HD-THEP-95-30

High temperature critical O(N) field models by LCE series

Thomas Reisz*
Institut für Theoretische Physik,
Universität Heidelberg,
Philosophenweg 16,
D-69120 Heidelberg, Germany

July 12, 1995

Abstract

The critical properties of renormalizable O(N) field models are determined by means of the high order (≥ 18) behaviour of convergent linked cluster series on finite temperature lattices. It is shown that those models become weakly coupled at the phase transition. The critical exponents agree to those of the corresponding superrenormalizable 3-dimensional models. Concerning critical amplitudes and subcritical behaviour, corrections induced by renormalizable couplings are measurable.

*Email address reisz@ThPhys.Uni-Heidelberg.DE

Quantum field theories at finite temperature are the target of recent intensive studies. This is not at least because they are supposed to allow for a quantitative investigation of the physics of the early universe, including early phase transitions, as well as of heavy ion collisions. There is increasing effort to develop and apply techniques and methods suitable for a serious understanding both of the theories themselves and of their direct physical (measurable) implications. For instance, standard weak coupling perturbative expansions suffer from various problems related to severe infrared singularities for theories with zero mass excitations. Various proposals exist to resolve or to circumvent those problems. Among those are hard thermal loop resummations [1], finite volume scaled perturbation theory [2, 3], and the techniques of scale dependent flow equations [4, 5] and block spin renormalization group [6].

A very promising way is to start with well defined lattice field theories. This in turn implies computer aided studies both numerically and analytically. Convergent expansion methods allow for arbitrary precise measurements within their convergence domain, limited only by available skill and resources. One of those convergent techniques is the (vertex renormalized) linked cluster expansion (LCE) (e. g. [7, 8, 9]). It is a series expansion of correlation functions with respect to the strength fields at different lattice sites are coupled. The weaker those couplings, the higher the disorder of the system. Those techniques have recently been generalized to apply properly to field theories on finite temperature lattices [10].

In this letter we present results obtained by applying the LCE to the high temperature phase of (low energy effective) O(N) symmetric scalar models on the lattice with renormalizable quartic interaction. The symmetry restoration for the O(4) case has been investigated earlier in [11]. Those models provide one of the fundamental building blocks of elektroweak standard models and serve as effective theories for the QCD chiral phase transition. The expansion parameter is the isotropic nearest neighbour coupling. The coefficients of susceptibility series depend on the quartic coupling in a non-trival way. They are uniform in sign by inspection, which implies convergence up to the critical temperature. Decoding the high order behaviour, detailed information on critical properties is obtained, such as numbers or bounds on universal critical exponents and amplitudes.

The standard canonical form of the action reads

$$S(\Phi,\kappa,\lambda) = \sum_{x\in\Lambda} \left(\Phi^2 + \lambda(\Phi^2-1)^2\right) - 2\kappa \sum_{<xy>} \Phi(x)\cdot\Phi(y), \tag{1}$$

where the last sum is over nearest neighbour lattice sites on the hypercube $\Lambda = L_0 \times \infty^3$ with periodic boundary conditions imposed, and $0 \le \lambda \le \infty$. We are interested in physical (renormalized) coupling constants as defined by the large scale behaviour of the vertex correlations

$$\widetilde{\Gamma}^{(2)}_{ab}(p,-p) = -\frac{1}{Z_R}\left(m_R^2 + p^2 + O(p^4)\right)\delta_{a,b} \quad \text{as } p=(p_0=0,\vec{p}\to 0), \tag{2}$$

and

$$\widetilde{\Gamma}^{(4)}_{abcd}(p_1 = \cdots = p_4 = 0) = -\frac{1}{Z_R^2}\frac{g_R}{3} C_4(a,b,c,d), \qquad (3)$$

$$\widetilde{\Gamma}^{(6)}_{a_1\cdots a_6}(p_1 = \cdots = p_6 = 0) = -\frac{1}{Z_R^3}\frac{h_R}{15} C_6(a_1,\ldots,a_6). \qquad (4)$$

