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Constraints on Chiral Perturbation Theory Parameters from QCD Inequalities

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Abstract

We explore some of the constraints imposed by positivity of the QCD measure (Weingarten's inequalities) on the parameters defining chiral perturbation theory. We find, in particular, that $2m_q \leq B_0$. The use of further properties of the exact fermion propagator yields information on some higher order parameters.

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1 QCD inequalitites

The long-distance realization of QCD is presently assumed to be described by chiral perturbation theory [1]. This idea is solidly based on the way QCD symmetries are implemented at low energies. In this letter we shall explore further quantitative first principle constraints on any-distance realization of QCD through some basic inequalities inherent to vector-like gauge theories.

To set up our approach we first briefly review Weingarten's original idea [2]. Let us consider an euclidean vector current, $V_\mu^a(x) = \mathrm{i}\bar{\psi}(x)\gamma_\mu \frac{\lambda^a}{2}\psi(x)$, where λ^a carries $SU(n_f)$ flavor indices. The euclidean two-current correlator is

$$\langle V_\mu^a(x)\ V_\nu^b(0)\rangle = \int d\mu\ \mathrm{Tr}\left(S_{x,0}\gamma_\mu \frac{\lambda^a}{2} S_{0,x}\gamma_\nu \frac{\lambda^b}{2}\right), \tag{1}$$

where $S_{x,0}$ corresponds to the exact fermionic propagator and $d\mu$ stands for the gluonic measure, including the fermionic determinant, which we now assume positive and postpone its discussion. Weingarten made the observation that the Cauchy-Schwarz matrix inequality, $\left|\mathrm{Tr}(UV^\dagger)\right|^2 \leq \mathrm{Tr}(UU^\dagger)\mathrm{Tr}(VV^\dagger)$, is applicable to the spinor trace. Thus, given a positive measure, the following chain of reasoning holds :

$$\left|\langle V_\mu^a(x)\ V_\nu^a(0)\rangle\right| \leq \int d\mu\ \left|\mathrm{Tr}\left(S_{x,0}\gamma_\mu \frac{\lambda^a}{2} S_{0,x}\gamma_\nu \frac{\lambda^a}{2}\right)\right| \leq \int d\mu\ \mathrm{Tr}\left(S_{x,0}\gamma_5 \frac{\lambda^a}{2} S_{0,x}\gamma_5 \frac{\lambda^a}{2}\right)\ , \tag{2}$$

where we have used the euclidean properties $\gamma_\mu^\dagger = \gamma_\mu$ and $S_{x,0}^\dagger = \gamma_5 S_{0,x}\gamma_5$. Note that no summation on a is implied and that the inequality holds no matter what space indices are taken. The last expression is manifestly positive as it is the square of the absolute value of a complex matrix. Noticing that this corresponds to the correlator of two pseudoscalar currents ($P^a(x) = \mathrm{i}\bar{\psi}\gamma_5\frac{\lambda^a}{2}\psi$), it follows that

$$\left|\langle V_\mu^a(x)\ V_\nu^a(0)\rangle\right| \leq \langle P^a(x) P^a(0)\rangle\ . \tag{3}$$

Weingarten showed that this result, when combined with a narrow resonance approximation, leads to

$$a(x)e^{-m_\rho|x|} \leq b(x)e^{-m_\pi|x|}\ , \tag{4}$$

where $a(x)$ and $b(x)$ are polynomials in x. Thus, an elegant inequality between masses follows

$$m_\rho \geq m_\pi, \tag{5}$$

since no power law can beat the exponential decay as $|x| \to \infty$.

The very same inequality may be applied to QED, yielding the result that ortopositronium is heavier than parapositronium. Furthermore, whatever quantum numbers are used at the outset, the inequality is always bounded by the pseudoscalar correlator, if just the correaltor can be written as a single trace of two propagators closing a fermion loop. This shows that the pion is the lowest lying resonance of the QCD spectrum.

Let us now come back to the discussion of the assumed positivity of the measure. Formally, the gluonic measure which includes the integration over fermionic variables is

$$d\mu \equiv \left[dA_\mu^a(x)\right] e^{-\frac{1}{4}\int F_{\mu\nu}^a F_{\mu\nu}^a} \left(\det(\not{D} + m)\right)^{n_f}\ , \tag{6}$$

where n_f corresponds to the number of flavors. This expression needs regularization. Weingarten [2] argued that the lattice provides a regularization which corresponds to a product of Haar positive measures on each link. Positivity would then hold uniformly in the continuum, infinite volume and zero quark mass limits.

