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A new ansatz: Fritzsch Mass Matrices with least modification

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ABSTRACT

We investigate how the Fritzsch ansatz for the quark mass matrices can be modified in the least possible way to accommodate the observed large top quark mass and the measured values of the CKM elements. As one of the solutions, we find that the {23} and the {32} elements of the up quark mass matrix are unequal. The rest of the assumptions are same as in Fritzsch ansatz. In this formalism we have an extra parameter i.e. the ratio of the {23} and the {32} element, which gets fixed by the large top quark mass. The predicted values for $\frac{V_{ub}}{V_{cb}}$, $\frac{V_{td}}{V_{ts}}$ from this new ansatz are in the correct experimental range even for the smaller values of $\tan\beta$. In the end, we write down the $SO(10)$ motivated superpotential for these new mass matrices

A simple approach for including the observed hierarchy of quark masses and mixing angles was suggested by Fritzsch[1]. It prescribes a form for the mass matrices which has certain amount of predictive power. It can predict the top mass based on the V_{cb} element of the CKM matrix as input. The top mass according to this ansatz can not be more than 90 GeV. The recent CDF data[2] however shows that the top mass (pole mass) is above 160 GeV ($m_t = 176 \pm 8\,(stat) \pm 10\,(syst)$GeV). So, the simplest version is clearly excluded.

This situation improves if one realizes these mass matrices at the GUT scale[3]. The effect of the running from the weak scale upto GUT scale is utilized to incorporate a somewhat heavier top in the theory. But the improvement is not enough to incorporate the recently discovered heavy top quark. If we try to use this top mass as input in this GUT scenario, the predicted value of V_{cb} is far off from the experimentally predicted range which is $V_{cb} = 0.0400 \pm 0.0025 \pm 0.0020$[4]. In Fig.1 we show the range of V_{cb} values for the range of the top mass m_t from 155 to 185 GeV , where m_t stands for the running mass. The relation between the running mass and the pole mass is given by

$$m_t = m_t\,(m_t)\left[1 + \frac{4}{3}\frac{\alpha_3}{\pi}\right] \tag{1}$$

Increasing the value of $\tan\beta$ improves the situation a little better as shown in Fig.1. However, to get the correct prediction for V_{cb}, one has to go beyond $\tan\beta \approx 65$, where the theory loses the perturbative nature [5].

In this letter we propose a modification of this Fritzsch ansatz in the least possible sense. We make the {23} and the {32} element asymmetrical in the up quark mass matrix. It then looks like:

$$M_u = P_u U Q_u \tag{2}$$

where

$$U = \begin{pmatrix} 0 & a_u & 0 \\ a_u & 0 & b_u \\ 0 & b_u^{'} & c_u \end{pmatrix}$$

The zeros are described later in the superpotential. The expressions $a, b, b^{'}, c$ are real numbers while P and Q are diagonal phase matrices. Among the quark phases contained in P and Q, only two are relevant for quark mixing, we denote them by ψ and ϕ .The down matrix is kept the same as in ref.[1], i.e.

$$M_d = P_d D Q_d \tag{3}$$

where

$$D = \begin{pmatrix} 0 & a_d & 0 \\ a_d & 0 & b_d \\ 0 & b_d & c_d \end{pmatrix}$$

We also observe the same hierarchical form as has been exercised in the Fritzsch ansatz, i.e. $c \gg b \sim b^{'} \gg a$. We realize these matrices at the GUT scale.

The real matrix U is diagonalized by the bi-orthogonal transformation $R_u U R_u^{'-1} = U^{diag}$ producing the eigenvalues $\{m_u, -m_c, m_t\}$.Using the hierarchy of masses $m_u \ll m_c \ll m_t$ we obtain

$$R_u = \begin{pmatrix} 1 & s_1^u - \chi_1^u s_2^u & s_1^u s_2^u + \chi_1^u \\ -s_1^u & 1 & s_2^u \\ -\chi_1^u & -s_2^u & 1 \end{pmatrix}$$

$$R_u^{'} = \begin{pmatrix} 1 & s_1^u - \chi_2^u s_2^{'u} & s_1 s_2^{'u} + \chi_2^u \\ -s_1^u & 1 & s_2^{'u} \\ -\chi_2^u & -s_2^{'u} & 1 \end{pmatrix}$$

