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Symplectic Quantization of the CP^1 model with the Chern-Simons Term

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ABSTRACT

The symplectic formalism is fully employed to study the gauge-invariant CP^1 model with the Chern-Simons term. We consistently accommodate the CP^1 constraint at the Lagrangian level according to this formalism.

Since Dirac [1] had introduced the consistent quantization method for constrained theories, there has been great progress on the subject about physical as well as mathematical properties of theories. Recently, Faddeev and Jackiw (FJ) [2] had suggested the first order Lagrangian method for the constrained Hamiltonian system. After their work, Barcelos-Neto and Wotzasek (BW) have proposed the symplectic formalism, which is really the improved version of the FJ's method for the case that the constraints are not completely eliminated, and applied this formalism to several models [3,4].

On the other hand, the CP^1 model with the Chern-Simons term [5,6], which becomes an archetype example of the field theory, was considered by Polyakov, and he found the Bose-Fermi statistics transmutation in the model [7]. Han [8] recently has analyzed this CP^1 model by using the Dirac formalism together with the first order Lagrangian method.

In this note, we analyze the CP^1 model with the Chern-Simons term by fully using the symplectic formalism [3], which is algebraically much simpler than that of Dirac method.

Our starting Lagrangian of the gauge-invariant CP^1 model with the Chern-Simons term [6,8] is given by

$$\mathcal{L} = \frac{\kappa}{2\pi}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho + (\partial_\mu + iA_\mu)z_a^*(\partial^\mu - iA^\mu)z_a; \quad a = 1,2 \tag{1}$$

with the CP^1 constraint

$$\Omega_1^{(0)} = |z_a|^2 - 1 = 0, \tag{2}$$

where our convention is $\epsilon^{012} = +1$. The Lagrangian (1) is invariant under the transformations $z(x) \to e^{-i\Lambda(x)}z(x)$, $A_\mu \to A_\mu - \partial_\mu\Lambda(x)$ up to a total divergence. In order to use the advantage of the symplectic formalism [2,3], which displays the whole set of symmetries in the symplectic two-form, let us first make the Lagrangian first-ordered introducing the auxiliary fields, which are their canonical momenta, for convenience, although it is not necessary,

$$\pi_a \equiv (\partial_0 + iA_0)z_a^*, \qquad \pi_a^* \equiv (\partial^0 - iA^0)z_a, \tag{3}$$

Then, the desired form of the first-ordered Lagrangian including the CP^1 constraint is given by

$$\mathcal{L}^{(0)} = \frac{\kappa}{2\pi}\epsilon^{ij}A_j\dot{A}_i + \pi_a\dot{z}_a + \pi_a^*\dot{z}_a^* + \Omega_1^{(0)}\dot{\alpha} - \mathcal{H}^{(0)}; \quad i,j = 1,2, \tag{4}$$

where the Hamiltonian is

$$\mathcal{H}^{(0)} = \pi_a\pi_a^* + iA_0(z_a\pi_a - z_a^*\pi_a^*) + (\vec{\nabla} - i\vec{A})z_a^* \cdot (\vec{\nabla} + i\vec{A})z_a - \frac{\kappa}{\pi}A_0\epsilon^{ij}\partial_i A_j, \tag{5}$$

and α a Lagrangian multiplier. Note that the canonical sector, which is the first four parts of Lagrangian (4), is understood up to a total time derivative from the usual symplectic conventions, and we have written the superscript to show the iteration properties of the procedure.

According to the symplectic formalism [2,3], we have the initial sets of symplectic variables and their conjugate momenta as follows

$$\begin{aligned}
(\xi^{(0)})^k &= (A^i, z_a, z_a^*, \pi_a, \pi_a^*, \alpha, A^0), \\
(a^{(0)})_k &= (\frac{\kappa}{2\pi}\epsilon^{ij}A^i, \pi_a, \pi_a^*, 0, 0, 0, 0, \Omega_1^{(0)}, 0).
\end{aligned} \tag{6}$$