Later, Vafa and Witten [3] used the fact that non-zero eigenvalues of the gauged Dirac operator are paired via multiplication by γ_5

$$\mathrm{i}\not{D}\,\phi = \lambda\phi \quad \longrightarrow \quad \mathrm{i}\not{D}\,\gamma_5\phi = -\lambda\gamma_5\phi \ . \tag{7}$$

As a consequence,

$$\det(\not{D}\, + m) = \prod_{\lambda>0}(\lambda^2+m^2)\prod_{\lambda=0} m \ , \tag{8}$$

which is formally positive definite. The need for a non-perturbative gauge-invariant parity-preserving regularization is likely to be supplied by higher-derivative proposals as in ref. [4].

Most remarkably, no violation of any of the inequalities derived from positivity of the measure is known. This includes a large number of different applications analyzed in the literature.

A second set of basic inequalities was put forward by Witten [5] using the property that E, the piece of the propagator that commutes with γ_5,

$$E \equiv S + \gamma_5\ S\ \gamma_5 = \frac{2m}{-\not{D}^2 + m^2} \quad , \quad S = \frac{1}{\not{D} + m} \quad , \tag{9}$$

is a positive operator. Witten used this fact to explain the positivity of $m_\pi^{+2} - m_\pi^{0\,2}$.

We shall try to exploit both kind of inequalities to put constraints on long-distance realizations of QCD.

2 Weingarten's inequalities in chiral perturbation theory

As a first exercise we consider the inequality given in Eq. (3) in the framework of $SU(2)$ chiral perturbation theory, with $m_q = m_u = m_d$. We need to recall that to lowest order

$$V_\mu^a(x) = \mathrm{i}f^{abc}\pi^b(x)\partial_\mu\pi^c(x) + \dots \quad , \quad P^a(x) = -\mathrm{i}B_0 f_\pi \pi^a(x) + \dots \tag{10}$$

An immediate coordinate space computation for the euclidean vector-vector correlator gives to first order (no sum over a)

$$\langle V_\mu^a(x)\ V_\nu^a(0)\rangle = \frac{1}{8\pi^4}\left(\frac{-4x_\mu x_\nu + 2\delta_{\mu\nu}x^2}{x^6}m_\pi^2 K_1^2 + \frac{\delta_{\mu\nu}}{x^3}m_\pi^3 K_1 K_0 + \frac{x_\mu x_\nu}{x^4}m_\pi^4\left(K_0^2 - K_1^2\right) + \dots\right) \tag{11}$$

and, for the pseudoscalar-pseudoscalar one,

$$\langle P^a(x) P^a(0)\rangle = \frac{1}{4\pi^2} f_\pi^2 B_0^2 \frac{m_\pi}{x} K_1 + \dots \tag{12}$$

where $K_n \equiv K_n(m_\pi|x|)$ are Bessel functions and the dots stand for contact terms and higher order contributions. We are now free to check Weingarten's inequality choosing at our best convenience any particular direction of x_μ or, if prefered, summing over $\mu = 1,..,4$. In all cases, we obtain

$$\alpha(x)e^{-2m_\pi|x|} \leq \beta(x)e^{-m_\pi|x|}, \tag{13}$$

where $\alpha(x)$ and $\beta(x)$ are polynomials in x. The inequality holds at $x \to \infty$ due to the correct description of the pion content of each current. The exponential decay associated to the two-pion threshold is bounded by the one-pion exchange observed in the pseudoscalar channel. It is also arguable that the r.h.s. of the inequality is order f_π^2 whereas the l.h.s. is order 1, thus subleading.

The massless limit of the above analysis yields

$$\frac{1}{4\pi^4}\frac{1}{x^6} \leq \frac{1}{4\pi^2} f_\pi^2 B_0^2 \frac{1}{x^2} \tag{14}$$

which is correctly obeyed as $x \to \infty$ due to the faster decay of the product of two propagators as compared to one. Nevertheless, neither the massive nor the massless inequalities are valid for any x. Both are violated at a distance of the order of the inverse of the pion mass, which is a hint at the need of higher order corrections when x is decreased.

