EXPLICIT RELATIVISTIC VORTEX SOLUTIONS FOR COOL TWO-CONSTITUENT SUPERFLUID DYNAMICS

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Abstract

We give a class of explicit solutions for the stationary and cylindrically symmetric vortex configurations for a "cool" two-component superfluid (i.e. superfluid with an ideal gas of phonons). Each solution is characterized only by a set of (true) constants of integration. We then compute the effective asymptotic contribution of the vortex to the stress energy tensor by comparison with a uniform reference state without vortex.

I. INTRODUCTION.

The subject of investigation in the present work is the class of vortex type (stationary cylindrically symmetric) configurations for the relativistic generalisation of Landau's two constituent superfluid theory, using the recently derived Lagrangian formulation [1] in which the independent variables are the superfluid phase scalar φ and the entropy current vector s^ρ. More particularly, it will be shown that the complete set of such vortex solutions is obtainable in analytic form in case of the "cool" phonon dominated limit regime. The required form of the Lagrangian for this "cool" regime has been found [2] to be simply given by

$$\mathcal{L} = P - 3\psi \ , \tag{1.1}$$

where P is the "cold" pressure function, depending just on the superfluid momentum covector

$$\mu_\rho = \hbar \nabla_\rho \varphi \ , \tag{1.2}$$

which governs the zero temperature limit for which the entropy current vanishes, while the thermal contribution ψ represents the generalised pressure of the phonon gas, which is given by a simple algebraic expression (4.3) involving s^ρ as well as μ_ρ and the "cold" sound speed $c_{\rm s}$ determined by the pressure function P.

It is to be remarked that whereas accurate treatment of non-stationary configurations with non-zero temperature would require allowance for viscosity of the entropy current, however, no loss of accuracy will be entailed by the use of the strictly conservative treatment as in Landau's original model [3] for treating the stationary equilibrium configurations under consideration here. Indeed under these circumstances there will in any case be no dissipation, the only effect of viscosity being to ensure that the "normal" part of the flow is constrained to have a configuration that is *rigid*.

Unlike the neglect of dissipation that is implicit in our use of a strictly Lagrangian formulation, the limitation that the states under consideration here be restricted to the "cool" regime governed by a Lagrangian of the particular form (1.1) in the sense explained above represents a significant physical limitation. It is provisionally necessary to postpone detailed quantitative analysis of states in the physically very interesting "warm" regime nearer to the phase transition (beyond which lies the regime of "hot" states in which superfluidity is absent altogether). The reason is not merely the expectation that the equation of state governing the explicit form of the dynamical equations in the "warm" regime would be too complicated to be easily tractible. A more compelling obstacle is the consideration that an appropriate "warm" generalisation of the "cool" equation of state (1.1) is not not yet available at all. Although extrapolation beyond the "cool" regime is not yet possible for all the explicit quantitative results obtained below, it will nevertheless be found possible to obtain many useful qualitative results that remain valid even in the "warm" regime, since as shown in Section III, the relevant conservation laws provide sufficiently many first integrals to solve the differential part of the vortex problem completely in the general case. One is thus left with a purely algebraic problem that remains intractible in the general case, but that is easily solved in the "cool" limit.

The formalism that will be used in the present work is fully covariant in the general, not just special, relativistic sense, but since the kinds of (neutron star or laboratory Helium) vortex that are envisaged are of limited scale it will be justifiable to ignore gravitation, i.e. to use a simple Minkowski space background, in the actual application to explicit solutions. As well as being obviously important for applications such as neutron star interiors [4] in which relativistic effects are actually quite large, the use of a covariant formulation of superfluidity theory is also advantageous as a source of supplementary physical insight even for the familiar laboratory example of liquid Helium-4 in which relativistic corrections are quantitatively negligible [5].

II. THE EQUATIONS OF MOTION FOR TWO-CONSTITUENT SUPERFLUID DYNAMICS.

Quite generally (not just in the "cool" regime) the dynamical equations for the generalised Landau theory are determined by the Lagrangian $\mathcal{L}$ in terms of the entropy vector s^ρ and the momentum covector μ_ρ given by (1.1) together with their dynamical conjugates, namely the thermal 4-momentum covector Θ_ρ and the particle number current vector n^ρ that are constructed [1] from the Lagrangian according to the infinitesimal variation formula

$$d\mathcal{L} = \Theta_\rho ds^\rho - n^\rho d\mu_\rho \ . \tag{2.1}$$

In terms of these quantities, the equations of motion consist just of the thermal momentum evolution equation

$$s^\rho \nabla_{[\rho} \Theta_{\sigma]} = 0 \tag{2.2}$$

(using square brackets to denote index antisymmetrisation) together with the usual particle and entropy conservation laws

$$\nabla_\rho n^\rho = 0 \tag{2.3}$$

and

$$\nabla_\rho s^\rho = 0 \ . \tag{2.4}$$

The formulation that has just been summarised has the technical advantage of being particularly economical in so much as it involves only 5 independent component variables, namely the phase scalar φ and the 4 independent components of the entropy current vector s^ρ. This feature of economy has recently been exploited for the purpose of setting up a correspondingly economical Hamiltonian formulation of the theory [6], and it has also been exploited as a guide to the formulation of an analagously economical theory for describing thermal effects in superconducting cosmic strings [7].

The dynamic equations (1.2), (2.2), (2.3), (2.4) entail the corresponding pseudo (in flat space strict) energy momentum conservation law

$$\nabla_\rho T^{\rho\sigma} = 0 \tag{2.5}$$

where the stress momentum energy density tensor is given in terms of the corresponding generalised pressure function

$$\Psi = \mathcal{L} - \Theta_\rho s^\rho \tag{2.6}$$

by

$$T^\rho_{\ \sigma} = n^\rho \mu_\sigma + s^\rho \Theta_\sigma + \Psi g^\rho_{\ \sigma} \ . \tag{2.7}$$

An obviously convenient way of choosing the 3 independent variables in a fundamental state function of the form characterised by (2.1) will be to take them to consist of the thermal rest frame entropy density s as given by

$$c^2 s^2 = -s_\rho s^\rho \ . \tag{2.8}$$

the new cross product variable y given by

$$c^2 y^2 = -\mu_\rho s^\rho \ , \tag{2.9}$$

together with the effective mass variable μ given by

$$c^2 \mu^2 = -\mu_\rho \mu^\rho \ . \tag{2.10}$$

It can then be seen that the secondary variables n^ρ and Θ_ρ will be given in terms of the primary variables μ_ρ and s^ρ of this formulation by

$$n^\rho = \Phi^2(\mu^\rho - \mathcal{A} s^\rho) \ , \qquad \Theta_\rho = \Phi^2(\mathcal{K} s_\rho + \mathcal{A}\mu_\rho) \ , \tag{2.11}$$

where the relevant dilation, determinant and anomally coefficients Φ^2, $\mathcal{K}$, and $\mathcal{A}$ are given by the partial differentiation formulae

$$c^2\Phi^2 = \frac{1}{\mu}\frac{\partial \mathcal{L}}{\partial \mu} \ , \qquad c^2\Phi^2\mathcal{K} = -\frac{1}{s}\frac{\partial \mathcal{L}}{\partial s} \ , \qquad c^2\Phi^2\mathcal{A} = -\frac{1}{2y}\frac{\partial \mathcal{L}}{\partial y} \ . \tag{2.12}$$

In terms of these variables, (2.7) can be rewritten as

$$T^{\rho\sigma} = \Phi^2(\mu^\rho\mu^\sigma + \mathcal{K} s^\rho s^\sigma) + \Psi g^{\rho\sigma} \ . \tag{2.13}$$

III. THE CLASS OF STRONG EQUILIBRIUM STATES OF A CYLINDRICAL VORTEX.

The class of vortex configurations to be dealt with here is of the maximally symmetric type, as characterised by both stationarity and cylindrical symmetry. This means that there are three independent commuting Killing vector symmetry generators, k^μ, ℓ^μ, and m^μ say, which may be taken to be the generators of time translations, longitudinal space translations (parallel to the axis) and axial rotations, respectively corresponding to ignoreable coordinates t, z, and ϕ say, where the latter (like the superfluid phase variable φ) is periodic with period

2π. Within this stationary cylindrical category, the class of solutions that will be dealt with here is characterised by the condition of equilibrium in the *strong* sense, which is to be understood here as implying exclusion of the (experimentally possible, but under natural conditions unlikely) presence of any source or sink of energy or fluid flux in the central core of the vortex (where the superfluidity conditions inevitably breakdown).

