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Classical Duality in Gauge Theories

by

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Abstract

A dual action is obtained for a general non-abelian and non-supersymmetric gauge theory at the classical level. The construction of the dual theory follows steps similar to those used in pure abelian gauge theory. As an example we study the spontaneously broken $SO(3)$ gauge theory and show that the electric and the magnetic fields get interchanged in the dual theory.

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The understanding of electric-magnetic duality is certainly one one of the challenges of four-dimensional field theories. This duality was conjuctured by Montonen and Olive [1] and shown by Osborn [2] to be plausible for $N = 4$ supersymmetric gauge theories. In fact in the $N = 4$ supersymmetric Yang-Mills case the electric-magnetic duality was firmly tested by Vafa and Witten [3].

Recently, dramatic new evidence for the validity of this conjecture has emerged from the work of A. Sen [4] and a version of Montonen-Olive duality was surprisingly found by Seiberg and Witten in $N = 2$ supersymmetric gauge theory in four dimensions [5]. This duality has many similar features with the target space duality encountred in string theories (see ref.[6] for a review).

In this paper we examine, at the classical level, the procedure of constructing a dual theory of a non-abelian and non-supersymmetric gauge theory.

The Dual Variables

Our starting point is the action for a general gauge theory in four dimensions. This takes the form

$$S = \int \mathrm{d}^4x\sqrt{\gamma}\left[\gamma^{\mu\rho}\gamma^{\nu\sigma}g_{ab}\left(\alpha F^a_{\mu\nu}F^b_{\rho\sigma} + \beta F^a_{\mu\nu}\widetilde{F}^b_{\rho\sigma}\right) + J^\mu_a(\phi)A^a_\mu + K^{\mu\nu}_{ab}(\phi)A^a_\mu A^b_\nu + S_0(\phi)\right] \quad . \quad (1)$$

Here α and β are two coupling constants and ϕ is any generic matter field on which J^μ_a, $K^{\mu\nu}_{ab}$ and S_0 depend. The field strength is $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \lambda f^a_{bc}A^b_\mu A^c_\nu$, where λ is related to α, and its dual is $\widetilde{F}^a_{\mu\nu} = \frac{1}{2}\epsilon^{\alpha\beta}{}_{\mu\nu}F^a_{\alpha\beta}$.

The action is invariant under the gauge transformations

$$\delta A^a_\mu = -\partial_\mu\epsilon^a + \lambda f^a_{bc}A^c_\mu\epsilon^b \quad (2)$$

provided that

$$\begin{aligned}
\delta S_0 &= J^\mu_a\partial_\mu\epsilon^a \\
\delta J^\mu_a &= 2K^{\mu\nu}_{ab}\partial_\nu\epsilon^b + \lambda f^c_{ab}J^\mu_c\epsilon^b \\
\delta K^{\mu\nu}_{ab} &= +\lambda\left(f^d_{ac}K^{\mu\nu}_{db} + f^d_{bc}K^{\mu\nu}_{ad}\right)\epsilon^c \quad .
\end{aligned} \quad (3)$$

Now in terms of the self-dual and anti-self-dual projectors $F^{\pm a}_{\mu\nu} = \frac{1}{2}\left(F^a_{\mu\nu} \pm \widetilde{F}^a_{\mu\nu}\right)$, the action takes the form

$$S = \int \mathrm{d}^4\mathrm{x}\sqrt{\gamma}\left(\tau_+\left(\mathrm{F}^+\right)^2 + \tau_-\left(\mathrm{F}^-\right)^2 + \mathrm{J}^\mu_\mathrm{a}\mathrm{A}^\mathrm{a}_\mu + \mathrm{K}^{\mu\nu}_{\mathrm{ab}}\mathrm{A}^\mathrm{a}_\mu\mathrm{A}^\mathrm{b}_\nu + \mathrm{S}_0\right) \quad , \quad (4)$$

where $\tau_\pm = \alpha \pm \beta$.

Our aim is to find an action which is classically equivalent to the above action. This we find by introducing some new degrees of freedom $G^{\pm\, a}_{\mu\nu}$ and the action we are after is given by

$$I = \int \mathrm{d^4x}\sqrt{\gamma}\left(\mathrm{a}\left(\mathrm{G}^+\right)^2 + \mathrm{b}\left(\mathrm{G}^-\right)^2 + \mathrm{cF^+G^+} + \mathrm{dF^-G^-} + \mathrm{J^\mu_a A^a_\mu} + \mathrm{K^{\mu\nu}_{ab}A^a_\mu A^b_\nu} + \mathrm{S_0}\right) \quad . \qquad (5)$$

Here a, b, c and d are some parameters that we are going to determine. In Maxwell's theory and in the absence of matter fields one arrives at this action by enlarging the gauge symmetry [7].