A word is needed about subtractions. The process of renormalization can be synthesized saying that bare amplitudes are transformed into distributions by correcting just its singular points. In our case, the pseudoscalar channel produces right away a *bona fide* distribution whereas the vector one does not. The product of two propagators is not a distribution, due to the $x = 0$ singularity. Any regularization takes care of this problem by adding the subtraction of a contact term. Subtractions are thus of the form, *e.g.* in dimensional regularization,

$$\frac{1}{\epsilon}\Box\delta^4(x) \quad , \frac{m_\pi^2}{\epsilon}\delta^4(x) \ , \ldots \tag{15}$$

Therefore, our discussion on the long-distance exponential decay of two-point correlators is clean and free of subtraction ambiguities. We are discussing the physics of the non-local part of the amplitude which remains unchanged along the renormalization process at this order.

For the sake of completeness, let us note that the analog of a standard momentum space renormalized amplitude[§]

$$\lambda + \lambda^2 \log\frac{p^2}{\nu^2} + \ldots \tag{16}$$

takes the following shape in coordinate space [6]

$$\lambda\delta^4(x) + \lambda^2\Box\frac{\log x^2\nu^2}{x^2} + \ldots \tag{17}$$

Changes of renormalization scheme, $\nu \to \nu'$, are absorbed by redefinition of the coupling constant. A massless amplitude issuing from a non-renormalizable perturbation theory will read an expression of the kind

$$\delta^4(x) + \frac{1}{f_\pi^2}\Box\delta^4(x) + \Box\frac{\log x^2\nu^2}{x^2} + \frac{1}{f_\pi^2}\Box\Box\frac{\log x^2\nu^2}{x^2} + \ldots \tag{18}$$

Only away from contact, boxes are allowed to act on the logs and the expansion parameter takes the form of $\frac{1}{x^2 f_\pi^2}$. In chiral perturbation theory, contact terms associated with L's do appear. Again, those are cleanly decoupled of our discussion above.

We now return to our exploitation of Weingarten's inequalities. As seen in our first example, the long-distance two-pion decay gives the clue to understand the fulfillment of the $VV < PP$

[§]Note that $\log p^2$ is to be understood as $p^2 \frac{\log p^2}{p^2}$ in the sense of distributions, the first p^2 acting by parts when necessary.

inequality. It is clear that a more constraining result can only emerge from an inequality involving two channels which are mediated by only one pion. This is the case of

$$\left|\langle A_\mu^a(x) A_\nu^a(x)\rangle\right| \leq \langle P^a(x) P^a(0)\rangle \ , \tag{19}$$

(a narrow resonance approximation would tell us that the A_1 meson is heavier than the pion). In chiral perturbation theory we obtain ($A_\mu^a(x) = f_\pi \partial_\mu \pi^a(x)$)

$$\left| f_\pi^2 \partial_\mu \partial_\nu \left(\frac{m_\pi}{x} K_1(m_\pi x) \right) \right| \leq f_\pi^2 B_0^2 \frac{m_\pi}{x} K_1(m_\pi x) \ . \tag{20}$$

Taking $\mu = \nu = 1$ and $x_2 = x_3 = x_4 = 0$, $x_1 = y$, we observe that both channels decay at the same exponential and leading power rate, differing only in the way dimensions are given. The constraint we obtain is, thus, an inequality between

$$m_\pi^2 \leq B_0^2 \ . \tag{21}$$

Before entering the phylosophical discussion of the result, let us note that the inequality (20) is again violated when y approaches $1/m_\pi$, a consistent sign of the need for higher order corrections at shorter distances. The massless limit is also verified since the axial channel is then reduced to a contact term. At variance with the $VV < PP$ case, both sides of the inequality are of the same order in f_π.

How should Eq. (21), and alike, be interpreted?

If the chiral expansion reproduces, order by order, a good approximation to the QCD correlators at large distances, the inequalities become constraints among the parameters in the effective theory, which, in principle, are calculable in QCD (*e.g.* on the lattice). Observables such as m_π^2 are expressed as functions of these parameters. In general, the inequalities we found are not among observables ¶.