In view of the conservation laws (2.3) and (2.4), the condition of equilibrium in this strong sense evidently entails that the flow has no radial components and so is necessarily *helical* : this means that the current vectors will be confined to the timelike hypersurface generated by the Killing vectors, so that they will be expressible in the form

$$n^\rho = \nu(k^\rho + v\ell^\rho + \omega m^\rho) \ , \tag{3.1}$$

$$s^\rho = \sigma(k^\rho + V\ell^\rho + \Omega m^\rho) \ . \tag{3.2}$$

The class of such helical flow configurations includes the specially simple and important subclass of *circular* flow configurations, namely those for which the longitudinal translation velocities v and V can be taken to be zero (for a suitably adjusted choice of k^ρ and ℓ^ρ), as is the case in the kinds of vortex that occurs most commonly in practice. Whereas both the amplitude factors, ν and σ, and in the particle current case also the translation velocity v (if relevant) and the angular velocity ω, can be expected to be radially dependent (though they must of course be independent of the ignoreable coordinates, t, z, ϕ), it is however implicit in the "strong" equilibrium condition that in so far as the entropy constituent is concerned the corresponding "normal" translation velocity V (if relevant), and the corresponding "normal" angular velocity Ω have to be strictly *uniform*, i.e. constant in the radial direction as well. This latter requirement expresses the condition that the "normal" constituent flux should be *rigid* in the sense of having its current vector aligned with a Killing vector field, $\bar{k}^\rho$ say, whose explicit form will be given by

$$\bar{k}^\rho = k^\rho + V\ell^\rho + \Omega m^\rho = \sigma^{-1} s^\rho \ . \tag{3.3}$$

The reason why it is implicit in the "strong" equilibrium condition that the "normal" constituent of the flow should be rigid, is that in a realistic treatment "normality" implies the presence of at least a small amount of viscosity which for non-rigid motion would cause dissipation. This could be compatible with stationarity only in the presence of a radial flux of entropy and hence also of energy, which would need to be supplied by the core even if the only heat sink were at the outer boundary.

As well as the simplifications involved in (3.1) and more particularly in (3.3), it is to be remarked that the restriction that we are only considering states of equilibrium in the strong sense – requiring a non dissipative flow configuration – has the practical advantage that there is no loss of realism in our use of a strictly conservative theory, of the kind summarised in Section 1: there would be no point in including the "normal" viscosity terms that are relevant in other contexts, because in the states treated here their presence would have no effect.

The condition that the "normal" part of the flow is rigid makes it subject to a relativistic generalisation [8] of a (Jacobi type) variant of the classical Bernoulli theorem. The condition that a Killing vector, $\bar{k}^\rho$ say, generates a symmetry of flow means that for any physical well defined field (though not of course for a gauge dependent quantity such as the superfluid phase variable φ) the corresponding Lie derivative should vanish. In the case of the thermal momentum covector Θ_ρ this symmetry condition is expressible as

$$2\bar{k}^\rho\nabla_{[\rho}\Theta_{\sigma]} + \nabla_\sigma(\bar{k}^\rho\Theta_\rho) = 0 \ . \tag{3.4}$$

The condition that the flow satisfies a rigidity condition of the form (3.3) can now be used to rewrite this Lie invariance condition in terms of the relevant current vector in the more specialised form

$$2s^\rho\nabla_{[\rho}\Theta_{\sigma]} = -\sigma\nabla_\sigma(\bar{k}^\rho\Theta_\rho) \ . \tag{3.5}$$

Up to this point the reasonning has been purely kinematic. However if we now use the fact that the momentum covector in question is subject to a dynamical equation of the standard form (2.2) it can be seen that the left hand side of (3.5) will simply vanish, leaving us with a uniformity condition that provides us with a first integral of the motion in the form

$$\bar{k}^\rho \Theta_\rho = -\bar{\Theta} \ , \tag{3.6}$$

where $\bar{\Theta}$ is a Jacobi-Bernouilli type constant, which is interpretable as an *effective temperature* that (like the quantities V and Ω introduced above) is *uniform* throughout. The existence of this uniform effective temperature (as measures with respect to the rigidly corotating frame of the "normal" flow) is interpretable as the relevant application of the "zeroth law of thermodynamics".

Although the particle current does not satisfy a rigidity condition, the fact that the associated dynamical equation is not just of the standard form but has the more restrictive form of the irrotationality condition (1.2) provides us with not merely one but three more independent Bernouilli type constants, corresponding respectively to the three independent Killing vectors. One way to see this is to start by substituting μ_ρ in the place of Θ_ρ in the Lie invariance condition (3.4). Since it is obvious that the dynamic condition (1.2) will then immediately anullate the first term, one is again left with a uniformity condition that provides another Bernouilli type constant, $\bar{E}$, which is interpretable as the *energy* per particle with respect to the rigidly rotating "normal" frame, and which is given by

$$\bar{k}^\rho \mu_\rho = -\bar{E}. \tag{3.7}$$

Furthermore, since this latter result is not of the Jacobi type in that its derivation did not actually depend on the rigid corotation property that singles out the combination $\bar{k}^\rho$, it is clear that an analogous, ordinary rather than Jacobi type, Bernouilli constant can be obtained by replacing $\bar{k}^\rho$ with any other symmetry generating Killing vector. We thus obtain not merely one but three more independent Bernouilli type constants, E, L, and M say, corresponding to the three independent Killing vectors, k^ρ, ℓ^ρ, and m^ρ, and interpretable respectively as representing the *energy, longitudinal momentum*, and *angular momentum*, per particle, which will be given by

$$k^\rho \mu_\rho = -E \ , \qquad \ell^\rho \mu_\rho = L \ , \qquad m^\rho \mu_\rho = M \ . \tag{3.8}$$

The prototype example (3.7) is of course not independent of these, but is given in terms of them as the constant coefficient linear combination

$$\bar{E} = E - VL - \Omega M \ . \tag{3.9}$$

Up to this point everything that has been done is fully general relativistic in the sense of being applicable with respect to an arbitrary system of coordinates $\{x^\rho\}$ for an arbitrarily curved space time metric $g_{\rho\sigma}$. If we wished to allow for self-gravitation we would still be left with a non-trivial differential system of Einstein type equations to be solved. However if, as will be sufficient in all the most obvious applications, we are willing to treat the gravitational spacetime background as fixed in advance, independently of any feedback from the superfluid system, then it is apparent that the work carried out already above has been sufficient to fully integrate all the relevant differential equations. To see this, it suffices to notice that, as the helicality property implies the absence of any radial components, each of the pair of currents involved has only three instead of four independent components, which means that the system is characterised locally by a total of only 6 independent field components. Therefore, in order to solve the corresponding differential system, it suffices to obtain a corresponding set of just 6 first integral constants. Precisely such a set is provided by the work that has just been described: assembling them all together the complete sextet of constants can be listed as the temperature $\bar{\Theta}$ and the linear and angular velocities V and Ω associated with the "normal" part of the system, together with the energy E (or its corotating counterpart $\bar{E}$) and the linear and angular momenta L and M associated with the superfluid part.

It is to be remarked that as far as the physical characterisation of the solution is concerned only 5 of these constants are independent, since by a longitudinal Lorentz adjustment either V or L given any desired value without loss of generality, the most mathematically convenient choice in the present formulation being to use the "superfluid frame" in which $L = 0$.