From the definition of the symplectic two form matrix

$$f_{ij}(x,y) = \frac{\partial a_j(y)}{\partial \xi^i(x)} - \frac{\partial a_i(x)}{\partial \xi^j(y)}, \tag{7}$$

we have the following singular symplectic matrix

$$f_{ij}^{(0)}(x,y) = \begin{pmatrix} A^{(0)} & B^{(0)} & C^{(0)}(y) \\ -B^{(0)^T} & 0 & 0 \\ -C^{(0)^T}(x) & 0 & 0 \end{pmatrix} \delta^2(x-y), \tag{8}$$

where

$$A^{(0)} = \begin{pmatrix} 0 & -\frac{\kappa}{\pi} & 0 & 0 & 0 & 0 \\ \frac{\kappa}{\pi} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, B^{(0)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, C^{(0)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ z_1^*(y) & 0 \\ z_2^*(y) & 0 \\ z_1(y) & 0 \\ z_2(y) & 0 \end{pmatrix}. \tag{9}$$

The above singular matrix $f_{ij}^{(0)}(x,y)$ has a zero eigenvalue and its eigenfunction. This is called a zero mode, $(\tilde{\nu}^(0))^k = (0,0,0,0,0,0,z_a^*v_{11}, z_a v_{11}, v_{11}, v_{12})$, where $v_{11}(x)$ and

$v_{12}(x)$ are independent and arbitrary functions. Furthermore, this zero mode $(\tilde{\nu}^{(0)})^k$ generates two constraints $\Omega_2^{(1)}$ and $\Omega_3^{(1)}$ such that

$$
\begin{aligned}
0 &= \int dx\ (\tilde{\nu}^{(0)})^k(x)\frac{\partial}{\partial(\xi^{(0)})^k}(x)\int dy\ \mathcal{H}^{(0)}(\xi(y)) \\
&= \int dx\ \{\ v_{11}(z_a^*\pi_a^* + z_a\pi_a) - v_{12}[i(z_a^*\pi_a^* - z_a\pi_a) + \frac{\kappa}{\pi}\epsilon^{ij}\partial_i A_j]\} \\
&\equiv \int dx\ [v_{11}\Omega_3^{(1)} - v_{12}\Omega_2^{(1)}]. \qquad (10)
\end{aligned}
$$

This is because of the fact that the Hamilton's equation has written by the form of the coupled velocities of their symplectic variables, *i.e.*, $f_{ij}^{(0)}\dot{\xi}^{j(0)} = \frac{\partial H^{(0)}}{\partial \xi^i}$. Then, the constraints $\Omega_2^{(1)}$ and $\Omega_3^{(1)}$ are incorporated into the Lagrangian to make the first-iterated Lagrangian [3] as follows

$$
\mathcal{L}^{(1)} = \left(\frac{\kappa}{2\pi}\epsilon^{ij}A_j\right)\dot{A}_i + \pi_a\dot{z}_a + \pi_a^*\dot{z}_a^* + \Omega_1^{(0)}\dot{\alpha} + \Omega_2^{(1)}\dot{\beta} + \Omega_3^{(1)}\dot{\gamma} - \mathcal{H}^{(1)}, \qquad (11)
$$

where the first-iterated Hamiltonian is given by

$$
\begin{aligned}
\mathcal{H}^{(1)}(\xi) &= \mathcal{H}^{(0)}(\xi)\ |_{\Omega_2^{(1)},\Omega_3^{(1)}=0} \\
&= \pi_a\pi_a^* + (\vec{\nabla} - i\vec{A})z_a^* \cdot (\vec{\nabla} + i\vec{A})z_a, \qquad (12)
\end{aligned}
$$

and β, γ are Lagrange multipliers. Through this process, we have reduced the original Hamiltonian using the constraints $\Omega_2^{(1)}$ and $\Omega_3^{(1)}$.