In Eq. (21), m_π^2 is not a parameter of the chiral lagrangian. If one asumes that the order parameter B_0 gives the main contribution to $m_\pi^2 = 2m_q B_0 + O(m_q^2)$, it sets $m_q = O(m_\pi^2)$, hence second order in chiral power counting. Therefore, Eq. (21) reads

$$2m_q \leq B_0 \ . \tag{22}$$

This is a constraint of QCD on the relative strength of the explicit breaking of chiral symmetry, driven by the quark mass m_q, versus the spontaneous breaking. It is no longer an assumption as it is imposed by the vector-like structure of the theory.

Other scenarios are possible in which a different chiral-counting for m_q is required. In the framework of Generalized Chiral Perturbation Theory[7][8] one could have assumed that the main contribution to m_π^2 comes from terms up to quadratic order in m_q. In the limit of isospin symmetry

$$m_\pi^2 = 2m_q B_0 + 4m_q^2 (A_0 + 2Z_0^s) + O(m_q^3) \tag{23}$$

(see refs. [7][8] for definitions of A_0 and Z_0^s) and, here, m_q does not count as m_π^2. If B_0 were small enough, then $m_q = O(m_\pi)$, what amounts to a re-shuffling of chiral orders in the standard expansion in the explicit breaking sector. The axial-axial correlator is also modified and Eq. (22) reads

$$\left| \frac{B_0}{2m_q} + A_0 + 2Z_0^s \right| \geq 1 \tag{24}$$

¶We thank A. Manohar for an observation which triggered this discussion.

The constraint is nontrivial. If B_0 is small, it still relates the allowed values of A_0 and Z_0^s in a way independent of m_q.

We wish to emphasize that the relative importance of the order parameters B_0, A_0, Z_0^s and possibly others has a unique answer in QCD, and it should be properly incorporated in order to have a meaningful expansion. It is QCD what sets the chiral power counting of m_q, which, if not properly taken into acount, may result in a violation of the inequality, order by order in the expansion. A re-sumation to all orders is required to verify the inequality again.

Similarly, if m_q is very big, *e.g.* as the charm quark mass, a chiral symmetric theory incorporating the charmed flavor is not expected to be a good approximation. The violation of Eq. (22) is apparent. Consider, however, a situation where m_q is reasonably small so that a chiral expansion is expected to work. If B_0 turns out to be even smaller, a violation of Eq. (22) is a sign from QCD that the usual expansion is not the appropriate one. Yet, on physical grounds, a generalized expansion is expected to exist. Then Eq. (24) may not be violated and should be regarded as a rigorous constraint.

We consider now the renormalization of our inequality to one loop. At this point, we have to understand that the mass of the pion we have been using so far is bare, $m_\pi = m_0$, and that the inequality we were discussing is $m_0^2 \leq B_0^2$. The one-loop renormalization turns out to be particularly simple as all one-loop graphs come from three different sources: i) tadpoles associated to the composite operator structure of the currents, ii) tadpoles coming through the expansion of the $\mathcal{L}_2$ term in the chiral lagrangian to next order and iii) insertions of L's, coming form $\mathcal{L}_4$. None of these renormalizations change the spatial behavior of the correlators but only its parameters. All contributions are finite due to the non-renormalization associated to the partial conservation of the axial current. When finite parts are gathered we get

$$m_\pi^2 \leq B_0^2 Z_m \tag{25}$$

where (for SU(2))

$$Z_m = \frac{m_\pi^2}{m_0^2} = 1 - \frac{8m_\pi^2}{f_\pi^2}\left(2L_4^r + L_5^r - 4L_6^r - 2L_8^r + \frac{m_\pi^2}{32\pi^2 f_\pi^2}\log\frac{m_\pi^2}{\mu^2}\right) \tag{26}$$

and, therefore, the bare inequality remains unaltered.

At two-loop order, we do encounter a change of the behavior of the correlators leading to the first appearance of three-pion thresholds. All other diagrams are combinations of tadpoles that again will only renormalize the parameters of the inequality.

Nothing prevents us to play at will with non-diagonal correlators in Weingarten's setting. We have done so for $\langle AP\rangle \leq \langle PP\rangle$ and obtained identical results to the above ones. Full consistency of Ward Identities demanded so. We have also performed a number of checks involving three-point amplitudes. Using a combination of Cauchy-Schwarz and Hölder inequalities we have proven *e.g.*

$$\left|\langle V^a_{\mu,x} P^b_y P^c_z\rangle\right| \;\leq\; \sqrt{2}\,\left|\varepsilon^{abc}\right|\;\left(\langle P^a_z P^a_x\rangle\langle P^a_z P^a_y\rangle\langle P^a_y P^a_x\rangle\right)^{\frac{1}{2}}\;, \tag{27}$$

where we have used a shorter but obvious notation. This inequality demands that the amplitude must decay exponentially at least as half of the mass of the pion in each pair of points. Although non-trivially, this is automatically fulfilled in chiral perturbation theory as we have checked. Colinear configuration of the three space points do saturate the inequality between exponential decays but the power laws manage to keep the result safe.