This action is invariant under two gauge transformations. The first is given by

$$\begin{aligned} \delta G^{+\, d}_{\mu\nu} &= \lambda f^d_{bc} G^{+\, c}_{\mu\nu} \epsilon^b \\ \delta G^{-\, a}_{\mu\nu} &= \lambda f^a_{ec} G^{-\, c}_{\mu\nu} \epsilon^e \end{aligned} \qquad (6)$$

and the second gauge transformation is

$$\begin{aligned} \delta G^{+\, d}_{\mu\nu} &= \frac{\lambda c}{2a} f^d_{cb} F^{+\, c}_{\mu\nu} \epsilon^b \\ \delta G^{-\, a}_{\mu\nu} &= \frac{\lambda d}{2b} f^a_{ce} F^{-\, c}_{\mu\nu} \epsilon^e \end{aligned} \qquad (7)$$

To see that this new action is classically equivalent to the original gauge theory, we eliminate the independent fields $G^\pm$ by their equations of motion

$$G^+ = -\frac{c}{2a} F^+ \quad , \quad G^- = -\frac{d}{2b} F^- \quad . \qquad (8)$$

Upon replacing these values for $G^\pm$ in the action I, we get back our original action S provided that

$$\tau_+ = -\frac{c^2}{4a} \quad , \quad \tau_- = -\frac{d^2}{4b} \quad . \qquad (9)$$

Notice that the two gauge transformations for $G^\pm$ are consistent with the equations of motion for $G^\pm$.

The dependence of the action I on the gauge field A^a_μ is at most quadratic and therefore A^a_μ can be classically integrated out. The equation of motion for A^a_μ is given by

$$A^a_\mu = -\widetilde{R}^{ab}_{\mu\nu} V^\nu_b \quad , \qquad (10)$$

where $\widetilde{R}^{ab}_{\mu\nu}$ and V^μ_a are defined below

$$\begin{aligned} R^{\mu\nu}_{ab} &= 2K^{\mu\nu}_{ab} + 2\lambda \gamma^{\mu\alpha}\gamma^{\nu\beta} g_{rs} f^s_{ab}\left(cG^{+\, r}_{\alpha\beta} + dG^{-\, r}_{\alpha\beta}\right) \\ V^\mu_a &= J^\mu_a - 2\gamma^{\alpha\beta}\gamma^{\mu\nu} g_{ab}\left(c\partial_\alpha G^{+\, b}_{\beta\nu} + d\partial_\alpha G^{-\, b}_{\beta\nu}\right) \end{aligned} \qquad (11)$$

and $\widetilde{R}^{ab}_{\mu\nu}$ is the inverse of $R^{\mu\nu}_{ab}$

$$R^{\mu\nu}_{ab}\widetilde{R}^{bc}_{\nu\alpha} = \delta^{\mu}_{\alpha}\delta^{a}_{c} \quad . \tag{12}$$

Replacing for A^a_μ in the action I we obtain the dual action

$$I_{\text{dual}} = \int \mathrm{d}^4x\sqrt{\gamma}\left[-\frac{c^2}{4\tau_+}\left(G^+\right)^2 - \frac{d^2}{4\tau_-}\left(G^-\right)^2 - \frac{1}{2}\widetilde{R}^{ab}_{\mu\nu}V^\mu_a V^\nu_b + S_0\right] \quad . \tag{13}$$

As expected, the couplings $\tau_\pm$ have been inverted. This action is a generalisation of a model put forward by Freedman and Townsend [8] and by Sugamoto [9] and which appeared recently in connection with treating duality using loop space variables [10].

One can verify that this dual action has inherited the two gauge invariances (6) and (7), up to a total derivative, where the fields $F^\pm$ are now built from the gauge field A^a_μ as given in (10). It is a straightforward calculation to show that under the transformation (6) we have

$$\begin{aligned}
\delta V^\mu_a &= \lambda f^c_{ab}V^\mu_c\epsilon^b + R^{\mu\nu}_{ab}\partial_\nu\epsilon^b \\
\delta R^{\mu\nu}_{ab} &= \lambda f^d_{ac}R^{\mu\nu}_{db}\epsilon^c + \lambda f^d_{bc}R^{\mu\nu}_{ad}\epsilon^c \quad .
\end{aligned} \tag{14}$$

This leads to the expetected gauge transformation for A^a_μ in (10); $\delta A^a_\mu = -\partial_\mu\epsilon^a + \lambda f^a_{bc}A^c_\mu\epsilon^b$.