where $s_1^u \equiv \sin\varphi_1^u = \sqrt{\frac{m_u}{m_c}}$ and $\mathrm{s}_2^u \equiv \sin\varphi_2^u = -\sqrt{\frac{m_c}{m_t}}$ and we have set $\cos\varphi_i^u \approx 1$. Similarly $\chi_1^u \equiv \frac{m_c s_1^u s_2^u}{r m_t}$, $\chi_2^u = \chi_1^u r$, $s_2^{'u} = \frac{s_2^u}{r}$, and $r \equiv \frac{b}{b^{'}}$.

The down quark mass matrix D is diagonalized by the orthogonal transformation $R_d D R_d^{-1} = D^{diag}$ producing the eigenvalues $\{m_d, -m_s, m_b\}$. Using the heirarchy of masses $m_d \ll m_s \ll m_b$, we obtain

$$R_d = \begin{pmatrix} 1 & s_1^d - \chi_1^d s_2^d & s_1^d s_2^d + \chi_1^d \\ -s_1^d & 1 & s_2^d \\ -\chi_1^d & -s_2^d & 1 \end{pmatrix}$$

$s_1^d \equiv \sin\varphi_1^d = \sqrt{\frac{m_d}{m_s}}$ and $s_2^d \equiv \sin\varphi_2^d = -\sqrt{\frac{m_s}{m_b}}$ and $\chi_1^d = \frac{m_s s_1^d s_2^d}{m_b}$. V_{CKM} at the unification scale in terms of the mass ratios is given by:

$$R_u \begin{pmatrix} 1 & & \\ & e^{i\sigma} & \\ & & e^{i\tau} \end{pmatrix} R_d^{-1} \tag{4}$$

We find the expressions for $V_{cb}^0, V_{ub}^0, V_{td}^0$, V_{ts}^0 as :

$$\left|V_{cb}^0\right| = \left|\sqrt{\frac{m_s}{m_b}} - e^{i\phi}\sqrt{\frac{m_c r}{m_t}}\right| \tag{5}$$

where $\phi = \tau - \sigma$,

$$\left|V_{ub}^0\right| = \left|\frac{m_s}{m_b}\sqrt{\frac{m_d}{m_b}} + e^{i\psi}\left(\sqrt{\frac{m_s}{m_b}}\sqrt{\frac{m_u}{m_c}} - e^{i\phi}\left(\sqrt{\frac{m_u r}{m_t}} - \frac{m_c}{m_t}\sqrt{\frac{m_u}{m_t r}}\right)\right)\right| \tag{6}$$

where $\psi = \sigma$,

$$\left|V_{td}^0\right| = \left|-\frac{m_c}{m_t}\sqrt{\frac{m_u}{m_t r}} - e^{i\psi}\sqrt{\frac{m_d}{m_s}}\left(\sqrt{\frac{m_s}{m_b}} - e^{i\phi}\sqrt{\frac{m_c r}{m_t}} + \left(\frac{m_s}{m_b}\right)^{\frac{3}{2}}\right)\right| \tag{7}$$

$$\left|V_{ts}^0\right| = \left|\sqrt{\frac{m_c r}{m_t}} - e^{i\phi}\sqrt{\frac{m_s}{m_b}}\right| \tag{8}$$

The zeros in the superscript indicates the GUT value. All the masses are at the GUT scale ($\sim 10^{16}$). We run them to the top scale ($\sim 170 GeV$), and also assume that the $\mathrm{M}_{SUSY} = m_t$.