The C_n denote the O(N)-invariant tensors, $C_2(a,b) = \delta_{a,b}$, and

$$C_{2n}(a_1,\ldots,a_{2n}) = \sum_{i=2}^{2n} \delta_{a_1,a_i}\, C_{2n-2}(a_2,\ldots,\widehat{a}_i,\ldots,a_{2n}), \quad n \geq 2, \qquad (5)$$

where $(\widehat{\ })$ implies omission. In terms of the standard (connected) susceptibilities,

$$\frac{C_{2n}(a_1,\ldots,a_{2n})}{(2n-1)!!}\,\chi_{2n} = \sum_{x_2,\ldots,x_{2n}} < \Phi_{a_1}(0)\Phi_{a_2}(x_2)\cdots\Phi_{a_{2n}}(x_{2n}) >^{\mathrm{conn}}, \qquad (6)$$

$$\mu_2 = \sum_x \left(\sum_{i=1}^{3} x_i^2\right) < \Phi_1(0)\Phi_1(x) >^{\mathrm{conn}} \qquad (7)$$

(spatial weight), they are given by

$$m_R^2 = 6\frac{\chi_2}{\mu_2} = \frac{Z_R}{\chi_2}, \qquad (8)$$

$$g_R = -\left(\frac{6}{\mu_2}\right)^2 \chi_4, \qquad (9)$$

$$h_R = -\left(\frac{6}{\mu_2}\right)^3 \left(\chi_6 - 10\frac{\chi_4^2}{\chi_2}\right). \qquad (10)$$

It is the set of correlations such as (6),(7) that LCEs are applied to in the first place. We do not give technical or graph theoretical details here. For those the interested reader is referred to [10]. One major point to be noticed that goes far beyond a purely graph theoretical rearrangement is the introduction of "1-particle irreducible" susceptibilities[1]. It is not just that the series representation of the latter are constructed explicitly. Their explicit use allows for a more precise measurement of renormalized coupling constants. This is particularly useful for the computation of couplings that are related to χ_n with $n > 2$, such as are g_R and h_R in the present case. They normally suffer from the problem that their critical exponents are differences of those of the higher χ_n of about the same magnitude. This adds to the equally severe problem that at fixed order increasing n implies decreasing accuracy due to potentiated singularities. Fortunately, having the (LCE) 1PI susceptibility series at hand, we can avoid this problem by defining appropriate ratios and expressing the latter in terms of the 1PI series directly.

[1] They are to be distinguished from the vertex functions leading to (2)-(4), cf. eg. [9, 10].

To this end, we first notice that connected and 1PI susceptibilities are related by

$$\begin{aligned}
\chi_2 &= \frac{\chi_2^{\rm 1PI}}{1-16\kappa\chi_2^{\rm 1PI}}, &(11)\\
\mu_2 &= \frac{\mu_2^{\rm 1PI,mod}}{(1-16\kappa\chi_2^{\rm 1PI})^2}, &(12)\\
\chi_4 &= \frac{\chi_4^{\rm 1PI}}{(1-16\kappa\chi_2^{\rm 1PI})^4}, &(13)\\
\chi_6 &= \frac{1}{(1-16\kappa\chi_2^{\rm 1PI})^6}\,\chi_6^{\rm 1PI,mod}, &(14)
\end{aligned}$$

where

$$\begin{aligned}
\mu_2^{\rm 1PI,mod} &= \mu_2^{\rm 1PI}+12\kappa(\chi_2^{\rm 1PI})^2, &(15)\\
\chi_6^{\rm 1PI,mod} &= \chi_6^{\rm 1PI}+120\kappa\,\frac{(\chi_4^{\rm 1PI})^2}{1-16\kappa\chi_2^{\rm 1PI}}. &(16)
\end{aligned}$$

In turn, we define the following intermediate functions.