It is also to be remarked that whereas the first five members of the sextet that has just been listed can take values in a continuous range, the quantisation condition on the phase variable φ in (1.2) entails of course that the last one, M, should be restricted to integral multiples of $\hbar$. Moreover, since vortices with higher values of the relevant winding number will typically be unstable, it is in common circumstances sufficient to consider only the almost trivial case with $M = 0$ (which might be deemed unworthy of description as proper vortex) and the first proper vortex possibility as given (subject to a judicious choice of sign convention) by $M = \hbar$.

Although, in a given background, the differential part of the problem is in principle solved by the foregoing derivation of the sextet of constants $\bar{\Theta}$, V, Ω, E, L, M, there remains what – depending on the complexity of the relevant equation of state – may be a highly non-trivial algebraic problem that still has to be solved if one wishes to obtain all the other important non-uniformly distributed physical field quantities involved in explicit form as variable functions of a suitably defined cylindrical radial coordinate r say, which can be taken to be the only independent variable in each solution.

Since such superfluid vortex configurations as can in practice be set up artificially under laboratory conditions, or as can be expected to occur naturally in neutron stars, are all characterised by dimensions that are small compared with the length scales over which self gravitational effects become important, they should be adequately describable in terms of a local gravitational background that is not merely given in advance (which is all that is needed for complete integrability as described above) but that can be taken more particularly to be *flat*. This allows us to work with a system of the standard cylindrical form $\{x^0, x^1, x^2, x^3\} = \{t, z, \phi, r\}$, such that only the radius coordinate r is non-ignoreable, while the others are the ignoreable coordinates t, z, ϕ, mentionned above in association with the Killing vectors k^ρ, l^ρ, m^μ, so that the metric will be given by the familiar formula

$$g_{\rho\sigma} dx^\rho dx^\sigma = -c^2 dt^2 + dz^2 + r^2 d\phi^2 + dr^2 \ . \tag{3.10}$$

With respect to this coordinate system the independent Killing vector symmetry generators will be given by

$$k^\mu \leftrightarrow (1,0,0,0) \ , \qquad \ell^\mu \leftrightarrow (0,1,0,0) \ , \qquad m^\mu \leftrightarrow (0,0,1,0) \ . \tag{3.11}$$

The corresponding components of the primary momentum covector and current vector in terms of which the formulation based on the Lagrangian $\mathcal{L}$ is formulated will be given by

$$\mu_\rho \leftrightarrow (-E, L, M, 0)\ , \qquad s^\rho \leftrightarrow \sigma(1, V, \Omega, 0)\ . \tag{3.12}$$

It is thus apparent that, in order to obtain the complete solution for all the relevant dynamical variables, all that is still needed is to find the functional dependence of the single variable σ on r.

It can immediately be seen that the required function σ is directly proportional to the quantity y^2 as defined by (2.9), which can be evaluated using (3.12) simply as

$$y^2 = \frac{\bar{E}}{c^2}\sigma\ . \tag{3.13}$$

It can similarly be seen that σ is also related by a simple though radially variable proportionality factor to the quantity s as defined by (2.8), which can be evaluated using (3.12) as

$$s^2 = \Big(1 - \frac{V^2}{c^2} - \frac{\Omega^2 r^2}{c^2}\Big)\sigma^2\ . \tag{3.14}$$

It follows that although it is still necessary to find out how σ depends on r in order to obtain the corresponding radial dependence of the separate scalar state variables s and y, the ratio y^2/s has a radial dependence that is immediately expressible in terms of the Lorentz factor

$$\Gamma = \Big(1 - \frac{V^2}{c^2} - \frac{\Omega^2 r^2}{c^2}\Big)^{-1/2}\ , \tag{3.15}$$

associated with the rigid velocity distribution of the "normal" flow by the formula

$$\frac{y^2}{s} = \frac{\bar{E}\Gamma}{c^2}\ . \tag{3.16}$$

Although (3.12) provides only the ratio (3.16), but not the absolute values, of s and y, it is more helpful for the third of the scalar fields that is needed for evaluating the Lagrangian function, namely the effective mass function μ as defined by (2.10), which can be seen to be given explicitly as a function of r by the easily memorable formula

$$c^2\mu^2 = \frac{E^2}{c^2} - L^2 - \frac{M^2}{r^2} \ . \tag{3.17}$$

It is evident that in the "cold" limit for which the entropy current vanishes the explicit solution is directly available in terms of just the 3 constants involved in (3.17) namely E, M and L (of which the last is physically redundant, being adjustable to zero by a longitudinal Lorentz transformation) since in that case the Lagrangian depends only on the single scalar μ. It is noteworthy that while the very simple formula (3.17) provides the complete solution to the (strong) equilibrium problem for a relativistic vortex in the cold limit, it remains formally valid without change, although it is no longer sufficient to provide the complete solution all by itself, in the generic "warm superfluid" case.

To complete the solution of the vortex problem in the "warm" case, we must make use of the remaining (sixth) constant of integration, which (unlike the other five) does not appear in (3.12), namely the corotating "normal frame" temperature $\bar{\Theta}$ as introduced in (3.6). With the aid of (2.11) it can be seen that (3.6) can be rewritten in the form

$$\bar{\Theta}\, c^2 y^2 = \bar{E}(\Psi - \mathcal{L}) \ , \tag{3.18}$$

where the bracketted function on the right is given explicitly by

$$\Psi - \mathcal{L} = -s\frac{\partial \mathcal{L}}{\partial s} - \frac{y}{2}\frac{\partial \mathcal{L}}{\partial y} \ . \tag{3.19}$$

In principle, since μ is given directly by (3.17), all that remains to be done to obtain the other two state variables y and s (and hence everything else) as functions of r is to carry out the simultaneous solution of the pair of algebraic equations (3.16) and (3.18) using the explicit radial dependence given by (3.15) and (3.17). In practice however, whereas the solution is immediate in the cold limit (for which y and s vanish so that (3.17) suffices by itself), in the generic "warm" case it will not even be possible to begin to tackle the problem until the form of the state is explicitly known, and even then it might reasonably be feared that the ensuing non-linear form of the functional form of the expression given by (3.19) would make it hopelessly intractible. What will be shown in the next section is that in the "cool" limit as described by the equation of state obtained in the preceeding work [2], the particular form of the function given by side of (3.19) will be such that the solution will, rather surprisingly, be obtainable in an explicit analytic form.

IV. THE EXPLICIT SOLUTION FOR THE COOL LIMIT CASE.

In the "cool" limit for which the entropy current is describable simply as a gas of phonons, the corresponding limit form of the equation of state that is obtained [2] as the relativistic generalisation of the corresponding classical formula of Landau [9] is given by an expression of the form

$$\mathcal{L} = P - 3\psi \ , \tag{4.1}$$

in which P is the zero temperature pressure function depending only on the single variable μ and determining the corresponding zero temperature sound speed $c_{\rm s}$ by the formula

$$\frac{c^2}{c_{\rm s}^2} = \mu \frac{d\mu}{dP}\frac{d^2 P}{d\mu^2} \ , \tag{4.2}$$

and ψ is the generalised pressure function of the phonon gas, which is expressible as a funcion of all three of the independent scalar variables s, y, and μ by the formula

$$\psi = \frac{\tilde\hbar}{3} c_{\rm s}^{-1/3}\Big(c^2 s^2 + (c_{\rm s}^2 - c^2)\frac{y^4}{\mu^2}\Big)^{2/3} \ , \tag{4.3}$$

where $\tilde\hbar$ is identifiable with sufficient accuracy for most purposes with the usual Dirac Plank constant $\hbar$, its exact value being given by

$$\tilde\hbar = \frac{9}{4\pi}\Big(\frac{5\pi}{6}\Big)^{1/3}\hbar \simeq 0.99\hbar \ . \tag{4.4}$$

It can be seen to follow immediately, just from the fact that this function ψ is homogeneous, of order 4/3 in the variables s and y^2, that the function given by (3.18) will work out in this "cool" limit case simply as

$$\Psi - \mathcal{L} = 4\psi \ . \tag{4.5}$$

Using (3.16), it can be seen that the equation obtained by substituting (4.5) in (3.18) can be solved explicitly to give the formula

$$y = \frac{\bar{E}^{1/2}}{c}\Big(\frac{3\bar{\Theta}}{4\tilde{\hbar}}\Big)^{3/2} c_{\rm s}^{1/2}\Big(\frac{c^2}{\Gamma^2} + \frac{(c_{\rm s}^2 - c^2)\bar{E}^2}{c^4\mu^2}\Big)^{-1} , \tag{4.6}$$

whose right hand side is interpretable as a function just of the pair of variables Γ and μ, since the latter determines $c_{\rm s}^2$ by the relation (4.2) that is obtained from the zero temperature equation of state. The formula (4.6) thereby gives y directly as an explicit function of the radius r since both Γ and μ are already known by (3.15) and (3.17) as functions of r. Having thus evaluated y one can then immediately obtain the remaining state variable s as a function of r using (3.15) and (3.16), thereby completing the solution of the vortex problem.