To accomplish the running, we write down the Yukawa sector which has the generic form

$$\mathcal{L}_Y = \bar{q}_L H_u \phi_u u_R + \bar{q}_L H_d \phi_d d_R + \bar{\ell}_L H_\ell \phi_d e_R + h.c. \tag{9}$$

where H_u, H_d, H_ℓ denote the 3×3 Yukawa coupling matrices for the up quarks, down quarks and charged leptons. The one loop evolution equations for the Yukawa matrices take the form ($t \equiv \ell n(\mu/M_G)$)[6]:

$$\begin{aligned}
16\pi^2 \frac{dH_u}{dt} &= \Big[Tr(3H_u H_u^\dagger + 3aH_d H_d^\dagger + aH_\ell H_\ell^\dagger) \\
&\quad + \frac{3}{2}(bH_u H_u^\dagger + cH_d H_d^\dagger) - G_U\Big] H_u \\
16\pi^2 \frac{dH_d}{dt} &= \Big[Tr(3aH_u H_u^\dagger + 3H_d H_d^\dagger + H_\ell H_\ell^\dagger) \\
&\quad + \frac{3}{2}(bH_d H_d^\dagger + cH_u H_u^\dagger) - G_D\Big] H_d \\
16\pi^2 \frac{dH_\ell}{dt} &= \Big[Tr(3aH_u H_u^\dagger + 3H_d H_d^\dagger + H_\ell H_\ell^\dagger) \\
&\quad + \frac{3}{2}bH_\ell H_\ell^\dagger - G_E\Big] H_\ell \quad .
\end{aligned} \tag{10}$$

For the minimal supersymmetric standard model (MSSM) under consideration the coefficients a, b, c are given by

$$(a, b, c) \quad = \quad (0, 2, \tfrac{2}{3}) \tag{11}$$

and the quantities G_U, G_D and G_E are given by:

$$\begin{aligned}
G_U &= \tfrac{13}{15}g_1^2 + 3g_2^2 + \tfrac{16}{3}g_3^2 \\
G_D &= \tfrac{7}{15}g_1^2 + 3g_2^2 + \tfrac{16}{3}g_3^2 \\
G_E &= \tfrac{9}{5}g_1^2 + 3g_2^2 \quad ;
\end{aligned} \tag{12}$$

The gauge couplings g_i (above) obey the standard one loop renormalization group equations:

$$8\pi^2 \frac{dg_i^2}{dt} = b_i g_i^4, \quad i = 1, 2, 3 \tag{13}$$

where

$$(b_1, b_2, b_3) \;=\; (\tfrac{33}{5}, 1, -3) \quad for \mathrm{MSSM} \tag{14}$$

From eqn.(9), one can compute the evolution equations for the eigenvalues of the Yukawa coupling matrices[7][8]:

$$\begin{aligned}
16\pi^2 \frac{df_i}{dt} &= f_i \left[3 \sum_{j=u,c,t} f_j^2 + 3a \sum_{\beta=d,s,b} f_\beta^2 + a \sum_{b=e,\mu,\tau} f_b^2 - G_U \right. \\
&\quad \left. + \frac{3}{2} b g_i^2 + \frac{3}{2} c \sum_{\beta=d,s,b} f_\beta^2 \mid V_{i\beta} \mid^2 \right] \\
16\pi^2 \frac{df_\alpha}{dt} &= f_\alpha \left[3a \sum_{j=u,c,t} f_j^2 + 3 \sum_{\beta=d,s,b} f_\beta^2 + \sum_{b=e,\mu,\tau} f_b^2 - G_D \right. \\
&\quad \left. + \frac{3}{2} b f_\alpha^2 + \frac{3}{2} c \sum_{j=u,c,t} f_j^2 \mid V_{j\alpha} \mid^2 \right] \\
16\pi^2 \frac{df_a}{dt} &= f_a \left[3a \sum_{j=u,c,t} f_j^2 + 3 \sum_{\beta=d,s,b} f_\beta^2 + \sum_{b=e,\mu,\tau} f_b^2 - G_E \right. \\
&\quad \left. + \frac{3}{2} b f_a^2 \right]
\end{aligned} \tag{15}$$

where $i = (u, c, t)$, $\alpha = (d, s, b)$, $a = (e, \mu, \tau)$.