$$\begin{aligned}
\chi_2^{\rm red}\equiv\frac{1}{d_2} &= \frac{\chi_2}{\chi_2^{\rm 1PI}}=\frac{1}{1-16\kappa\chi_2^{\rm 1PI}}\simeq\mathcal{A}_2^{\rm red}\left(1-\frac{\kappa}{\kappa_c}\right)^{-\gamma}, &(17)\\
\mu_2^{\rm red} &= \frac{\mu_2 d_2}{\chi_2^{\rm 1PI}}\simeq\mathcal{A}_\mu^{\rm red}\left(1-\frac{\kappa}{\kappa_c}\right)^{-2\nu}, &(18)\\
\chi_4^{\rm red} &= \frac{\chi_4^{\rm 1PI}}{(\mu_2^{\rm 1PI,mod})^2 d_2}\simeq\mathcal{A}_4^{\rm red}\left(1-\frac{\kappa}{\kappa_c}\right)^{-(\gamma+\omega_g)}, &(19)\\
\chi_6^{\rm red} &= \frac{\chi_6^{\rm 1PI,mod}}{(\mu_2^{\rm 1PI,mod})^3 d_2}\simeq\mathcal{A}_6^{\rm red}\left(1-\frac{\kappa}{\kappa_c}\right)^{-(\gamma+\omega_h)}. &(20)
\end{aligned}$$

We have indicated the expected leading singular behaviour. Possible multiplicative logarithmic corrections are included in the amplitudes $\mathcal{A}^{\rm red}$. From this we infer that

$$\begin{aligned}
m_R^2 &= \frac{6}{\mu_2^{\rm red}}\sim\left(1-\frac{\kappa}{\kappa_c}\right)^{2\nu}, &(21)\\
Z_R &= \frac{6\chi_2^{\rm 1PI}}{\mu_2^{\rm red}d_2}\sim\left(1-\frac{\kappa}{\kappa_c}\right)^{\nu\eta}, &(22)\\
g_R &= -36\chi_4^{\rm red}d_2\sim\left(1-\frac{\kappa}{\kappa_c}\right)^{-\omega_g}, &(23)\\
h_R &= -216\chi_6^{\rm red}d_2\sim\left(1-\frac{\kappa}{\kappa_c}\right)^{-\omega_h}. &(24)
\end{aligned}$$

Introducing the lattice spacing $a(\kappa,\lambda)$ such that $T=1/(L_0 a)$ and $T\to T_c$ as $a(\kappa,\lambda)\to a(\kappa_c,\lambda)$ for fixed L_0, we get in terms of dimensionfull (physical) quantities

$$\begin{aligned}
\frac{g_{\rm ph}T}{m_{\rm ph}} &\simeq \mathcal{A}_4\,(T-T_c)^{-\omega_g-\nu}, &(25)\\
h_{\rm ph}T^2 &\simeq \mathcal{A}_6\,(T-T_c)^{-\omega_h}. &(26)
\end{aligned}$$

The results presented below are obtained from the 2- and 4-point suscecptibility series to 18th order (and the 6-point correlation to 16th order) available for the complete range of quartic couplings. We expect the accuracy of an nth order computations on the $L_0 \times \infty^3$ lattice to be at least comparable to that one of the order $n - L_0$ on the symmetric ∞^4 lattice, if in both cases the same methods are used. So at least lattices with $L_0 = 4, 6$ are covered by our computation.

The determination of the critical couplings has been done along the following lines (cf. also [12, 13, 14]). The above functions are obtained as series representations in the hopping parameter 2κ, with coefficients depending nontrivially on the bare coupling constants λ. Let us denote by f_L the truncation of the series representation of

$$f(\lambda, \kappa) = \sum_{m_f \leq \nu} c_\nu(\lambda)(2\kappa)^\nu \tag{27}$$

at order $L \geq 0$,

$$f_L(\lambda, \kappa) = \sum_{m_f \leq \nu \leq L} c_\nu(\lambda)(2\kappa)^\nu. \tag{28}$$

The susceptibilitiy series are convergent within a circle of radius $2\kappa_c(\lambda)$. By inspection, the coefficients are of uniform sign, which implies the (strongest) singularity closest to the origin lies on the positive real axis. Using this, the leading critical behaviour of f, assumed to be given by

$$f \simeq \mathcal{A} \left(1 - \frac{\kappa}{\kappa_c}\right)^{-\omega} \quad \text{as } \kappa \to \kappa_c-, \tag{29}$$

is determined as follows. The critical coupling $\kappa_c(\lambda)$ where the 2-point susceptibility diverges is computed as the smallest positive root of d_2, using that $\chi_2^{\mathrm{1PI,c}}(\lambda) = \lim_{\kappa \to \kappa_c(\lambda)-} \chi_2^{\mathrm{1PI}}(\lambda, \kappa)$ stays finite at κ_c. That is,

$$d_{2,L}(\lambda, \kappa_c(\lambda, L)) = 0, \tag{30}$$

$$\kappa_c(\lambda, L) = \kappa_c(\lambda) + \frac{\delta}{L} + o(L^{-1}). \tag{31}$$