The simplest kind of zero temperature equation of state that can be envisaged for illustrating the application of this formula is the polytropic kind which, for a given "rest" mass m per particle, takes the form $P \propto (\mu - m)^{1+\alpha}$ where the index α is a positive constant (taking the value $\alpha = 3$ in the standard case of a relativistic gas with kinetic energy large compared with the rest mass energy), so that, by (4.2), one obtains $c_{\rm s}^2 = (c^2/\alpha)(1 - m/\mu)$, which conveniently reduces to a constant in the high density limit for which μ becomes large compared with m.

V. THE NATURAL CUT OFF RADIUS FOR A VORTEX CELL.

The validity of the solutions obtained in the previous sections is of course limited to a finite range of the radial coordinate r which cannot exceed the critical null corotation radius at which the rigid rotation velocity reaches the speed of light so that the Lorentz factor Γ defined by (3.15) becomes infinite. There is no danger of approaching such a singularity in the usual laboratory experiments on exactly cylindrical superfluid vortices for which wall of the container limits the radius to a value that is very small compared with the value. The occurrence of such a singularity is also avoided in more natural contexts (such as that of the neutron star material whose analysis is the ultimate purpose of the present work) under conditions such that the local superfluid flow can be represented by a honeycomb lattice of hexagonal vortex cells. In such an application a cylindrically symmetric solution of the kind obtained in the previous sections can be used as approximate description of the motion within an individual hexagonal vortex cell. In a stationary state the honeycomb lattice will be in a state of rigid rotation with an angular velocity Ω that must be the same as that of the normal constituent (if any) in order to avoid the presence of dissipation (which would be incompatible with strict stationarity). In such a configuration, each approximately cylindrical vortex cell will have a natural outer cut off radius,

$$r = \bar{r} \tag{5.1}$$

say, where $\bar{r}$ is just *half* the distance between neighbouring vortex cores. (The axial symmetry approximation should be extremely accurate in the inner region $r << \bar{r}$ where most of the vortex energy is concentrated, so the errors due to neglect of the breakdown from axial to hexagonal symmetry in the outer regions should be relatively small.) The symmetry between neigbouring vortices of the lattice implies that at the midpoint between nearest neighbour vortex cores the angular velocity, not just of the normal flux s^μ, but also of the particle flux n^μ should should agree with that angular velocity of the lattice, with respect to the frame in which the longitudinal velocity V of the rigid normal flow vanishes. This means that provided we fix the choice of the longitudinal Killing vectors k^μ and ℓ^μ by imposing the requirement

$$V = 0 \tag{5.2}$$

then the appropriate cut off radius (5.1) can be characterised simply by the condition

$$\omega = \Omega \, , \tag{5.3}$$

where ω is the radially variable angular velocity of the particle current as introduced in (3.1). Since it can be seen from (2.11) and (3.12) that this angular velocity variable will be given by

$$\frac{\omega r^2}{c^2} = \frac{M - \Omega r^2 \mathcal{A}\sigma}{E - c^2 \mathcal{A}\sigma} \ , \tag{5.4}$$

the equation (5.3) is easily soluble. The natural cut off radius value is thereby found to be given by the simple formula

$$\bar{r}^2 = \frac{c^2 M}{E\Omega} \ . \tag{5.5}$$

This formula can be used to relate the angular velocity ω of the rigidly rotating frame to the corresponding circumferential value $\bar{w}$ of of the circulation w per unit area. Since the momentum circulation κ say will be given in terms of the angular momentum constant M by $\kappa = 2\pi M$, the corresponding mean vorticity, w for a circuit of radius r will be given by

$$w = \frac{2M}{r^2} \ , \tag{5.6}$$

so that for the corresponding effective macroscopic vorticity, as given by the momentum circulation per unit area for each entire vortex cell, one obtains the formula

$$\bar{w} = 2E\Omega/c^2 \ . \tag{5.7}$$

For the purpose of applications to the analysis of macroscopic properties of a vortex lattice, it is of particular interest to evaluate total integrated quantities, such as the total effective longitudinal energy momentum tensor of a vortex as functions of the corresponding total longitudinal currents, within the relevant cut off radius (5.1).

It is shown in the appendix how the asymptotic deviation of the sectionally averaged value $\overline{Q}$ of any suitable physical quantity Q from its value $Q_\ominus$ in an appropriately chosen homogeneous reference state goes like $r^{-2}\ln r$ for large cut-off distances r, the multiplicative factor depending on a corresponding "net asymptotic deviation coefficient" $\widehat{Q}$. In order for the averaging process to be meaningful, it is not necessary that the physical quantity Q under consideration should be a scalar in the 4-dimensional sense, but is sufficient that it should be a scalar with respect to the coordinates ϕ and r of the section at constant t and z over which the integrals are taken, as is the case for tensor components as longitudinally projected onto the symmetry axis whose coordinates are the subset $\{x^i\} \leftrightarrow \{t, z\}$, for $\{i\} = \{0, 1\}$ within the full set $\{x^\mu\}$ introduced in (3.10). This applies in particular to the corresponding longitudinally projected components of the stress tensor, whose sectionally integrated total will be given in terms of the corresponding averaged values $\overline{T^j{}_k}$ by

$$\langle T^j{}_k \rangle = \pi r^2 \overline{T^j{}_k} \ . \tag{5.8}$$

The foregoing formulae are also valid for the trace of the orthogonally projected part of the stress tensor, which is twice what is known as the the *lateral* pressure $\varPi$ say, whose definition is expressible equivalently by

$$2\varPi = T^\rho{}_\rho - T^i{}_i \ , \tag{5.9}$$

so that its average will be expressible in terms of the scalar functions introduced in Section 1 by

$$2\overline{\varPi} = 3\overline{\Psi} - \overline{\Lambda} - \overline{T^i{}_i}. \tag{5.10}$$

According to the general formula obtained in the appendix, the averages in (5.8) will be given asymptotically by an expression of the form

$$\overline{T^j{}_k} - T^j_{\ominus k} \sim \widehat{T^j{}_k} \frac{Mw}{4c^2\mu^2} \ln\Big\{\frac{w_\odot}{w}\Big\}, \tag{5.11}$$

as $w \to 0$ (using the standard notation convention according to which the symbol $\sim$ relates quantities whose ratio tends to unity in the limit under consideration) where $T^j_{\ominus k}$ are the corresponding reference state values, and $w_\odot$ is a fixed vorticity value determined according to (5.6) by a suitably chosen fixed "sheath" radius $r_\odot$ whose precise specification is unimportant when w is sufficiently small.

VI. STRESS - ENERGY COEFFICIENTS FOR A COOL VORTEX CELL.

The purpose of this section is to use the procedure described in the appendix to evaluate the coefficient $\widehat{T^j{}_k}$, taking the variables q that fix the reference state to be simply the longitudinal components μ^i of the particle momentum (the other components being irrelevant since the axial symmetry ensures that they average to zero) and the longitudinal components s^i of the entropy current. This choice for the reference state is not at all compelling, and we shall consider for instance in the next section, in the cold limit, a reference state defined by the longitudinal components of the particle current. Our choice here is simply motivated by the computational simplicity when one will wish to apply the sectional averaging to a vortex solution, for which the most convenient variables are the particle momentum and the entropy current as illustrated in Section III.