We will also need the evolution equations for the elements of the CKM matrix[7][8]:

$$\begin{aligned}
16\pi^2 \frac{d}{dt} \mid V_{i\alpha} \mid^2 &= 3c \left[\sum_{j \neq i} \sum_{\beta=d,s,b} \frac{f_i^2 + f_j^2}{f_i^2 - f_j^2} f_\beta^2 Re\left(V_{i\beta} V_{j\beta}^* V_{j\alpha} V_{i\alpha}^* \right) \right. \\
&\quad \left. + \sum_{\beta \neq \alpha} \sum_{j=u,c,t} \frac{f_\alpha^2 + f_\beta^2}{f_\alpha^2 - f_\beta^2} f_j^2 Re\left(V_{j\beta}^* V_{j\alpha} V_{i\beta} V_{i\alpha}^* \right) \right] \quad .
\end{aligned} \tag{16}$$

The above expressions would simplify considerably if we exploit the hierarchy in the Yukawa couplings ($f_b \gg f_s \gg f_d$, etc) and in the CKM matrix elements. If only the leading terms are kept, one obtains the following approximate expressions for the evolution of the various mass ratios and the mixing angles:

$$\begin{aligned}
16\pi^2 \frac{d}{dt} \left(\frac{m_\alpha}{m_b} \right) &= -\frac{3}{2} \left(\frac{m_\alpha}{m_b} \right) \left(b f_b^2 + c f_t^2 \right), \quad \alpha = d, s \\
16\pi^2 \frac{d}{dt} \left(\frac{m_i}{m_t} \right) &= -\frac{3}{2} \left(\frac{m_i}{m_t} \right) (b f_t^2 + c f_b^2), \quad i = u, c
\end{aligned}$$

$$
\begin{aligned}
16\pi^2 \frac{d}{dt}\left(\frac{m_d}{m_s}\right) &= -\frac{3}{2}\left(\frac{m_d}{m_s}\right)\left(bf_s^2 + cf_c^2 + cf_t^2 \mid V_{ts} \mid^2\right) \\
16\pi^2 \frac{d}{dt}\left(\frac{m_u}{m_c}\right) &= -\frac{3}{2}\left(\frac{m_u}{m_c}\right)\left(bf_c^2 + cf_s^2 + cf_b^2 \mid V_{cb} \mid^2\right) \\
16\pi^2 \frac{d}{dt} \mid V_{i\alpha} \mid &= -\frac{3}{2} c \mid V_{i\alpha} \mid \left(f_t^2 + f_b^2\right) \qquad (i\alpha) = (ub), (cb), (td), (ts) \\
16\pi^2 \frac{d}{dt} \mid V_{us} \mid &= -\frac{3}{2} c \mid V_{us} \mid \left(f_c^2 + f_s^2 + f_t^2 \frac{\mid V_{td} \mid^2 - \mid V_{ub} \mid^2}{\mid V_{us} \mid^2}\right) \\
16\pi^2 \frac{d}{dt} \mid V_{cd} \mid &= -\frac{3}{2} c \mid V_{cd} \mid \left(f_c^2 + f_s^2 + f_b^2 \frac{\mid V_{ub} \mid^2 - \mid V_{td} \mid^2}{\mid V_{cd} \mid^2}\right) \quad . \qquad (17)
\end{aligned}
$$

One comment is necessary here:
$\frac{|V_{ub}|}{|V_{cb}|}$ and $\frac{|V_{td}|}{|V_{ts}|}$ do not run in one loop.
The low energy i.e. the m_t scale values of the Yukawa couplings are :

$$
\lambda_b(m_t) = \frac{\sqrt{2} m_b(m_b)}{\eta_b v \cos\beta}, \lambda_\tau(m_t) = \frac{\sqrt{2} m_\tau(m_\tau)}{\eta_\tau v \cos\beta}, \lambda_t(m_t) = \frac{\sqrt{2} m_t(m_t)}{v \sin\beta} \qquad (18)
$$

where $\eta_f = m_f(m_f)/m_f(m_t)$ gives the running of the masses below $\mu = m_t$, obtained from 3-loop QCD and 1 loop QED evolution, for heavy flavors $f = t, b, c, \tau$. For light flavors $f = s, d, e, \mu$ we stop at $\mu =1$ GeV and define $\eta_f = m_f(1GeV)/m_f(m_t)$. For $\alpha_3 = 0.118$, $\eta_b \simeq 1.5$, $\eta_c \simeq 2.1, \eta_s = \eta_d = \eta_u \simeq 2.4$. The running mass values are $m_b(m_b) = 4.25 \pm 0.15GeV, m_\tau(m_\tau) = 1.7777GeV, m_c(m_c) = 1.2GeV, m_s(1GeV) \simeq 0.175GeV, m_u(1GeV) \simeq 0.006GeV, m_d(1GeV) \simeq 0.008GeV$[9].