However, under the gauge transformation (7) we have the following transformations for V^μ_a and $R^{\mu\nu}_{ab}$

$$\begin{aligned}
\delta V^\mu_a &= \lambda f^c_{ab}V^\mu_c\epsilon^b + R^{\mu\nu}_{ab}\partial_\nu\epsilon^b - 2c\lambda\gamma^{\mu\nu}\gamma^{\alpha\beta}g_{rc}f^r_{ab}\left(G^{+\,c}_{\nu\alpha} + \frac{c}{2a}F^{+\,c}_{\nu\alpha}\right)\partial_\beta\epsilon^b \\
&- 2d\lambda\gamma^{\mu\nu}\gamma^{\alpha\beta}g_{rc}f^r_{ab}\left(G^{-\,c}_{\nu\alpha} + \frac{d}{2b}F^{-\,c}_{\nu\alpha}\right)\partial_\beta\epsilon^b + 2c\lambda\gamma^{\mu\nu}\gamma^{\alpha\beta}g_{rc}f^c_{ab}\partial_\alpha\left(G^{+\,r}_{\beta\nu} + \frac{c}{2a}F^{+\,r}_{\beta\nu}\right)\epsilon^b \\
&+ 2d\lambda\gamma^{\mu\nu}\gamma^{\alpha\beta}g_{rc}f^c_{ab}\partial_\alpha\left(G^{-\,r}_{\beta\nu} + \frac{d}{2b}F^{-\,r}_{\beta\nu}\right)\epsilon^b \\
\delta R^{\mu\nu}_{ab} &= \lambda f^d_{ac}R^{\mu\nu}_{db}\epsilon^c + \lambda f^d_{bc}R^{\mu\nu}_{ad}\epsilon^c - 2c\lambda^2\gamma^{\mu\alpha}\gamma^{\nu\beta}g_{sc}f^s_{dr}f^r_{ab}\left(G^{+\,d}_{\alpha\beta} + \frac{c}{2a}F^{+\,d}_{\alpha\beta}\right)\epsilon^c \\
&- 2d\lambda^2\gamma^{\mu\alpha}\gamma^{\nu\beta}g_{sc}f^s_{dr}f^r_{ab}\left(G^{-\,d}_{\alpha\beta} + \frac{d}{2b}F^{-\,d}_{\alpha\beta}\right)\epsilon^c \quad .
\end{aligned} \tag{15}$$

We see that the two transformations coincide when the equations of motion for $G^\pm$ are satisfied. Notice also that varying I_{dual} with respect to $G^\pm$ leads to the equations (8) where A^a_μ is as given by (10).

It is worth mentioning that the two terms in I involving the parameters c and d can be written as

$$cF^+G^+ + dF^-G^- = \frac{1}{2}(c+d)\,FG + \frac{1}{2}(c-d)\,\widetilde{F}G \tag{16}$$

and in the pure abelian gauge theory one usually takes $c=-d=1$. This choice is crucial in this case because integrating out the gauge field from I leads to $\partial_\mu \widetilde{G}_{\mu\nu}=0$, and by virtue of Poincaré's lemma this in turn implies that $G_{\mu\nu}=\partial_\mu B_\nu-\partial_\nu B_\mu$, where B_μ is an abelian one-form [10].

The SO(3) Gauge Theory

Let us now consider an example and examine the consequences of this change of variables. The model we have in mind is the $SO(3)$ gauge theory with a Higgs triplet. The corresponding Lagrangian is given by

$$\mathcal{L}=-\frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu}+\frac{1}{2}\left(D_\mu\phi\right)^a\left(D_\mu\phi\right)^a+\frac{\mu^2}{2}\phi^a\phi^a-\frac{\lambda}{4}\left(\phi^a\phi^a\right)^2 \quad , \tag{17}$$

where $\mu^2>0$, $F^a_{\mu\nu}=\partial_\mu A^a_\nu-\partial_\nu A^a_\mu+e\epsilon^{abc}A^b_\mu A^c_\nu$ and $D_\mu\phi^a=\partial_\mu\phi^a+e\epsilon^{abc}A^b_\mu\phi^c$. The flat space-time metric has the signature $(+,+,+,-)$. A quick comparison of this action and S in (1) gives the different quantities used in S. In particular $\beta=0$ and $\alpha=-1/4$.