The reference state is thus chosen to satisfy the condition that the corresponding reference momentum components $\mu_{i\ominus}$ and reference current components $s^i_{\ominus}$ agree respectively with the average momentum components $\overline{\mu_i}$ and the current components $\overline{s^i}$ of the vortex, so that in the notation of (A9) one has

$$\delta_{\ominus}\overline{\mu_i} = 0, \qquad \delta_{\ominus}\overline{s^i} = 0\ . \tag{6.1}$$

According to (A23), this means that the required "net" asymptotic deviation coefficients $\widehat{T^j_{\ k}}$ will be given in terms of the corresponding "gross" asymptotic deviation coefficients $\widetilde{T^j_{\ k}}$ by

$$\widehat{T^j_{\ k}} = \widetilde{T^j_{\ k}} - \widetilde{s^i}\,\frac{\partial}{\partial s^i} T^j_{\ k}\ , \tag{6.2}$$

using the fact that

$$\widetilde{\mu_i} = 0, \tag{6.3}$$

which follows obviously from (3.12).

To apply the formula (6.2) we first need to use (A19) to obtain the required "gross" asymptotic deviation coefficients $\widetilde{s^i}$ for the entropy current itself, whose "net" asymptotic deviation coefficients will automatically vanish $\widehat{s^i} = 0$ as an automatic consequence of the choice (6.1). In the case of the vortex solution, the dependence of s^i on μ is completely confined to the coefficient σ as can be seen from (3.12), which is itself related to the "cross" scalar y^2 according to (3.13), so the deviation formula (A19) gives

$$\widetilde{s^i} = -\frac{\mu}{y^2}\frac{dy^2}{d\mu}s^i \ , \tag{6.4}$$

where the limit $r \to \infty$ is implicit in the right hand side, as will be the case in all the formulas giving asymptotic coefficients.

We are now ready to consider the application to the stress tensor which can be written in the form

$$T^\rho{}_\sigma = \Phi^2\left(\mu^\rho\mu_\sigma + \mathcal{K}s^\rho s_\sigma\right) + \left[\mathcal{L} + c^2\Phi^2\left(s^2\mathcal{K} + y^2\mathcal{A}\right)\right] g^\rho{}_\sigma \ . \tag{6.5}$$

It is now straightforward to compute $\widetilde{T^j{}_k}$ by using systematically the definition (A19). We skip here the details of the calculation, inviting the reader to consult the similar but simpler procedure for the cold limit that we will detail in the next section. The result involves many terms. It turns out that when one adds to the expression for $\widetilde{T^j{}_k}$ the extra term in order to get the net asymptotic coefficient $\widehat{T^j{}_k}$, a lot of cancellations occur, some due to the relation

$$ds = \frac{s}{y^2}dy^2 + \frac{1}{2}s\Gamma^2 d\left(\Gamma^{-2}\right) \ , \tag{6.6}$$

so that the final result can be simply expressed as

$$\widehat{T^j{}_k} = -\mu\frac{\partial\Phi^2}{\partial\mu}\mu^j\mu_k + \frac{\mu^2}{s}\frac{\partial\Phi^2}{\partial s}s^j s_k + \mu^2c^2\left(-\Phi^2 + s\frac{\partial\Phi^2}{\partial s} + y^2\frac{\partial\Phi^2}{\partial y^2}\right)g^j{}_k \ . \tag{6.7}$$

We now consider the asymptotic behaviour of the lateral pressure contribution (5.10) which can be seen to be given locally by

$$\mathit{\Pi} = \mathcal{L} + c^2\Phi^2\left[s^2\mathcal{K} + y^2\mathcal{A} + \frac{1}{2}(\mu_\infty^2 - \mu^2) + \frac{1}{2}\mathcal{K}\sigma^2(\Gamma_\infty^{-2} - \Gamma^{-2})\right] \ . \tag{6.8}$$

Although the expression that one obtains "naively" for $\widetilde{\mathit{\Pi}}$, i.e. by just applying the definition (A16), involves a lot of terms, it can be checked, by using (6.6) as well as the differential of the relation (3.18) written in the form

$$\frac{s^2}{y^2}\Phi^2\mathcal{K} + \Phi^2\mathcal{A} = \frac{\bar{\Theta}}{\bar{\bar{E}}} \ , \tag{6.9}$$

whose right hand side is constant, that the "gross" asymptotic deviation coefficient cancels out altogether, i.e. one simply gets

$$\widetilde{\Pi} = 0 \ . \tag{6.10}$$

This result could of course have been predicted in advance from the requirement of overall stress balance on the outer boundary.

Despite of the stress balance condition (6.10), the lateral pressure will nevertheless have a non vanishing "net" asymptotic deviation coefficient that will be given by

$$\widehat{\Pi} = \mu \left(s \frac{\partial \Phi^2}{\partial s} + y^2 \frac{\partial \Phi^2}{\partial y^2} \right) . \tag{6.11}$$

The preceeding results are valid for any equation of state $\mathcal{L}$. We now specialize these results to the "cool" equation of state (4.1) for which we have explicit vortex solutions. In this case, the dilation coefficient is given by

$$\Phi^2 = \frac{1}{\mu c^2} \frac{dP}{d\mu} - \frac{3}{\mu c^2} \frac{\partial \psi}{\partial \mu}, \tag{6.12}$$

the second term on the right hand side being homogeneous of order 4/3 in the variables s and y^2. It can thus be checked easily that the lateral pressure is simply given by

$$\widehat{\Pi} = -\frac{4}{c^2} \frac{\partial \psi}{\partial \mu} \ . \tag{6.13}$$

whereas the coefficient $\widehat{T^j_{\ k}}$ is given by substituting (6.12) in (6.7).

VII. STRESS - ENERGY COEFFICIENTS FOR A COLD VORTEX CELL.

In this final section, we restrict our attention to the case of the cold limit in which there is no entropy vector. The superfluid in this zero temperature limit reduces to a particular case of perfect fluid and the Lagrangian $\mathcal{L}$ reduces simply to the pressure function $P(\mu)$. In this simple case the current will by given, in terms of the "dilatonic" amplitude Φ that plays a key role in the vorticity variational formulation [10] of the zero temperature fluid model, just by

$$n^\rho = \Phi^2 \mu^\rho \ , \tag{7.1}$$

where Φ^2 and its derivative are given by

$$\Phi^2 = \frac{n}{\mu} \ , \qquad \mu \frac{d\Phi^2}{d\mu} = \Phi^2 \Big(\frac{c^2}{c_{\rm s}^2} - 1 \Big) \ . \tag{7.2}$$

and where the particle number density n is defined by

$$n^\rho n_\rho = -c^2 n^2 \tag{7.3}$$

and can also be derived from the pressure function $P(\mu)$ by

$$n = c^{-2} \frac{dP}{d\mu} \ . \tag{7.4}$$

One can also introduce the energy density of the fluid, $\rho(n)$, as the Legendre transform of the pressure function $P(\mu)$,

$$\rho(n) = \mu n - c^{-2} P(\mu) \ . \tag{7.5}$$

It follows from (3.12) that the longitudinal and angular velocities v and ω introduced in (3.1) and the corresponding amplitude factor ν will be given by

$$v = \frac{c^2 L}{E} \ , \qquad \omega = \frac{c^2 M}{E r^2} \ , \qquad \nu = \frac{E}{c^2} \Phi^2 . \tag{7.6}$$

Finally, in this zero temperature limit, the stress tensor has the simple perfect fluid form

$$T^\rho{}_\sigma = n^\rho \mu_\sigma + P g^\rho{}_\sigma \ . \tag{7.7}$$

It is evident from (6.2) that in the cold limit for which the entropy current vanishes the "gross" deviations (with respect to the large distance limit) are the same as the "net" deviations with respect to a reference state of the kind postulated in the preceeding section. This is because the cold limit of such a reference state limit is characterised just by the longitudinal momentum components, which are radially uniform, so that deviations from their large distance limit values simply vanish:

$$\delta_\infty \overline{\mu_i} = 0 \ . \tag{7.8}$$

Using (7.2) the "gross" variation coefficients for the longitudinal stress energy tensor in the cold limit can therefore be obtained simply by setting the entropy current s^i to zero in (6.7) which gives

$$\widetilde{T^j_{\ k}} = \left(1 - \frac{c^2}{c_{\rm S}^2}\right) n^j \mu_k - (\rho c^2 + P) g^j_k, \tag{7.9}$$

while by (6.9) the corresponding "gross" lateral pressure simply vanishes.