We solve for $r \left(= \frac{b}{b'}\right)$ from V_{cb}^0 using eqn.(5) for a range of values of φ as shown in Table 1. The values of φ are chosen so that r is real. We use these values of r to predict V_{ub}^0(eqn.(6)),V_{td}^0(eqn.(7)) and V_{ts}^0(eqn.(8)). We then calculate the values of V_{ub}, V_{cb}, V_{td} and V_{ts} at the low scale using eqn.(17). In Table 1 , we present the values of $\frac{V_{ub}}{V_{cb}}$ and $\frac{V_{td}}{V_{ts}}$ from our ansatz and compare them with the experimental values. The dependence of any prediction on ψ is negligible, ψ is kept fixed at $\pi/2$. As shown in the Table 1, the prediction of the model is in excellent agreement with the current experimental ranges for an wide range of $\tan\beta$. The model will be tested further as the experimental ranges are narrowed down in the future.

Motivated by the supersymmetric $SO(10)$ Grand Unifying group, we can write down the superpotential for the mass matrices

$$
W = f_{12}^{(10'')} \psi_1 \psi_2 \varphi_{10''} + f_{23}^{(10')} \psi_2 \psi_3 \varphi_{10'} + f_{23}^{(120)} \psi_2 \psi_3 \varphi_{120} + f_{33}^{10} \psi_3 \psi_3 \varphi_{10} + h.c. \qquad (19)
$$

Here, φ_{120} has the vev only in the up direction. The zeros are produced by the discrete symmetry. From eqn.(19), we obtain the following form of the mass matrices.

$$U = \begin{pmatrix} 0 & 10'' & 0 \\ 10'' & 0 & 10', 120 \\ 0 & 10', 120 & 10 \end{pmatrix}, D = \begin{pmatrix} 0 & 10'' & 0 \\ 10'' & 0 & 10' \\ 0 & 10' & 10 \end{pmatrix} \tag{20}$$

Here, the entries in an element correspond to the Higgs fields contributing to that element. Since $\{120\}$ is an antisymmetric representation , the $\{23\}$ and $\{32\}$ elements are asymmetric.

In conclusion we summarize that if $\{23\}$ and $\{32\}$ elements of the up quark sector of the Fritzsch mass matrices are unequal, one can predict $\frac{V_{ub}}{V_{cb}}$ and $\frac{V_{td}}{V_{ts}}$ in the correct experimental range even with a heavy top in the theory. Moreover, for this ansatz, $\tan\beta$ need not be very high, it can lie anywhere between 0.6 and 65. Also, it is possible to write a superpotential for this ansatz.

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TABLE CAPTION

Table 1

The predicted values of $\frac{V_{ub}}{V_{cb}}$ $\frac{V_{td}}{V_{ts}}$ from the new ansatz are compared with their experimental values for different values of $\tan\beta$ and $\tan\phi$

FIGURE CAPTION

Figure 1

The predicted values of V_{cb} from Fritzsch ansatz is plotted as functions of m_t, for values of $\tan\beta$=3 and 60. The experimental range of V_{cb} is also shown. To be conservative, we have taken the deviation from the central value of V_{cb} to be 0.006 instead of 0.003.

Table 1:

tan β	**tan** ϕ	**predicted** $\frac{V_{ub}}{V_{cb}}$	**predicted** $\frac{V_{td}}{V_{ts}}$	**Exptal.range of** $\frac{V_{ub}}{V_{cb}}$	**Exptal.range of** $\frac{V_{td}}{V_{ts}}$
3	0	0.069	0.23	0.03-0.137	0.11-0.36
	0.18	0.082	0.23		
	0.23	0.084	0.22		
20	0	0.069	0.23		
	0.18	0.080	0.22		
	0.24	0.084	0.22		
40	0	0.068	0.23		
	0.18	0.078	0.23		
	0.24	0.082	0.22		
60	0	0.067	0.22		
	0.18	0.072	0.22		
	0.26	0.075	0.22		