3 Weingarten's inequalitites in momentum space

The inequalities we have found among chiral lagrangian parameters are independent on the technique used to derived them. We here digressed momentarily to reobtain our results in momentum space so as to make an easiest contact with the way results are often presented in the literature.

The proof of Weingarten's inequalities, as sketched in the introduction, does not go through in momentum space for a good reason: unlike in coordinate space, the two fermion propagators S_k and S_{q+k} differ in their arguments by a momentum insertion q, and are thus unrelated. Naive manipulation of Cauchy-Schwarz inequality will not reconstruct physical traces.

There exists, however, an inequality in momentum space at any value of euclidean Q^2 for each inequality in coordinate space. Let us start from the observation [10] that for a given operator B

$$\int d^3x \langle 0|B(-\mathrm{i}\tau,\vec{x})B(0,\vec{0})|0\rangle = \int_0^\infty dE e^{-E\tau}\rho(E^2) \ , \tag{28}$$

which holds for any value of euclidean time $\tau > 0$ and where ρ is the spectral function defined (in minkowski space) as

$$\rho(q^2) = \sum_\Gamma (2\pi)^3\delta^4(q-p_\Gamma)\langle 0|B(0,\vec{0})|\Gamma\rangle\langle\Gamma|B(0,\vec{0})|0\rangle \ , \tag{29}$$

where the sum is extended to all possible intermediate states with suitable quantum numbers.

Since the integral in $\vec{x}$ has positive measure, Weingarten's inequalities in euclidean coordinate space translate into inequalities between transforms of spectral functions if the spatial integral exists. The generic Weingarten's inequality for B bilinear in quark fields $|\langle B(x)B(0)\rangle| \leq \langle P(x)P(0)\rangle$ becomes

$$|\Phi(\tau)| \leq \Phi_P(\tau) \ , \tag{30}$$

where $\Phi(\tau)$ is the laplace transform of the spectral function $\rho(E^2)$, as in Eq. (28).

The momentum (minkowski) space correlator

$$\Pi(q^2) = \mathrm{i}\int d^4x \ e^{\mathrm{i}qx}\langle 0|TB(x)B(0)|0\rangle \ , \tag{31}$$

verifies a dispersion relation which may need subtractions. For instance, a twice subtracted dispersion relation is of the form ($q^2 = -Q^2 < 0$)

$$\Pi(Q^2) = \Pi(0) + \Pi'(0)Q^2 + (Q^2)^2\int_0^\infty dE^2 \ \frac{\rho(E^2)}{(E^2)^2(E^2+Q^2)} \ . \tag{32}$$

Our aim is to establish inequalities among the functions $\Pi(Q^2)$ at euclidean momenta. The strategy is to convolute inequality (30) with positive functions $F(\tau) > 0$. It turns out that the functions $F_0 = 1+\cos Q\tau$, $F_q = 1-\cos Q\tau$, $F_2 = (\frac{Q\tau}{2})^2 - \sin^2(\frac{Q\tau}{2})$ and $F_3 = \frac{1}{3}(\frac{Q\tau}{2})^4 - (\frac{Q\tau}{2})^2 + \sin^2(\frac{Q\tau}{2})$ are positive and lead to

$$\begin{aligned}
\left|\Pi(Q^2)+\Pi(0)\right| &\leq \Pi_P(0)+\Pi_P(Q^2) \ , \\
\left|\Pi(0)-\Pi(Q^2)\right| &\leq \Pi_P(0)-\Pi_P(Q^2) \ , \\
\left|\Pi(Q^2)-\Pi(0)-Q^2\Pi'(0)\right| &\leq \Pi_P(Q^2)-\Pi_P(0)-Q^2\Pi'_P(0) \ , \\
\left|\tfrac{1}{2}(Q^2)^2\Pi''(0)+Q^2\Pi'(0)+\Pi(0)-\Pi(Q^2)\right| &\leq \tfrac{1}{2}(Q^2)^2\Pi''_P(0)+Q^2\Pi'_P(0)+\Pi_P(0)-\Pi_P(Q^2) \ ,
\end{aligned} \tag{33}$$

which are inequalities that apply when zero, one, two and three subtractions are required. The generalization to any number of subtractions is immediate. In the cases where $\langle B(x)B(0)\rangle$ is positive for all x, the inequalities hold without the need for absolute values in the l.h.s. of (33).