For the regression, we always choose $L \geq 9$. The values obtained agree within the error bars to those ones got by the standard ratio criterions on the series representation of χ_2. In this way, however, the error bars themselves are considerably smaller. A possible antiferromagnetic singularity at $-\kappa_c(\lambda)$ is shifted to $-\infty$ by changing to the new variables $z = \frac{2\kappa}{1 - \frac{\kappa}{\kappa_c(\lambda)}}$, leading to

$$\overline{f}_L(\lambda, z) = \sum_{\nu \leq L} \overline{c}_\nu(\lambda) z^\nu. \tag{32}$$

Knowing the critical point κ_c or z_c, the critical exponent ω is obtained by the large order ν behaviour of the ratio

$$r_\nu = \frac{\overline{c}_\nu}{\overline{c}_{\nu-1}} = \frac{1}{z_c}\left(1 + \frac{\omega - 1}{\nu} + R_\nu\right), \tag{33}$$

where $R_\nu = o(\nu^{-1})$ as $\nu \to \infty$. The decay strength of R_ν is sensitive to the presence of multiplicative logarithmic corrections. A constant amplitude $\mathcal{A}$ as above, (29), implies that the large order behaviour of R_ν is determined by the subleading power like behaviour on the boundary of the convergence domain, $R_\nu = O(\nu^{-1-\Delta})$, where typically $1/2 \leq \Delta \leq 1$ [13, 15]. On the other hand, if $\mathcal{A}$ diverges or approaches 0 logarithmically as κ approaches κ_c,

$$\mathcal{A} \sim \left(\log|1 - \frac{\kappa}{\kappa_c}|\right)^\tau, \tag{34}$$

with $\tau \neq 0$, the decay is much weaker, $R_\nu = O(\frac{1}{\nu\psi(\nu)}) = O(\frac{1}{\nu \ln \nu})$. Direct measurement of such logarithmic exponents can hardly be done from the LCE series. But the χ^2/df can strongly hints this property.

The critical amplitude is obtained by comparing the series of f to the right hand side of (29) and linear regression $\mathcal{A}_\nu = \mathcal{A} + O(\nu^{-1})$.

We don't use Pade' approximants. They give hints to the values of critical exponents, but the error bars never happened to be smaller.

Tab. 1 and 2 show the resulting critical line and the critical exponent γ both for the 1-component and the O(4) model, as obtained on the $4 \times \infty^3$ lattice. The coupling $\overline{\lambda}$ as defined by $\overline{\lambda} = -\frac{N+2}{6} \frac{v_4 - 3(v_2)^2}{(v_2)^2}$ for O(N), with

$$v_{2n} = \frac{\int d^N\Phi \Phi_1^n e^{-(\Phi^2 + \lambda(\Phi^2-1)^2)}}{\int d^N\Phi e^{-(\Phi^2+\lambda(\Phi^2-1)^2)}}, \tag{35}$$

is monotonically increasing with λ and $0 \leq \overline{\lambda} \leq 1$ as $0 \leq \lambda \leq \infty$. Beyond the gaussian (zero temperature) fixed point at $\overline{\lambda} = 0$ there is definite non-mean field behaviour at larger couplings. We identify the finite temperature phase transition as the strong coupling part, including a range $\overline{\lambda}_{\min} \leq \overline{\lambda} \leq 1$ where the critical exponents are stable. Tab. 3 and 4 summarize the critical data on the exponents and amplitudes for $N = 1, \ldots, 4$, obtained from the high T part of the phase diagram. They are compared to the data of the corresponding superrenormalizable 3-dimensional models with the numbers that we have determined along the same lines as outlined above[2].