Introducing the longitudinal unit flow vector, u^ρ say, and the corresponding spacial projection tensor

$$\gamma^\rho_\sigma = g^\rho_{\ \sigma} + \frac{1}{c^2} u^\rho u_\sigma \ , \qquad\qquad u^\rho u_\rho = -c^2, \tag{7.10}$$

according to the component prescription $\mu_\ominus \{u^0, u^1, u^2, u^3\} = \{\overline{\mu^0}, \overline{\mu^1}, 0, 0\}$ (in the coordinate system introduced in (3.10) that we have been using) so that the uniform asymptotic limit state is characterised by the condition that its momentum should simply have the form

$$\mu_{\ominus\rho} = \mu_\ominus u_\rho \ , \tag{7.11}$$

the meaning of the formula (7.9) can be interpreted as follows. In terms of the corresponding pressure $P_\ominus$ and energy density $\rho_\ominus$ given, via the equation of state (2.1) by

$$P_\ominus = P\{\mu_\ominus\} \ , \qquad\qquad \rho_\ominus = \mu_\ominus \frac{dP}{d\mu}_\ominus - P_\ominus \ , \tag{7.12}$$

the stress energy tensor of the uniform asymptotic limit state will be given by

$$T^{\ \rho}_{\ominus\ \sigma} = \rho_\ominus u^\rho u_\sigma + P_\ominus \gamma^\rho_{\ \sigma} \ . \tag{7.13}$$

The significance of (7.9) is that that it specifies the coefficient in the asymptotic formula

$$\langle T^j_{\ k} \rangle - \langle T^{\ j}_{\ominus k} \rangle \sim \frac{\pi M^2}{2c^2\mu^2} \widetilde{T^j_{\ k}} \ \ln\left\{\frac{w_\odot}{w}\right\} \tag{7.14}$$

for the difference of the total sectionally integrated stress momentum energy density components from the values they would have for the uniform asymptotic limit state (if the vortex were absent) as a function of the cut off radius r as given in terms of the corresponding circulation per unit area w by (5.6). The components $T^{\ j}_{\infty k}$ in this expression are just the longitudinal subset of the 4-dimensional set given by (7.13) while the formula (7.9) for the coefficients $\widetilde{T^j_{\ k}}$ can be rewritten equivalently, in a form more directly comparable with (6.13), as

$$\widetilde{T^j_{\ k}} = (\rho c^2 + P)\Big((2c^{-2} - c_{\rm S}^{-2})u^j u_k - \gamma^j_{\ k}\Big). \tag{7.15}$$

Substituting this in (7.14) gives the simple explicit formula

$$\langle T^j_{\ k}\rangle - \langle T^{\ j}_{\ominus k}\rangle \sim \frac{\pi M^2}{2}\Phi^2\Big((2c^{-2} - c_{\rm S}^{-2})u^j u_k - \gamma^j_{\ k}\Big)\ \ln\Big\{\frac{w_\odot}{w}\Big\}\ . \tag{7.16}$$

The final formula (7.16) is interpretable as meaning that relative to the reference state labelled by $_\ominus$, with the same longitudinal momentum component values $\overline{\mu}_i$, the vortex cell has an effective energy per unit length, $[U]_\ominus$ say, that is given by

$$[U]_\ominus \sim \left(2 - \frac{c^2}{c_{\rm S}^2}\right)[T]_\ominus \tag{7.17}$$

where, $[T]_\ominus$ is the corresponding effective tension, which, for the usual case with the lowest quantum value, $M = \hbar$, will be given as a function of the mean vorticity w defined by (5.6) and the amplitude Φ defined by (6.2) as

$$[T]_\ominus \sim \frac{\pi\hbar^2}{2}\Phi^2\ \ln\Big\{\frac{w_\odot}{w}\Big\}\ , \tag{7.18}$$

wherein – in view of the fact that higher order corrections have been neglected – it is sufficiently accurate to take the quantities involved to be those characterising the simple vortex free reference state.

In the non-relativistic limit, in which one can make the substitution $\Phi^2 \sim \rho/m^2$ where m is the relevant fixed Newtonian mass per particle, the expression (7.18) can be seen to agree with the well known formula given by Hall [11] for the effective vortex tension defined as the integrated longitudinal pressure deficit relative to the asymptotic value. Hall pointed out that this value is the same as that given by the Feynman formula for the energy per unit length, as evaluated in the incompressible case. It is to be remarked that the incompressible

Newtonian case corresponds to the non-relativistic limit of the "stiff" case characterised by $c_{\rm s}^2 = c^2$, and that for a"softer" material the energy per unit length given by (7.16) will be relatively diminished, so much so that it will actually be negative whenever $c_{\rm s}^2 < c^2/2$, as will typically be the case in realistic applications, including all except perhaps the most central regions of neutron stars. This energy deficit is simply due to the central matter deficit resulting, unless the material is sufficiently stiff, from the centrifugal effect. It will be seen in the next section that, when normalised with respect to a fixed amount of conserved matter, the energy per unit length will always be positive, and furthermore that its value will remain equal to that given by (7.18) even for material that is quite "soft".

VIII. STRESS - ENERGY WITH RESPECT TO THE CURRENT FLUX REFERENCE STATE.

Rather than the prescription of Section VI, which as we have seen would give no difference between "gross" and "net" deviations in the cold limit, let proceed with the analysis of the cold limit on the base of a prescription that is less mathematically trivial. Instead of merely choosing the reference state to be specified by the longitudinal components of the momentum covector, it is more useful for many physical purposes to choose a reference state to be fixed by the longitudinal components of the particle current flux (the other components being irrelevant since the axial symmetry ensures that they average to zero). To avoid confusion with the previous choice of the reference state that was labelled by the symbol $_\ominus$, we shall use a slightly modified symbol, $_\oslash$, to label this new reference state which is chosen so as to satisfy the condition that the corresponding reference current components $n^i_\oslash$ agree with average current components $\overline{n^i}$ of the vortex, so that in the notation of (A8) one has

$$\delta_\oslash \overline{n^i} = 0 \ . \tag{8.1}$$

According to the appendix, in the same way as in the expression (6.2), the relation between the "net" and "gross" asymptotic deviation coefficients will be given in this case by

$$\widehat{T^j_{\ k}} = \widetilde{T^j_{\ k}} - \widetilde{n^i}\,\frac{\partial}{\partial n^i} T^j_{\ k} \ . \tag{8.2}$$

To use this formula we need to the "gross" asymptotic deviation coefficients $\widetilde{n^i}$ for the current itself. Using the last of the expressions (7.6), it can be seen that the asymptotic deviation formula (A16) will give

$$\widetilde{n^i} = \Big(1 - \frac{c^2}{c_{\rm S}^2}\Big) n^i \ , \tag{8.3}$$

in which the ratio of the light and sound speeds is given by (4.2).

In view of the fact that, by (3.12), the relevant components of the momentum covector μ_ρ are constant, as also are those of the metric, they can be taken outside the averaging process so as to allow us to write

$$\widetilde{T^j_{\ k}} = \widetilde{n^j}\mu_k + \widetilde{P} g^j_{\ k} \ . \tag{8.4}$$

Therefore to complete the evaluation of $\widetilde{T^j_{\ k}}$ all that remains to be done is to obtain the asymptotic deviation coefficient of the pressure scalar P. Applying (A16) and using (7.4) and (7.5) it can be seen that the required result is given simply by

$$\widetilde{P} = -(\rho c^2 + P) \ . \tag{8.5}$$

Putting this together with (8.3) in (8.4) we obtain a direct derivation of the complete expression (7.9) for the "gross" asymptotic deviation coefficient of the stress momentum energy density tensor.