Once again, let us set $(\xi^{(1)})^k$ and $(a^{(1)})_k$ for the first-iterated symplectic variables and their conjugate momenta

$$
\begin{aligned}
(\xi^{(1)})^k &= (A^i, z_a, z_a^*, \pi_a, \pi_a^*, \alpha, \beta, \gamma), \\
(a^{(1)})_k &= (\frac{\kappa}{2\pi}\epsilon^{ij}A^j, \pi_a, \pi_a^*, 0, 0, 0, 0, \Omega_1^{(0)}, \Omega_2^{(1)}, \Omega_3^{(1)}), \qquad (13)
\end{aligned}
$$

respectively. Then the symplectic two form matrix is written as

$$
\begin{aligned}
f_{ij}^{(1)}(x,y) &= \begin{pmatrix} A^{(0)} & B^{(0)} & C^{(1)}(y) \\ -B^{(0)^T} & 0 & D^{(1)}(y) \\ -C^{(1)^T}(x) & -D^{(1)^T} & 0 \end{pmatrix}\delta^2(x-y) \\
&\equiv F_{ij}^{(1)}(x,y)\delta^2(x-y). \qquad (14)
\end{aligned}
$$

where

$$C^{(1)}(y)=\begin{pmatrix} 0 & +\frac{\kappa}{\pi}\partial_2^y & 0 \\ 0 & -\frac{\kappa}{\pi}\partial_1^y & 0 \\ z_1^*(y) & -i\pi_1(y) & \pi_1(y) \\ z_2^*(y) & -i\pi_2(y) & \pi_2(y) \\ z_1(y) & i\pi_1^*(y) & \pi_1^*(y) \\ z_2(y) & i\pi_2^*(y) & \pi_2^*(y) \end{pmatrix}, D^{(1)}(y)=\begin{pmatrix} 0 & -iz_1(y) & z_1(y) \\ 0 & -iz_2(y) & z_2(y) \\ 0 & iz_1^*(y) & z_1^*(y) \\ 0 & iz_2^*(y) & z_2^*(y) \end{pmatrix}.$$

This matrix is still singular, and there is also a zero mode $(\tilde{\nu}^{(1)})^k=(-\partial_i v_{12}, iz_a v_{12}, -iz_a^* v_{12}, -i\pi_a v_{12}, i\pi_a^* v_{12}, 0, v_{12}, 0)$ at this stage of iteration. However, this zero mode does not generate any additional constraint, because it leads to the following trivial identity

$$\int dx(\tilde{\nu}^{(1)})^k(\xi)\frac{\partial}{\partial(\xi^{(1)})^k(x)}\int dy\mathcal{H}^{(1)}(\xi)=0. \tag{15}$$

This is exactly the result of having a gauge symmetry in the symplectic formalism. In fact, we can easily check that the first-iterated Lagrangian (11) is invariant up to the total divergence under the following transformation

$$\delta(\xi^{(1)})^k=(\tilde{\nu}^{(1)})^k\eta, \tag{16}$$

where η is only a function of time, or equivalently

$$\begin{aligned} \delta A^i &= -\eta\partial_i v_{12}, \quad \delta z_a = i\eta v_{12} z_a, \\ \delta z_a^* &= -i\eta v_{12} z_a^*, \quad \delta\pi_a = -i\eta v_{12}\pi_a, \\ \delta\pi_a^* &= i\eta v_{12}\pi_a^*, \quad \delta\alpha = 0, \\ \delta\beta &= \eta v_{12}, \quad \delta\gamma = 0. \end{aligned} \tag{17}$$

Now, in order to obtain the desired Dirac brackets, we impose the well-known Coulomb gauge condition, $\nabla\cdot\vec{A}=0$, at the Lagrangian level by using the consistent gauge fixing procedure in the symplectic formalism [3]. With this constraint, we can directly obtain the gauge fixed first-order Lagrangian from the first-iterated Lagrangian (10) as follows

$$\mathcal{L}_{GF}^{(2)} = \frac{\kappa}{2\pi}\epsilon^{ij}A_j\dot{A}_i+\pi_a\dot{z}_a+\pi_a^*\dot{z^*}_a+\Omega_1^{(0)}\dot{\alpha}+\Omega_2^{(1)}\dot{\beta}+\Omega_3^{(1)}\dot{\gamma}+\Omega_4^{(2)}\dot{\sigma}-\mathcal{H}^{(2)}, \tag{18}$$

where σ is a Lagrange multiplier, and $H^{(2)}$ is naturally the second-iterated Hamiltonian

$$\begin{aligned}\mathcal{H}^{(2)}(\xi) &= \mathcal{H}^{(1)}(\xi)\mid_{\Omega^{(2)}=0} \\ &= \pi_a\pi_a^* + (\vec{\nabla} - i\vec{A})z_a^* \cdot (\vec{\nabla} + i\vec{A})z_a. \end{aligned} \tag{19}$$

