A New Mass Formula for NG Bosons in QCD

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Abstract

An often used mass formula for Nambu-Goldstone (NG) bosons in QCD, such as the pions, involves the condensate $<\overline{q}q>$, f_π and the quark current masses. We show that this expression is wrong. Analysis of the interplay between the Dyson-Schwinger equation for the constituent quark effect and the Bethe-Salpeter equation for the NG boson results in a new mass formula.

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The computation of the low energy properties of QCD is a difficult non-perturbative problem in quantum field theory. However one species of hadron, the (almost) Nambu-Goldstone (NG) bosons, such as the pions, have always played a key role. Because they are directly associated with the dynamical breaking of chiral symmetry their properties

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are strong indicators of the nature of the underlying quark-gluon dynamics in QCD. However the small current masses of the u and d quarks means that the pions are not strictly massless, as for true NG bosons, but acquire small masses. They are therefore also significant as low mass hadronic excitations.

One expects that there should be some perturbative expression for the pion mass in terms of the quark current masses which is built upon the underlying non-perturbative chiral-limit quark-gluon dynamics. While the relation of the low pion mass to the breaking of chiral symmetry dates back to the current algebra era and PCAC [1], the often used implementation in QCD has the form (see for example [2, 3]), where m is the average current mass of the quark and antiquark, (with $s = q^2$),

$$M_\pi^2 = \frac{48m}{f_\pi^2}\int \frac{d^4q}{(2\pi)^4}\sigma_s(q^2) = \frac{48m}{f_\pi^2}.\frac{\pi^2}{(2\pi)^4}\int_0^\infty sds\sigma_s(s) = \frac{2m\rho}{f_\pi^2} \tag{1}$$

Here the integral $\rho =< \overline{q}q >$ is the so called condensate parameter,

$$\rho = 24\int \frac{d^4q}{(2\pi)^4}\sigma_s(q^2), \tag{2}$$

and f_π is the usual pion decay constant. In (2) $\sigma_s(s)$ is the chiral limit ($m \to 0$) scalar part of the constituent quark propagator

$$G(q) = (iA(s;m)q.\gamma + B(s;m) + m)^{-1} = -iq.\gamma\sigma_v(s;m) + \sigma_s(s;m). \tag{3}$$

We note that the expression for ρ in (2) is slowly convergent in QCD, because for large $s \to \infty$ $B(s)$ decreases like $1/s\ln[s/\Lambda^2]^{1-\lambda}$ where $\lambda = 12/(33 - 2N_f)$ and Λ is the QCD scale parameter. Nevertheless some integration cutoff is usually introduced. In NJL type models [3] this cutoff is mandatory because forms for σ_s are used for which ρ is divergent. In either case the values of m and $< \overline{q}q >$ are then usually quoted as being relative to some cutoff energy, often $1GeV$. An alternative approach [4] is to use finite energy sum rules and Laplace sum rules.

Here we present a new analysis of the chiral symmetry breaking in QCD. We extract a new expression for the pion mass, which essentially replaces (1). Nevertheless the new result, see equation (14) below, is very similar to (1) except that it contains a naturally arising cutoff function $c(s)$, and also a dynamical enhancement function $\epsilon_s(s)$ for the quark current mass m. This means that the pion mass is dominated by IR processes, and not UV prcesses as in (1).

The analysis will be done in the context of the Global Colour Model (GCM) of QCD [5, 6]. This model is basically a truncation of QCD. From the GCM various other models and phenomenologies may be derived by further approximation. These include NJL, CPT and various mean-field soliton models. Importantly the GCM does not introduce any new divergences that were not already in QCD.

For the purpose of the NG mass formula we consider here only the essential equations. A more general insight may be obtained within the context of the functional integral formulation of the GCM [6]. The first equation is the Dyson-Schwinger (DSE) equation for the constituent quark propagator,

$$B(p^2;m) = \frac{16}{3}\int \frac{d^4q}{(2\pi)^4} D(p-q).\frac{B(q^2;m)+m}{q^2A(q^2;m)^2+(B(q^2;m)+m)^2}, \tag{4}$$

$$[A(p^2;m)-1]p^2 = \frac{8}{3}\int \frac{d^4q}{(2\pi)^4} q.pD(p-q).\frac{A(q^2;m)}{q^2A(q^2;m)^2+(B(q^2;m)+m)^2}, \tag{5}$$

using a Feynman-like gauge and the perturbative quark-gluon vertex function. Here $D(q)$ is an effective gluon propagator which may be extracted from meson data [7].