Proceeding towards the evaluation of the "net" deviation coefficients in which we are ultimately interested, the next step is to use (8.3) again for working out the second term required for the application of (8.2). The contribution from the pressure scalar P, which (in view of the asymptotic agreement $\Pi \sim P$ as $\mu \to \mu_\infty$) is the same as that for Π, is given simply by

$$\frac{\partial P}{\partial n^i} = -\frac{c_{\rm S}^2}{c^2}\mu_i \ , \Rightarrow \widetilde{n^i}\,\frac{\partial P}{\partial n^i} = \widetilde{n^i}\,\frac{\partial \Pi}{\partial n^i} = \Big(\frac{c_{\rm S}^2}{c^2} - 1\Big)(\rho c^2 + P) \ . \tag{8.6}$$

As an immediate corollary we see from (8.2) that in to contrast to the case where the reference state is specified by the momentum, the lateral pressure will have a non vanishing "net" asymptotic deviation coefficient that will be given by

$$\widehat{\Pi} = \Big(1 - \frac{c_{\rm s}^2}{c^2}\Big)(\rho c^2 + P) \ . \tag{8.7}$$

After analogously evaluating the other relevant contributions their combination in the complete longitudinally projected stress tensor is obtained as

$$\widetilde{n^i} \, \frac{\partial}{\partial n^i} T^j{}_k = \Big(\frac{c_{\rm s}^2}{c^2} - \frac{c^2}{c_{\rm s}^2}\Big) n^j \mu_k + \Big(\frac{c_{\rm s}^2}{c^2} - 1\Big)(\rho c^2 + P) g^j{}_k \ . \tag{8.8}$$

When this last, rather unweildy, expression is subtracted off from (7.9) in accordance with the prescription (8.2), we end up with a comparitively simple expression for the required "net" deviation coefficient of the stress tensor, which is expressible just by

$$\widehat{T^j_k} = \Big(1 - \frac{c_{\rm s}^2}{c^2}\Big) n^j \mu_j - \frac{c_{\rm s}^2}{c^2}(\rho c^2 + P) g^j{}_k \ , \tag{8.9}$$

in which, as in preceeding formulae, the relevant values of the quantities involved are those of the large distance limit.

This result is to be interpreted with respect to the reference state as characterised by a density $n_\oslash$ given in terms of the average flow by the component prescription $n_\oslash\{u^0, u^1, u^2, u^3\} = \{\overline{n^0}, \overline{n^1}, 0, 0\}$ where u^ρ is the same unit vector as was introduced in (7.10) so that the corresponding current has the form

$$n^\rho_\oslash = n_\oslash u^\rho \ . \tag{8.10}$$

In terms of the corresponding density $\rho_\oslash$ and pressure $P_\oslash$ given, via the equation of state (7.5), by

$$\rho_\oslash = \rho\{n_\oslash\} \ , \qquad\qquad P_\oslash = n_\oslash \frac{d\rho}{dn_\oslash} - \rho_\oslash \ , \tag{8.11}$$

The stress energy tensor of the uniform asymptotic limit state will be given in terms of these quantities an expression of the perfect fluid form that is the formal analogue of (7.13) namely

$$T^{\rho}_{\oslash\sigma} = \rho_{\oslash} u^{\rho} u_{\sigma} + P_{\oslash} \gamma^{\rho}{}_{\sigma} \ . \tag{8.12}$$

With respect to this reference state, the expression for the total sectionally integrated stress tensor will be given by

$$\langle T^{j}{}_{k} \rangle - \langle T^{j}_{\oslash k} \rangle \sim \frac{\pi M^2}{2} \Phi^2 \Big(\frac{1}{c^2} u^j u_k - \frac{c_{\rm s}^2}{c^2} \gamma^{j}_{k} \Big) \ \ln \Big\{ \frac{w_{\odot}}{w} \Big\}. \tag{8.13}$$

This last formula is interpretable as meaning that relative to the reference state labelled $\oslash$, with the same average longitudinal current vector $\overline{n}^i$, the vortex cell has an effective tension, $[T]_{\oslash}$ say, and an effective energy density per unit length, $[U]_{\oslash}$ say, that are related by

$$[T]_{\oslash} \sim \frac{c_{\rm s}^2}{c^2} [U]_{\oslash} \ , \tag{8.14}$$

where $[U]_{\oslash}$ is given by

$$[U]_{\oslash} \sim \frac{\pi \hbar^2}{2} \Phi^2 \ln \Big\{ \frac{w_{\odot}}{w} \Big\} \ , \tag{8.15}$$

for the lowest quantum value $M = \hbar$.

It is to be remarked that relative to the reference state used in the present section, as defined in terms of a fixed quantity of conserved matter, the effective tension can be seen from (8.14) to tend to zero in the limit $c_{\rm s}^2 << c^2$. It is also to be noted that – as a generalisation of the non-relativistic Hall [11] equality – the last formula (8.15) for the energy per unit length is the same as the formula (7.18) that was obtained for the effective tension, albeit relative to a different reference state. However it is only in the "stiff" case characterised by $c_{\rm s}^2 = c^2$ that, as in the incompressible non-relativistic case, the effective tension will be equal to the energy per unit length with respect to the *same* reference state.

APPENDIX: ASYMPTOTIC FORMS OF SECTIONAL INTEGRALS AND AVERAGES.

The purpose of this appendix is to consider sectional integrals and averages of a generic local radially dependent physical quantity Q in the asymptotic limit as the outer cut off radius $\bar{r}$ of the section tends to infinity, i.e. in the limit as the mean vorticity,

$$\bar{w} = \frac{2M}{\bar{r}^2}, \tag{A1}$$

tends to zero. We use angle brackets to denote the sectional integral of Q over the entire two dimensional section Σ say out to the exterior cut off radius $\bar{r}$ of the cylindrically symmetric vortex region under consideration, i.e. we write

$$\langle Q \rangle = \int_\Sigma Q\, d\Sigma = 2\pi \int^{\bar{r}} Q\, r dr \ . \tag{A2}$$

Such an integral will typically diverge in the asymptotic limit $\bar{r} \to \infty$, unlike the corresponding average,

$$\overline{Q} = \frac{\langle Q \rangle}{\pi \bar{r}^2} = \frac{2}{\bar{r}^2} \int^{\bar{r}} Q\, r dr. \tag{A3}$$

which will typically tend to a well defined finite limit. Provided that Q itself is asymptotically well behaved in the sense of tending smoothly to a well defined asymptotic limit

$$Q \to Q_\infty \tag{A4}$$

as $r \to \infty$, its average will obviously converge to the same asymptotic limit i.e. we shall have

$$\overline{Q} \to Q_\infty \ . \tag{A5}$$

Our interest will therefore focus for the time being on the non trivial difference between the average and its limit, as defined by

$$\delta_\infty \overline{Q} = \overline{Q} - Q_\infty \tag{A6}$$

and on the corresponding difference as defined for the total integral by

$$\delta_\infty \langle Q \rangle = \langle Q \rangle - \pi \bar{r}^2 Q_\infty = \pi \bar{r}^2 \delta_\infty \overline{Q} \ . \tag{A7}$$

Whereas it is evident from (A5) that $\delta_\infty \overline{Q}$ will tend to zero, on the other hand $\delta_\infty \langle Q \rangle$ will still diverge, albeit only logarithmicly, i.e. much less strongly than $\langle Q \rangle$, as $\bar{r} \to \infty$.