Note that this Hamiltonian, which is simply obtained, is exactly the well-known reduced physical Hamiltonian of the original CP^1 model with the Chern-Simons term, which may be obtained through the several steps with three definitions of the canonical, total, and reduced Hamiltonian in the usual Dirac formalism of the constrained systems [1]. Furthermore, this model has only four constraints in the symplectic formalism, while eight constraints are contained in the Dirac method. Therefore, the symplectic formalism is algebraically much simpler than that of Dirac's.

Now the symplectic procedure is straight-forwardly treated as just the second-iterated stage with the following symplectic variables and their conjugate momenta

$$\begin{aligned}(\xi^{(2)})^k &= (A^i, z_a, z_a^*, \pi_a, \pi_a^*, \alpha, \beta, \gamma, \sigma) \\ (a^{(2)})_k &= (\frac{\kappa}{2\pi}\epsilon^{ij}A^j, \pi_a, \pi_a^*, 0, 0, 0, 0, \Omega_1^{(0)}, \Omega_2^{(1)}, \Omega_3^{(1)}, \Omega_4^{(2)}). \end{aligned} \tag{20}$$

Following to the definition of the symplectic matrix, we then find the second-iterated non-singular symplectic two form matrix

$$f_{ij}^{(2)}(x,y) = \begin{pmatrix} F^{(1)}(x,y) & F^{(2)}(y) \\ -F^{(2)^T}(x) & 0 \end{pmatrix} \delta^2(x-y). \tag{21}$$

where

$$F^{(2)^T}(x) = (\, -\partial_i^x \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\,).$$

Then we have the inverse as follows

$$[f_{ij}^{(2)}]^{-1}(x,y) = \begin{pmatrix} G(y) & I(y) \\ -I^T(x) & J(y) \end{pmatrix} \delta^2(x-y). \tag{22}$$

where

$$G = \begin{pmatrix} 0 & -\frac{i\pi}{\kappa}z_b\epsilon^{ij}\frac{\partial_j}{\nabla^2} & \frac{i\pi}{\kappa}z_b^*\epsilon^{ij}\frac{\partial_j}{\nabla^2} & \frac{i\pi}{\kappa}\pi_b\epsilon^{ij}\frac{\partial_j}{\nabla^2} & -\frac{i\pi}{\kappa}\pi_b^*\epsilon^{ij}\frac{\partial_j}{\nabla^2} \\ +\frac{i\pi}{\kappa}z_a\epsilon^{ij}\frac{\partial_i}{\nabla^2} & 0 & 0 & \delta_{ab} - \frac{1}{2}z_a z_b^* & -\frac{1}{2}z_a z_b \\ -\frac{i\pi}{\kappa}z_a^*\epsilon^{ij}\frac{\partial_i}{\nabla^2} & 0 & 0 & -\frac{1}{2}z_a^* z_b^* & \delta_{ab} - \frac{1}{2}z_a^* z_b \\ -\frac{i\pi}{\kappa}\pi_a\epsilon^{ij}\frac{\partial_i}{\nabla^2} & -\delta_{ab} + \frac{1}{2}z_a^* z_b & -\frac{1}{2}z_a^* z_b^* & -\frac{1}{2}(z_a^*\pi_b - z_b^*\pi_a) & -\frac{1}{2}(z_a^*\pi_b^* - z_b\pi_a) \\ +\frac{i\pi}{\kappa}\pi_a^*\epsilon^{ij}\frac{\partial_i}{\nabla^2} & -\frac{1}{2}z_a z_b & -\delta_{ab} + \frac{1}{2}z_a z_b^* & \frac{1}{2}(z_b^*\pi_a^* - z_a\pi_b) & -\frac{1}{2}(z_a\pi_b^* - z_b\pi_a^*) \end{pmatrix},$$