The results can be summarized as follows. The renormalized quartic coupling vanishes at T_c proportional to the mass ($\omega_g = -\nu$). Within bounds, the 6-point coupling has a vanishing critical exponent ($\omega_h = 0$). The most precise data for the critical exponents γ and ν are obtained by (33) with $R_\nu = O(\frac{1}{\nu \ln \nu})$, indicating multiplicative logarithmic corrections to (29). All the critical exponents, i.e. γ, ν as well as ω_g, ω_h agree in value to those of the 3-dimensional O(N) Φ^4 models. The numbers on the critical couplings and exponents are the most precise ones as yet

[2] Cf. also e.g. [16, 17]. For an extended list of references on 3-dimensional critical numbers both for O(N) field models and N-vector spin models cf. [5] and [18]. For the latter, the best of those data are given in [18].

obtained. They are in reasonable agreement to earlier investigations both by Monte Carlo methods [11] and scale dependent flow equations [5].

For $\omega_g = -\nu$ and $\omega_h = 0$, the ansatz (29) yields values on the critical amplitudes (25),(26) summarized in Tab. 4. Although the error bars are rather large on those 3rd generation quantities (after the critical line and the critical exponents), we measure deviations of the 3d data. This is in conformity with the observation that multiplicative logarithmic corrections are present that are not provided by the 3d superrenormalizable models. Hence we have a twofold strong evidence that a quantitative description beyond the leading critical behaviour by a 3d effective theory needs non-superrenormalizable couplings such as Φ^6, regardless that the quartic coupling vanishes at T_c. This might become important outside of the critical region, and in particular for first order transitions.

All the 4d data presented above are obtained on the $4 \times \infty^3$ lattice. Consistent results are obtained on the larger $6 \times \infty^3$ hypercube. Yet the relative error bars become about twice as large due to a reduction of the effective order by two. As yet, the data for temporal extent $L_0 \geq 8$ are not very predictive to 18th order of computations. There are future ways to cure this. First, the construction algorithms are that efficient that the order of the susceptibility series can still be increased within available resources. Second, zero temperature "background" can be subtracted. Third, convergence behaviour can be improved by including "irrelevant" next to nearest neighbour interactions.

References

[1] E. Braaten, R. D. Pisarski, Nucl. Phys. **B337** (1990) 569.

[2] T. Reisz, Z. Phys. **C53** (1992) 169.

[3] B. Petersson, T. Reisz, Nucl. Phys. **B353** (1991) 757.

[4] C. Wetterich, Phys. Lett. **B301** (1993) 90.

[5] N. Tetradis, C. Wetterich, Nucl. Phys. **B422** [FS](1994) 541.

[6] U. Kerres, G. Mack, G. Palma, Desy 94-226 and hep-lat 9505008

[7] M. Wortis, "Linked cluster expansion", in Phase transition and critical phenomena, vol.3, eds. C. Domb and M.S. Green (Academic Press,London 1974).

[8] C. Itzykson, J.-M. Drouffe, "Statistical field theory", vol.2, Cambridge University Press, 1989.

[9] M. Lüscher, P. Weisz, Nucl. Phys. **B300**[FS22] (1988) 325.

[10] T. Reisz, "Advanced Linked Cluster Expansion. Scalar Fields at Finite Temperature", Heidelberg preprint May 24, 1995, HD-THEP-95-23, and hep-lat 9505023.

[11] K. Jansen, P. Seuferling, Nucl. Phys. **B343** (1990) 507.

[12] D.S. Gaunt, A.J. Guttmann, "Asymptotic analysis of coefficients", in Phase transition and critical phenomena, vol.3, eds. C. Domb and M.S. Green (Academic Press,London 1974).

[13] A.J. Guttmann, "Asymptotic analysis of Power-Series Expansions", in Phase transition and critical phenomena, vol.13, eds. C. Domb and J.L. Lebowitz (Academic Press).

[14] A. J. Guttmann, J. Phys. A **20** (1987) 1839.

[15] F. Wegener, Phys. Rev. **B5** (1972) 4529.

[16] K. Kanaya, S. Kaya, "Critical Exponents of a three-dimensional O(4) spin model", preprint UTHEP-284 and hep-lat 9409001.

[17] G. A. Baker, B. G. Nickel, D. I. Meiron, Phys. Rev. **B17** (1978) 1365.

[18] P. Butera, M. Comi, "New extended high temperature series for the N-vector spin models on three-dimensional bipartite lattices", hep-lat 9505027.