Using Fourier transforms (4) may be written in the form

$$D(x) = \frac{3}{16}\frac{B(x)}{\sigma_s(x)}, \tag{6}$$

which implies that knowledge of the quark propagator determines the effective gluon propagator. Multiplying (6) by $B(x)/D(x)$, and using Parseval's identity for the RHS, we obtain the identity

$$\int d^4x \frac{B(x)^2}{D(x)} = \frac{16}{3}\int \frac{d^4q}{(2\pi)^4} B(q)\sigma_s(q). \tag{7}$$

The second basic equation is the Bethe-Salpeter equation (BSE) for the pion mass-shell state at the level of approximation that matches (4) and (5) in the GCM analysis[8]

$$\Gamma^f(p,P) = \frac{8}{3}\int \frac{d^4q}{(2\pi)^4} D(p-q) tr_{SF}(G_+T^gG_-T^f)\Gamma^g(q,P) \tag{8}$$

where $G_\pm = G(\pm q - \frac{P}{2})$. This BSE is for isovector NG bosons, and only the dominant $\Gamma = \Gamma^f T^f i\gamma_5$ amplitude is retained; the spin trace arises from projecting onto this dominant amplitude. Here $\{T^b, b = 1,..,N_F^2-1\}$ are the generators of $SU(N_F)$, with

$tr(T^f T^g = \frac{1}{2}\delta_{fg})$. The BSE (8) is an implicit equation for the mass shell $P^2 = -M^2$. It has solutions *only* in the time-like region $P^2 \leq 0$. Fundamentally this is ensured by (4) and (5) being the specification of an absolute minima of an effective action [6]. Nevertheless the loop momentum is kept in the space-like region $q^2 \geq 0$; this mixed metric device ensures that the quark and gluon propagators remain close to the real space-like region where they have been most thoroughly studied. Very little is known about these propagators in the time-like region $q^2 < 0$. The GCM gives a detailed description of the pion properties, including its coupling to other states, in the language of effective non-local actions [6].

The non-perturbative quark-gluon dynamics is expressed here in (4) and (5). Even when $m = 0$ eqn. (4) can have non-perturbative solutions with $B \neq 0$. This is the dynamical breaking of chiral symmetry. When $m = 0$ eqn.(8) has a solution for $P^2 = 0$; the Goldstone theorem effect. For the at-rest state $\{P_0 = 0, \vec{P} = \vec{0}\}$ it is easily seen that eqn.(8) reduces to eqn.(4) with $\Gamma^f(q,0) = B(q^2)$. When $\vec{P} \neq \vec{0}$ then $\Gamma^f(q,P) \neq B(q)$, and (8) must be solved for $\Gamma^f(q,P)$.

We shall now determine an accurate expression for the mass of the pion when $m \neq 0$. This amounts to finding an analytic solution to the BSE (8), when the constituent quark propagators are determined by (4) and (5). The result will be accurate to order m. For small $m \neq 0$ eqns.(4) and (5) have solutions of the form

$$B(s;m) + m = B(s) + m.\epsilon_s(s) + O(m^2),$$

$$A(s,m) = A(s) + m.\epsilon_v(s) + O(m^2)$$

For large space-like s we find that $\epsilon_s \to 1$, but for small s we find that $\epsilon_s(s)$ can be significantly larger than 1 (see Fig.1). This is a dynamical enhancement of the quark current mass by gluon dressing in the infrared region. Even in the chiral limit the quark running mass $M(s) = B(s)/A(s)$ is essential for understanding any non-perturbative QCD quark effects. At $s = 0.3GeV^2$ we find [7] that $M(s) \approx 270MeV$.

Because the pion mass M_π is small when m is small, we can perform an expansion of the P_μ dependence in the kernel of (8). Since the analysis is Lorentz covariant we can, without loss of validity, choose to work in the rest frame with $P = (\vec{0}, iM_\pi)$, giving, for equal mass quarks for simplicity,

$$\Gamma(p) = \frac{2}{9}M_\pi^2 \int \frac{d^4q}{(2\pi)^4} D(p-q) I(s)\Gamma(q) +$$

$$+ \frac{16}{3}\int \frac{d^4q}{(2\pi)^4} D(p-q) \frac{1}{s(A(s)+\epsilon_v(s).m)^2 + (B(s)+m.\epsilon_s(s))^2}\Gamma(q) + ..., \quad (9)$$

where

$$I(s) = (\sigma_v^2 - 2(\sigma_s\sigma_s' + s\sigma_v\sigma_v') - s(\sigma_s\sigma_s'' - (\sigma_s')^2) - s^2(\sigma_v\sigma_v'' - (\sigma_v')^2). \quad (10)$$

By using Fourier transforms the integral equation (9), now with explicit dependence on M_π, can be expressed in the form of a variational mass functional,

$$M_\pi[\Gamma]^2 = -\frac{24}{f_\pi[\Gamma]^2}\int \frac{d^4q}{(2\pi)^4} \frac{\Gamma(q)^2}{s(A(s)+\epsilon_v(s).m)^2 + (B(s)+m.\epsilon(s))^2} + \frac{9}{2f_\pi[\Gamma]^2}\int d^4x \frac{\Gamma(x)^2}{D(x)} \quad (11)$$

in which

$$f_\pi[\Gamma]^2 = 6\int \frac{d^4q}{(2\pi)^4} I(s)\Gamma(q)^2. \quad (12)$$

The functional derivative $\delta M_\pi[\Gamma]^2/\delta\Gamma(q) = 0$ reproduces (9). The mass functional (11) and its minimisation is equivalent to the pion BSE in the near chiral limit. To find an estimate for the minimum we need only note that the change in M_π^2 from its chiral limit value of zero will be of 1st order in m, while the change in the rest-frame $\Gamma(q)$ from its chiral limit value $B(q^2)$ will be of 2nd order in m.