The $SO(3)$ gauge theory has a time-independent solution of the form [11, 12, 13, 14]

$$\begin{aligned} A^a_i &= \epsilon_{abi}x_b\left(\frac{K(r)-1}{er^2}\right) \\ A^a_0 &= x_a J(r)/er^2 \\ \phi^a &= x_a H(r)/er^2 \quad . \end{aligned} \tag{18}$$

This form solves the equations of motion if the functions K, J and H satisfy the differential equations [13]

$$\begin{aligned} r^2J'' &= 2JK^2 \\ r^2H'' &= 2HK^2-\mu^2r^2H\left(1-\frac{\lambda}{e^2\mu^2r^2}H^2\right) \\ r^2K'' &= K\left(K^2-J^2+H^2-1\right) \quad . \end{aligned} \tag{19}$$

After the symmetry breaking one needs to identify the physically observable fields, especially the photon $F_{\mu\nu}$. A gauge invariant definition for the electromagnetic field $F_{\mu\nu}$ is given by [11]

$$\mathcal{F}_{\mu\nu}=\frac{1}{\phi}\phi^aF^a_{\mu\nu}-\frac{1}{e\phi^3}\epsilon_{abc}\phi^aD_\mu\phi^bD_\nu\phi^c \quad , \tag{20}$$

where $\phi=\left(\phi^a\phi^a\right)^{1/2}$. This definition can be cast into the more appealing expression

$$\mathcal{F}_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu-\frac{1}{e\phi^3}\epsilon_{abc}\phi^a\partial_\mu\phi^b\partial_\nu\phi^c \quad , \tag{21}$$

where the gauge field A_μ is defined through

$$A_\mu = \frac{1}{\phi}\phi^a A^a_\mu \ . \tag{22}$$

This solution has both electric and magnetic fields. These are given by

$$\begin{aligned} \mathcal{E}_i &= \mathcal{F}_{i0} = \frac{x_i}{r}\frac{\mathrm{d}}{\mathrm{dr}}\left(J(r)/er\right) \\ \mathcal{B}_i &= \frac{1}{2}\epsilon_{ijk}\mathcal{F}_{jk} = -\frac{1}{e}\frac{x_i}{r^3} \ . \end{aligned} \tag{23}$$

One can also calculate the electric and the magnetic charges of this solution. In particular, the Dirac quantisation condition gives the magnetic charge $g = 1/e$.

After this brief summary of the solutions to the $SO(3)$ gauge theory let us express this solution in terms of the variables $G^a_{\mu\nu}$. We will take for this purpose $c = -d = 1$. This choice leads to $a = b = 1$ and to the following equation of motion for $G^a_{\mu\nu}$

$$G^a_{\mu\nu} = -\frac{1}{2}\widetilde{F}^a_{\mu\nu} \ . \tag{24}$$

In terms of the variables $G^a_{\mu\nu}$ an obvious gauge invariant quantity is given by

$$\mathcal{G}_{\mu\nu} = \frac{1}{\phi}\phi^a G^a_{\mu\nu} + \frac{1}{2}\frac{1}{e\phi^3}\epsilon_{abc}\epsilon_{\mu\nu\alpha\beta}\phi^a D_\alpha\phi^b D_\beta\phi^c \ , \tag{25}$$

where the covariant derivative is in terms of the gauge field as defined in (10). The $+\frac{1}{2}\epsilon_{\mu\nu\alpha\beta}$ tensor in the second term is needed to get $\mathcal{G}_{\mu\nu}$ in a familiar form when expressed in terms of the gauge filed A_μ as given in (22). Indeed upon using the equations of motion for $G^a_{\mu\nu}$ and the expression of A_μ we find that

$$\mathcal{G}_{\mu\nu} = -\frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\mathcal{F}_{\alpha\beta} \ . \tag{26}$$

Hence if we identify $\mathcal{G}_{\mu\nu}$ as the electromagnetic field in the dual theory, we see that what was previously called the electric field in the original theory has become the magnetic field in the dual theory and vice-versa.

In this paper we have found a way of constructing a classical dual action for a general non-abelian and non-supersymmetric gauge theory. The theory we obtained describes the dynamics of a rank-two antisymmetric tensor field and could be studied for its own right regardless of its origin.

Going from the action S in (1) to the action I in (5) can be made into a formal step in the path integral by noticing that

$$\begin{aligned} &\exp\left[\int \mathrm{d}^4x\left(\tau_+\left(F^+\right)^2 + \tau_-\left(F^-\right)^2\right)\right] \quad \propto \\ &\int \mathcal{D}G^+\mathcal{D}G^- \exp\left[\int \mathrm{d}^4x\left(a\left(G^+\right)^2 + b\left(G^-\right)^2 + cF^+G^+ + dF^-G^-\right)\right] \ . \end{aligned} \tag{27}$$

In this way one obtains a dual action at the quantum level and the properties of the partition function could be examined along the lines of refs.[7, 15].
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Note Added: I became aware, while this work was being proof-read, that some related work has been done in ref.[16] using a different method and specialising to the case of pure gauge theory only.

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