Table 1: The critical line $\kappa_c(\lambda)$ for the 1-component Φ^4 model on the $4 \times \infty^3$ lattice. Also shown is the critical exponent $\gamma(\lambda)$, that exhibits the high temperature phase transition as the strong coupling range of the phase diagram.

$\overline{\lambda}$	λ	κ_c	γ
1	∞	0.075773(21)	1.2407(83)
0.95	6.92952	0.085272(24)	1.2392(87)
0.9	4.3303	0.093339(27)	1.2406(102)
0.85	3.25072	0.10085(4)	1.2395(106)
0.8	2.5836	0.10789(4)	1.2372(97)
0.7	1.7320	0.12053(4)	1.2387(110)
0.6	1.1769	0.13094(4)	1.1977(85)
0.5	0.77841	0.13845(3)	1.1731(67)
0.4	0.48548	0.14244(3)	1.1570(58)
0.3	0.27538	0.14239(3)	1.1323(65)
0.2	0.13418	0.13860(2)	1.1007(44)
0.1	0.04770	0.13225(2)	1.0538(46)
0.05	0.01997	0.12866(1)	1.0311(24)
0.01	0.003457	0.12573(1)	1.0066(24)

Table 2: The same as Tab. 1, but for the O(4) symmetric model.

$\overline{\lambda}$	λ	κ_c	γ
1	∞	0.30967(16)	1.4473(202)
0.975	39.4673	0.30800(18)	1.4516(229)
0.95	19.4314	0.30614(19)	1.4417(243)
0.925	12.725	0.30418(20)	1.4310(258)
0.9	9.34653	0.30211(18)	1.4211(233)
0.8	4.12700	0.29204(17)	1.3780(236)
0.7	2.24446	0.27772(15)	1.3380(206)
0.6	1.25586	0.25802(13)	1.2982(197)
0.5	0.67905	0.23313(11)	1.2569(184)
0.4	0.34514	0.20532(8)	1.2164(153)
0.3	0.16348	0.17848(6)	1.1728(123)
0.2	0.07041	0.15583(4)	1.1247(101)
0.1	0.02370	0.13826(2)	1.0696(56)
0.05	0.00989	0.13116(1)	1.0346(25)

Table 3: The critical exponents of the Φ^4 O(N) models for $N = 1, \ldots, 4$, both in 4 dimensions (upper values) and in 3 dimensions at zero temperature (lower values).

N	$\overline{\lambda}_{\min}$	γ	ν	$\nu\eta$	ω_g	ω_h
1	0.85	1.2400(87)	0.6300(49)	0.0193(132)	-0.6296(95)	-0.0016(136)
	0.90	1.2406(36)	0.6301(18)	0.0183(52)	-0.6300(42)	-0.0008(72)
2	0.90	1.3238(139)	0.6694(64)	0.0206(150)	-0.6683(131)	0.0030(191)
	0.90	1.3250(52)	0.6734(28)	0.0199(76)	-0.6788(75)	0.0040(126)
3	0.90	1.4032(156)	0.7167(76)	0.0246(169)	-0.7132(164)	0.0010(262)
	0.90	1.4029(85)	0.7131(40)	0.0232(87)	-0.7165(115)	-0.0167(218)
4	0.95	1.4469(225)	0.7356(93)	0.0257(200)	-0.7285(276)	0.0075(397)
	0.90	1.4504(113)	0.7361(68)	0.0213(123)	-0.7268(134)	-0.0104(247)

Table 4: The corresponding critical amplitudes $\mathcal{A}_4$ and $\mathcal{A}_6$ for $\omega_g = -\nu$ and $\omega_h = 0$, both in 4 dimensions (upper values) and in 3 dimensions (lower values). The numbers hold under the assumption that no multiplicative logarithmic factors occur.

N	$\mathcal{A}_4$	$\mathcal{A}_6$
1	16.61 ± 1.46	590 ± 103
	23.72 ± 1.49	1386 ± 174
2	15.39 ± 2.24	464 ± 121
	20.33 ± 1.76	913 ± 177
3	14.99 ± 2.40	406 ± 114
	17.87 ± 1.68	672 ± 142
4	13.83 ± 2.37	293 ± 94
	15.29 ± 1.56	447 ± 105