Notice that these inequalities involved subtracted correlators. Mathematically, this is due to the fact that the inverse laplace transform of $\frac{1}{Q^2+E^2}$ is $2\cos Q\tau$, which is not a positive function. Remarkably enough, there exist positive functions $(F_0, ..., F_n)$ that enable the extraction of the desired inequalities by taking proper care of the subtractions needed to end up with a convergent integral of $\rho(E^2)$.

Convolutions with even powers of τ also furnish inequalities among derivatives of $\Pi(Q^2)$ at $Q^2 = 0$. Again, care must be taken of the subtractions: the more subtractions needed, the higher the derivative of Π is to be considered.

Let us consider the example of the vector correlator. In minkowski space one has, in the limit of exact isospin symmetry,

$$\mathrm{i}\int d^4x\; e^{\mathrm{i}qx}\langle 0|TV^a_\mu(x)V^a_\nu|0\rangle = \left(q_\mu q_\nu - q^2 g_{\mu\nu}\right)\Pi_V(q^2)\ . \tag{34}$$

We proceed defining the standard spectral function

$$\left(q_\mu q_\nu - q^2 g_{\mu\nu}\right)\rho_V(q^2) = \sum_\Gamma (2\pi)^3\delta^4(q-p_\Gamma)\langle 0|V_\mu(0,\vec{0})|\Gamma\rangle\langle\Gamma|V_\nu(0,\vec{0})|0\rangle\ . \tag{35}$$

Similar dispersion relations to that in Eq. (32) relate Π_V to ρ_V. In QCD, Π_V requires one subtraction [9]. Working out all inequalities is now straightforward,

$$Q^2\left|{\Pi'_A}^{(1)}(Q^2) - {\Pi'_A}^{(1)}(0)\right| \leq \Pi_P(Q^2) - \Pi_P(0) - Q^2\Pi'_P(0)\ . \tag{36}$$

Moreover, the application of an even-power convolution is even simpler, yielding the result

$$|\Pi'_V(0)| \leq \frac{1}{2}\Pi''_P(0)\ . \tag{37}$$

Let us note that the proof of Eq. (33) also involves the analiticity properties of the two-point functions in momentum space. Notice that this procedure removes contact terms related to subtractions. This reminds us that in coordinate space the same was automatically done by just analyzing correlators at non-zero distances.

Using a similar result to Eq. (37) for the non-transverse part of the axial correlator, the inequality $m_\pi^2 \leq B_0^2$ emerges again.

Once chiral perturbation theory is set up, the inequalities we get for the parameters of the lagrangian cannot depend on which space the inequalities are treated. It is a technicality to pass from one space to the other, the physical content remaining the same. Coordinate space allows a far simpler analysis due to the trivial decoupling of subtractions at large distances.

4 Witten's inequalities in chiral perturbation theory

As an initial remark we note that the positivity of the operator E is formally related to the sign of the condensate, $\langle\bar{q}q\rangle = -\frac{1}{V}\int d\mu\ \mathrm{Tr}E$, where V is the volume of space. This observation is further related to the infrared limit of the spectral density of the quark propagator (see ref. [11]).

Let us now concentrate again on two-point correlators. We follow Witten's argument [5] and define the matrix $E_{x,0} \equiv S_{x,0} + \gamma_5 S_{x,0} \gamma_5$ which commutes with γ_5 and corresponds to the matrix element of the positive operator E in Eq. (9). Then,

$$\int d\mu \ \mathrm{Tr}\left(EAEA^\dagger\right) \geq 0 \ . \tag{38}$$

In particular, for $A = \gamma_\mu e^{ik\hat{x}}$ and $A = e^{ik\hat{x}}$ we get respectively ($S^a(x) = \bar{\psi}(x)\frac{\lambda^a}{2}\psi(x)$)

$$\langle V_\mu^a(Q)V_\mu^a(-Q)\rangle - \langle A_\mu^a(Q)A_\mu^a(-Q)\rangle \geq 0 \tag{39}$$

$$\langle P^a(Q)P^a(-Q)\rangle - \langle S^a(Q)S^a(-Q)\rangle \geq 0 \tag{40}$$

The first inequality was analyzed in ref. [5] whereas the second one (which could have been deduced from Weingarten's setting) has not been explored in the way we shall use it.