$$I = \begin{pmatrix} 0 & -\frac{\pi}{\kappa}\epsilon^{ij}\frac{\partial_j}{\nabla^2} & 0 & -\frac{\partial_i}{\nabla^2} \\ -\frac{1}{2}z_a & 0 & 0 & iz_a\frac{1}{\nabla^2} \\ -\frac{1}{2}z_a^* & 0 & 0 & -iz_a\frac{1}{\nabla^2} \\ -\frac{1}{2}\pi_a & 0 & -\frac{1}{2}z_a^* & -i\pi_a\frac{1}{\nabla^2} \\ -\frac{1}{2}\pi_a^* & 0 & -\frac{1}{2}z_a & -i\pi_a^*\frac{1}{\nabla^2} \end{pmatrix}, J = \begin{pmatrix} 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{\nabla^2} \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & -\frac{1}{\nabla^2} & 0 & 0 \end{pmatrix}.$$

As results, since $\{\xi^{(2)i}(x), \xi^{(2)j)}(y)\} = (f^{(2)})^{-1}_{ij}(x,y)$ according to the FJ method [2,3], we directly read off the well-known results [6,8] of the nonvanishing Dirac brackets from this inverse matrix as follows

$$\begin{aligned}
\left\{A^i(x),\ z_a(y)\right\}_D &= -\frac{i\pi}{\kappa}z_a\epsilon^{ij}\frac{\partial_j^x}{\nabla^2}\delta^2(x-y), \\
\left\{A^i(x),\ z_a^*(y)\right\}_D &= \frac{i\pi}{\kappa}z_a^*\epsilon^{ij}\frac{\partial_j^x}{\nabla^2}\delta^2(x-y), \\
\left\{A^i(x),\ \pi_a(y)\right\}_D &= -\frac{i\pi}{\kappa}\pi_a\epsilon^{ij}\frac{\partial_j^x}{\nabla^2}\delta^2(x-y), \\
\left\{A^i(x),\ \pi_a^*(y)\right\}_D &= \frac{i\pi}{\kappa}\pi_a^*\epsilon^{ij}\frac{\partial_j^x}{\nabla^2}\delta^2(x-y), \\
\{z_a(x),\ \pi_b(y)\}_D &= (\delta_{ab} - \frac{1}{2}z_a z_b^*)\delta^2(x-y), \\
\{z_a(x),\ \pi_b^*(y)\}_D &= -\frac{1}{2}z_a z_b\delta^2(x-y), \\
\{z_a^*(x),\ \pi_b(y)\}_D &= -\frac{1}{2}z_a^* z_b^*\delta^2(x-y), \\
\{z_a^*(x),\ \pi_b^*(y)\}_D &= (\delta_{ab} - \frac{1}{2}z_a^* z_b)\delta^2(x-y), \\
\{\pi_a(x),\ \pi_b(y)\}_D &= -\frac{1}{2}\left(z_a^*\pi_b - z_b^*\pi_a\right)\delta^2(x-y), \\
\{\pi_a(x),\ \pi_b^*(y)\}_D &= -\frac{1}{2}\left(z_a^*\pi_b^* - z_b\pi_a\right)\delta^2(x-y), \\
\{\pi_a^*(x),\ \pi_b^*(y)\}_D &= -\frac{1}{2}\left(z_a\pi_b^* - z_b\pi_a^*\right)\delta^2(x-y).
\end{aligned} \tag{23}$$

It seems appropriate to comment on the quantization of the system. For the simple case that the operator ordering problem does not exist, we can directly replace the Dirac brackets with the quantum commutator : $\{\ ,\ \}_D \to -i[\ ,\ ]$. But we should carefully treat the CP^1 case having the ordering problem [6]. For this CP^1 model, Han has already found the correct Dirac Brackets [8]. On the other hand, there exist the other effective formalism [9,10] avoiding the ordering problem, which is a kind of the BFV-BRST method [11]. Recently this formalism has been successively applied to the CP^{N-1} model [12].

In conclusions we consistently accommodate the CP^1 constraint at the Lagrangian level according to this formalism. As a result, we have explicitly considered the gauge invariant CP^1 model with the Chern-Simons term by fully using the symplectic formalism comparing with the Dirac method.

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