Hence to lowest order in m we have that the pion mass is given by

$$M_\pi^2 = \frac{48m}{f_\pi[B]^2}\int \frac{d^4q}{(2\pi)^4} \frac{\epsilon_s(s)B(s) + s\epsilon_v(s)A(s)}{sA(s)^2 + B(s)^2} \frac{B(s)^2}{sA(s)^2 + B(s)^2}$$

$$-\frac{24}{f_\pi[B]^2}\int \frac{d^4q}{(2\pi)^4} \frac{B(s)^2}{sA(s)^2 + B(s)^2} + \frac{9}{2f_\pi[B]^2}\int d^4x \frac{B(x)^2}{D(x)} + O(m^2) \quad (13)$$

However the pion mass has been shown to be zero in the chiral limit. This is confirmed as the two $O(m^0)$ terms in (13) cancel because of the identity (7). Note that it might appear that f_π would contribute an extra m dependence from its kernel in (10). However because the numerator in (13) is already of order m, this extra contribution must be of higher order in m.

Hence we finally arrive at the analytic expression, to $O(m)$, for the NG boson $(\text{mass})^2$ from the solution of the BSE in (8) which includes the non-perturbative gluon dressing to give constituent quarks

$$M_\pi^2 = \frac{2m\rho_{eff}}{f_\pi^2} \quad \text{where} \quad \rho_{eff} = 24 \int \frac{d^4q}{(2\pi)^4}[\epsilon_s(s)\sigma_s(s) + s\epsilon_v(s)\sigma_v(s)]c(s) \tag{14}$$

defines an effective condensate parameter ρ_{eff}, which replaces the definition in (2). The ρ_{eff} integrand involves a naturally arising function

$$c(s) = \frac{B(s)^2}{sA(s)^2 + B(s)^2},$$

which acts as a smooth cutoff function. It is this function which causes the pion mass to be IR dominated. As well ρ_{eff} contains a contribution from the vector part of the chiral-limit quark propagator.

While (14) is exact to $O(m)$ it is instructive to not expand the quark propagator in (11) in powers of m. Using the identity (7) in (11) we obtain

$$M_\pi^2 \approx \frac{24}{f_\pi[B]^2} \int \frac{d^4q}{(2\pi)^4} B(s)(\sigma_s(s;m) - \sigma_s(s;0)). \tag{15}$$

which does not include any m dependence from the pion form factor $\Gamma(q,P)$. From this expression we see that it is the change in the condensate, $< \bar{q}q >_m - < \bar{q}q >_0$, which is induced by the current mass of the quarks, that determines the NG masses. This change is in the infrared region.

We now give examples of the functions occuring in (14). In ref. [7] the chiral-limit quark propagator was obtained by fitting a number of meson observables to meson data. The forms used for the quark propagator are

$$\sigma_s(s) = c_1 exp(-d_1 s) + c_2 exp(-d_2 s), \qquad \sigma_v(s) = \frac{2s - \beta^2(1 - exp(-2s/\beta^2))}{2s^2}. \tag{16}$$

where the form for σ_s is only valid in the non-asymptotic region. The parameter values are shown in Table 1. This determines $c(s)$ in (14). It remains to determine the enhancement function $\epsilon_s(s)$. Here we ignore the vector enhancement function $\epsilon_v(s)$ which appears to make only a small contribution in eqn.(14). This chiral-limit quark propagator determines the effective gluon propagator via eqn.(6). In principal we could imagine Fourier transforming $D(x)$ to obtain $D(q)$, which would then allow (4) and (5) to be solved, and for $\epsilon_s(s)$ to be determined from $B(s;m) - B(s)$ for small m. In practice a multi-rank separable expansion of $D(p-q)$ in the kernel of (4) is introduced [7], which greatly facilitates the computations. The resulting $\epsilon_s(s)$ is shown in Fig.1. We note that its value of approximately $\epsilon_s(s) \approx 4$ implies that in the infrared region, appropriate to the internal dynamics of hadrons, the quark current mass of $\sim 6MeV$ is enhanced by gluon dressing to some $24MeV$. Of course the major effect is the chiral-limit constituent mass of some $270MeV$. We compute using (14) that $\rho_{eff} = (0.233GeV)^3$, and from (12) that $f_\pi = 93.0MeV$. With $m = 6.5MeV$ (14) gives $M_\pi = 138.5MeV$ without cutoffs or renormalisation procedures.

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Table 1: Chiral-Limit Quark Propagator Parameters

c_1	0.5200GeV^{-1}	c_2	1.1794GeV^{-1}
d_1	2.0737GeV^{-2}	d_2	4.7214GeV^{-2}
β	0.5082GeV		

Figure 1 Caption

Shows (a) the scalar enhancement function $\epsilon_s(s)$, (b) the integrand of the effective condensate parameter ρ_{eff} (in arbitrary units), and (c) the cutoff function $c(s)$.

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