We consider now the evaluation of the above inequalities in chiral perturbation theory in the chiral limit and in leading $\frac{1}{N_c}$, when chiral logarithms are suppressed [1, 12]. One gets

$$f_\pi^2 + 4L_{10}Q^2 + O(Q^4) \geq 0 \ , \tag{41}$$

$$B_0^2\left(\frac{f_\pi^2}{Q^2} - 16L_8 + O(Q^2)\right) \geq 0 \ . \tag{42}$$

It is noteworthy that all contact terms which are related to external sources (H_1 and H_2) are cancelling in the VV-AA and PP-SS combinations. The magnituds of L_8 and L_{10} are such that the inequalities, computed to second order, are indeed obeyed for Q^2 up to m_ρ^2. To get more constraining information, we would need to obtain an inequality involving the derivatives of the momentum correlators. So far, we have not found such a property.

Playing around with small variations of Witten's inequalities, it is easy to prove

$$\langle P^a(Q)P^a(-Q)\rangle - \langle S^a(Q)S^a(-Q)\rangle \leq \langle P^a(0)P^a(0)\rangle - \langle S^a(0)S^a(0)\rangle \tag{43}$$

which is trivially verified due to the presence of the pion pole (also valid in the massive case). It is also simple to obtain the coordinate space inequality (where no sum over i is implied here)

$$\langle V_i^a(x)V_i^a(0)\rangle - \langle A_i^a(x)A_i^a(0)\rangle \leq \langle P^a(x)P^a(0)\rangle - \langle S^a(x)S^a(0)\rangle \ , \tag{44}$$

which magically transforms into Weingarten's inequality at long distances.

In general, the exploitation of Witten's inequalities is more subtle. Their information stems from unitarity constraints of the particular E operator. Physically they mingle oposite sign contributions from many resonances. It is reasonable to expect that they may lead to more constraining inequalities for the physical parameters of the chiral perturbation expansion, when corrections are considered.

Let us finish this section with a comment on Kaplan-Manohar symmetry. Both Eq. (22) and (42) do break the hidden symmetry of the order p^4 chiral lagrangian discussed in ref. [13]. This reflects that the inequalities stem from QCD and, thus, tell apart different values of, e.g., L_8.

5 Some extra results

The exploitation of QCD inequalities remains bounded to variations of Weingarten's and Witten's ideas. We here proposed a few new avenues for research.

Amplitudes involving path ordered Wilson lines are easily amenable to inequality analysis. By using a combination of Cauchy-Schwarz and Hölder inequalities we have found that

$$|\langle\bar{\psi}_x\Gamma\mathcal{U}_{x,0}\psi_0\rangle| \leq |\langle P^a(x)P^a(0)\rangle|^{\frac{1}{2}} \ , \tag{45}$$

where $\mathcal{U}_{x,0} = Pe^{\int_0^x dz^\mu A_\mu}$ and Γ stands for any combination of Dirac gamma matrices. At long distances hadronization will make the (gauge invariant) l.h.s. decay faster than any meson. Nevertheless, we find interesting that a sort of gauge invariant constituent mass associated to the quark line must be heavier than half of the pion mass.

A second example of new inequality can be obtained from arguing that the correlator of the trace of the fermionic stress tensor is bounded by the trace of the total stress tensor. This leads to

$$\langle\theta^{fermions}(x)\theta^{fermions}(0)\rangle \leq \langle\theta^{total}(x)\theta^{total}(0)\rangle \ . \tag{46}$$

If we use chiral perturbation theory we obtain

$$m_q B_0 \leq m_\pi^2 \ , \tag{47}$$

which is indeed obeyed at leading order.

Let us finish by stating that QCD inequalities constraint the values of the chiral perturbation theory parameters. All standard numerical values fall in the right place. More ingenuity is necessary to produce more severe constraints.

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