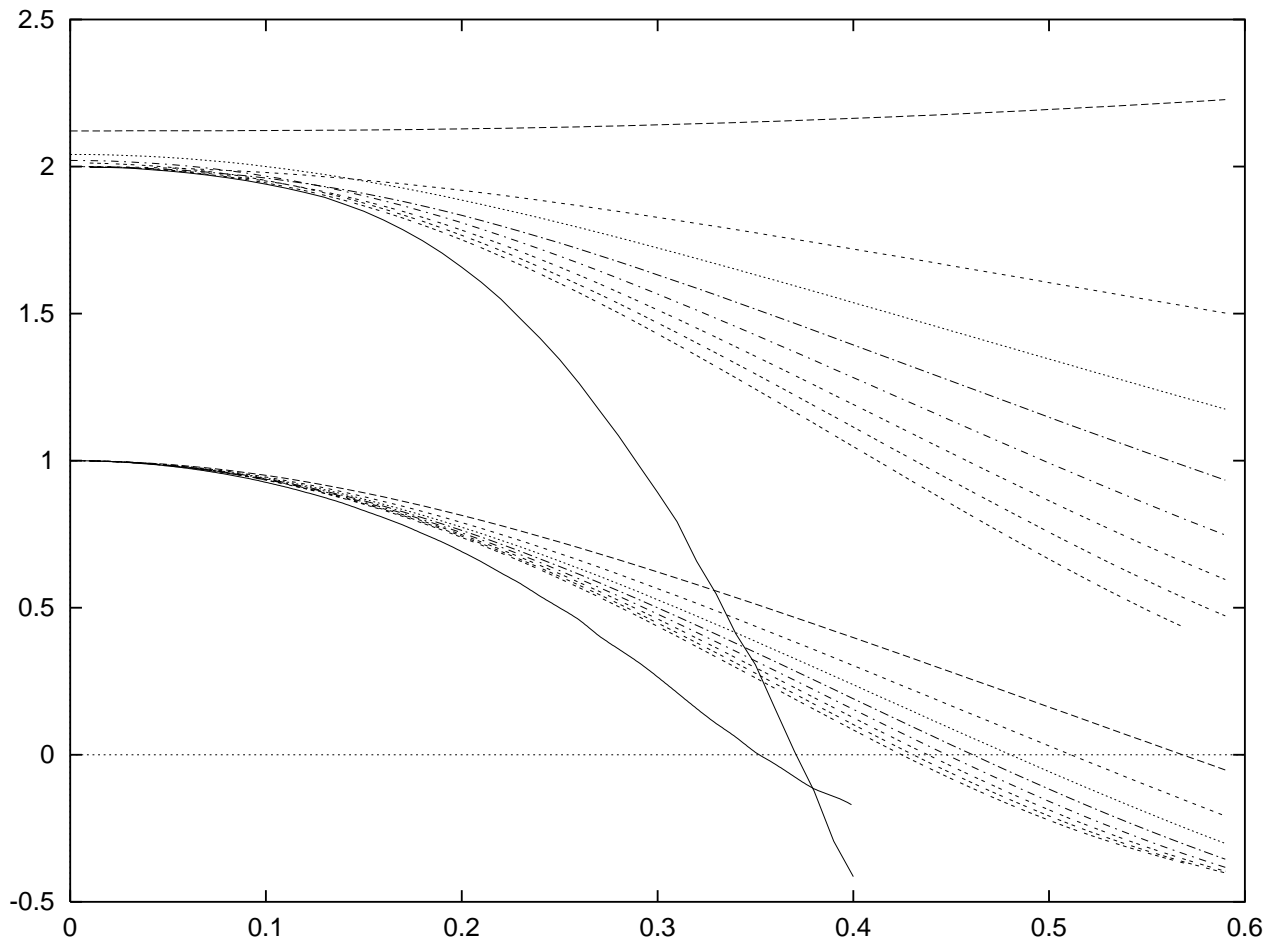


Figure 3: Spectrum of two lightest states for various K 's and their extrapolation. Dotted lines are for K from 3 to 10. The solid line is the extrapolation.

of our calculations to higher K . It would certainly be interesting to check the continuation of the trends we observed and perform more reliable extrapolations.

In conclusion, we have presented some numerical evidence which suggests that the spectrum of a supersymmetric large- N matrix field theory becomes continuous and massless at a critical value of the coupling constant, λ_c . A similar phenomenon occurs in the $c = 1$ matrix model where λ_c corresponds to the onset of the world sheet continuum limit, and the Liouville mode emerges as an additional target space coordinate. By analogy, we speculate that our 1+1 dimensional matrix model describes a 2+1 dimensional non-critical superstring in some background. If true, one might further consider an exciting possibility of finding a non-perturbative matrix model formulation of higher dimensional superstring theories.

By construction, the conjectured 2+1 theory will possess (1,1) supersymmetry in two of its dimensions. From the special nature of the Liouville mode, however, we find it extremely unlikely that this symmetry is extended to 2+1 dimensions. As such, it is similar in spirit to the model considered in [7]. It would be interesting to further explore the relationship between these two models. Perhaps more can be learned about the relation between our approach and its world sheet formulation by studying the renormalization group flow and

the operator algebra of the world sheet action (1) along the lines of [11]. It would also be interesting to explore the relation to the Green-Schwartz approach taken in [8, 9]. The NSR approach, the Green-Schwartz approach, and the matrix models offer complementary methods for the definition of supersymmetric non-critical strings. It would be very interesting to find a model which we understand in all three formulations.

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