In order to analyse this limit it is useful to decompose the range of integration into an "asymptotic" zone, $\bar{r} > r > r_\odot$, $\bar{w} < w < w_\odot$ and an inner "sheath" zone, $r < r_\odot$, $w > w_\odot$, taking the sheath radius $r_\odot$ to be large compared with the central core radius within which the superfluidity breaks down but small compared with the outer cut off radius $\bar{r}$. It can be seen that the average $\overline{Q}$ as taken over the entire section out to $\bar{r}$ will be given in terms of the sheath average $\overline{Q}_\odot$ as taken over just the inner zone extending only to $r_\odot$ by an expression of the form

$$\overline{Q} = \frac{\bar{w}}{w_\odot}\overline{Q}_\odot + \bar{w}\int_{\bar{w}}^{w_\odot} Q\frac{dw}{w^2} \ . \tag{A8}$$

The subject of our ultimate concern is not so much the total integral or average as given by the foregoing expressions but rather just the part thereof that is attributable to the vortex *per se*, after subtracting off a contribution that would have been there in its absence. What we are specially interested in are not the "gross" differences defined by (A4) but the corresponding "net" differences

$$\delta_\ominus \overline{Q} = \overline{Q} - \overline{Q}_\ominus \ , \tag{A9}$$

where $Q_\ominus$ is the value that the quantity under consideration would have in some appropriately chosen reference state. We take this reference state not just vortex free but more specifically *uniform*, which of course entails the exact equality $Q_\ominus = \overline{Q}_\ominus$. In practice we shall compute this "net" difference by starting from the "gross" difference and by using the relation

$$\delta_\ominus \overline{Q} = \delta_\infty \overline{Q} - \delta_\infty Q_\ominus \tag{A10}$$

where $\delta_\infty Q_\ominus$ is the difference between $Q_\ominus$ and the aymptotic value Q_∞ in the vortex state, i.e.

$$\delta_\infty Q_\ominus = \overline{Q}_\ominus - Q_\infty \ . \tag{A11}$$

It is to be noted that the possibility of using a reference state that is uniform is made possible by our neglect of gravity in the work from the end of Section III onwards.

The complete specification of the reference state and the corresponding difference $\delta_\infty Q_\ominus$ is not fixed just by the uniformity requirement but is is a matter of discretion, depending on the physical context under consideration. The most natural choice for many purposes is to fix the reference state by the requirement that that it give unchanged values for the total integrals and averages of the independent current variables, but for some purposes other choices might be more convenient. Before adopting any particular convention as to the choice of reference state let us first consider the "gross" difference denoted by $\delta_\infty Q$ which is already well defined *a priori.*

In order to see the nature of the dominant contribution to the "gross" differences in the large r limit, we exploit the fact that the equation (3.17) for the radial dependence of the chemical potential will be expressible in the form

$$\delta_\infty \mu^2 = -\frac{M}{2c^2} w \ . \tag{A12}$$

Assuming that the quantity $Q(r)$ under consideration has a dependence on the radial coordinate r only through the intermediary of either $\mu(r)$ or $\Gamma(r)$, one can use a Taylor expansion for Q considered as function of the two variables μ^2 and Γ^{-2},

$$Q(r) = Q(\bar{r}) + \left[\frac{dQ}{d\mu^2}\right]_{\bar{r}} \left(\mu^2(r) - \mu^2(\bar{r})\right) + \left[\frac{dQ}{d(\Gamma^{-2})}\right]_{\bar{r}} \left(\Gamma^{-2}(r) - \Gamma^{-2}(\bar{r})\right) + ... \tag{A13}$$

where we have written explicitly only the first terms of the Taylor expansion. Using (3.12) and (3.15), one gets

$$\mu^2(r) - \mu^2(\bar{r}) = -\frac{M}{2c^2}\left(w - \bar{w}\right), \tag{A14}$$

and

$$\Gamma^{-2}(r) - \Gamma^{-2}(\bar{r}) = \frac{2M\Omega^2}{c^2}\left(\frac{1}{\bar{w}} - \frac{1}{w}\right) = \frac{Mc^2}{2E^2}\bar{w}\left(1 - \frac{\bar{w}}{w}\right), \tag{A15}$$

where the second equality follows from (5.7).

Taking into account all the orders of the Taylor expansion, one can see easily from the previous expressions, that the contribution from any order β ($\beta \geq 1$) will be a sum of terms of the form $\varpi^\beta (w/\varpi)^\alpha$ with $\alpha \leq \beta$. When one substitutes the Taylor expansion (A13) of $Q(r)$ into the integral in (A8), such a term will yield the contribution of the form

$$\varpi \int_{\varpi}^{w} \varpi^\beta \left(\frac{w}{\varpi}\right)^\alpha \frac{dw}{w^2} = \varpi^{1+\beta-\alpha} \frac{w^{\alpha-1}}{\alpha - 1} - \frac{w^\beta}{\alpha - 1},$$

except the case $\alpha = 1$, in which case the contribution will have the form $\varpi^\beta \ln\{w/\varpi\}$. It can thus be seen that the expansion for (A8) will take the form

$$\overline{Q}(\bar{r}) = Q(\bar{r}) - \frac{M}{4c^2}\Big[\frac{1}{\mu}\frac{dQ}{d\mu}\Big]_{\bar{r}} \varpi \,\ln\{\frac{w_\odot}{w}\} + \mathcal{O}\{\varpi\} \ . \tag{A16}$$

whose dominant term is not afected be the rigid rotation whose contribution is proportional to Ω and thus at most $\mathcal{O}\{\varpi\}$.

Finally, using once more the Taylor expansion (A13), one arrives at the simpler form

$$\overline{Q}(\bar{r}) = Q_\infty - \frac{M}{4c^2}\Big[\frac{1}{\mu}\frac{dQ}{d\mu}\Big]_{\infty} \varpi \,\ln\{\frac{w_\odot}{\varpi}\} + \mathcal{O}\{\varpi\} \ . \tag{A17}$$

We finally obtain a limiting relation of the form

$$\delta_\infty \overline{Q} \sim \tilde{Q} \frac{Mw}{4c^2\mu^2} \,\ln\{\frac{w_\odot}{w}\} \ , \tag{A18}$$

where the "gross" asymptotic deviation coefficient is defined (so as to have the same dimensionality as Q itself) as the logarithmic derivative

$$\tilde{Q} = -\left[\mu \frac{dQ}{d\mu}\right]_\infty \ . \tag{A19}$$

We now come to the final step, which is to take account of the difference between the asymptotic limit value Q_∞ with respect to which the preceeding "gross" deviations are defined, and the corresponding reference state value $Q_\ominus$ with respect to which the desired "net" deviations are defined. In order to proceed, let us use the abbreviation q to indicates some chosen set (which might consist of the relevant independent longitudinal current components) of the independent field variables needed to specify a homogeneous state of the system, so that (with implicit sumation over the individual variables in the set q) the difference (A11) between the actual asymptotic value and its value in the as yet unspecified *homogeneous reference state* will have an asymptotic form given by an expression of the form

$$\delta_{\infty} Q_{\ominus} \sim \frac{\partial Q}{\partial q}\, \delta_{\infty} q_{\ominus} \sim \frac{\partial Q}{\partial q}\, (\delta_{\infty}\overline{q} - \delta_{\ominus}\overline{q}) \; . \tag{A20}$$

More specificly, let us suppose that the set of variables q is chosen so as to fix a corresponding the choice of the reference state in an obviously natural way by simply matching the corresponding average (and therefore also the total) values, so that the homogeneous reference state values $q_{\ominus}$ agree with average values $\overline{q}$ for the vortex state under consideration, which is expressible in terms of the notation of (A10) just by

$$\delta_{\ominus}\overline{q} = 0 \; . \tag{A21}$$

Subject to this choice, it evidently follows from (A20) that the "net" variation (A10) in which we are ultimately interested will be given for a generic quantity Q by

$$\delta_{\ominus}\overline{Q} \sim \delta_{\infty}\overline{Q} - \frac{\partial Q}{\partial q}\, \delta_{\infty}\overline{q} \; . \tag{A22}$$

It now follows from the application of (A18) to the q variables that there will be a modified *net asymptotic deviation coefficient* given by

$$\widehat{Q} = \widetilde{Q} - \widetilde{q}\, \frac{\partial Q}{\partial q} \; , \tag{A23}$$

in terms of which the "net" deviation (A24) will be given by the asymptotic formula

$$\delta_{\ominus}\overline{Q} \sim \widehat{Q} \frac{M w}{4 c^2 \mu^2}\, \ln\{\frac{w_{\odot}}{w}\} \; . \tag{A24}$$

APPENDIX: ACKNOWLEDGMENTS

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