Improving lattice perturbation theory

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Lepage and Mackenzie have shown that tadpole renormalization and systematic improvement of lattice perturbation theory can lead to much improved numerical results in lattice gauge theory. It is shown that lattice perturbation theory using the Cayley parametrization of unitary matrices gives a simple analytical approach to tadpole renormalization, and that the Cayley parametrization gives lattice gauge potentials gauge transformations close to the continuum form. For example, at the lowest order in perturbation theory, for SU(3) lattice gauge theory, at $\beta = 6$, the 'tadpole renormalized' coupling $\tilde{g}^2 = \frac{4}{3}g^2$, to be compared to the non-perturbative numerical value $\tilde{g}^2 = 1.7g^2$.

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1. Introduction

Lepage and Mackenzie[1] have made considerable advances in numerical simulations of lattice gauge theories by improving the lattice gauge theory action in two ways:
1. The action is improved in the sense of Symanzik's[2] program.
2. The action is improved by 'tadpole renormalization'—a procedure by which the gauge potential associated to each link variable is given not by the usual correspondence

$$U_\mu \leftrightarrow \exp(igaA_\mu)$$

but by

$$U_\mu \leftrightarrow u_0(1 + igaA_\mu).$$

It is observed by Lepage and Mackenzie[1] (and collaborators[4]) that these two improvements together lead to a much better convergence to a continuum limit, with orders of magnitude less computational effort.

Given these remarkable results, it is important to understand exactly what is analytically involved in these improvements, especially for tadpole renormalization, which is not as well-understood as the Symanzik-type improvements. One might hope that a better understanding of the continuum limit would be useful as well for understanding how to link the strong coupling expansion and continuum perturbation theory[5], for example.

That the Lepage-Mackenzie prescription needs some explication can be seen, for instance, by considering the gauge transformation properties of the above correspondence. A prime requirement for gauge invariance is $UU^\dagger = 1$, which is not consistent with $U_\mu \leftrightarrow u_0(1 + igaA_\mu)$ unless $u_0^{-2} = 1 + (ga)^2 A_\mu^2$. The intuition is that u_0 incorporates short distance fluctuations of the link variable, leaving A_μ as the continuum gauge field at long distances. However, it is obvious that the exponential parametrization that is manifestly unitary cannot be factorized in any simple manner for non-Abelian gauge theories, so as to lead to the required form. One is lead, then, to ask: Given that tadpole renormalization does appear to work, what is the systematic analytical approximation that underlies it? Clearly, from the nomenclature, one hopes to resum some subsets of graphs in lattice perturbation theory, so as to make the rest of the lattice perturbation theory closer to continuum perturbation theory. Consistent resummations of selected subsets of graphs in continuum relativistic gauge theories are rare. On the lattice, of course, one has a great deal more freedom, and reorganizing the perturbation theory to make the approach to the continuum simpler is a good thing to consider. The basic idea, then, is that the reorganized lattice perturbation theory should be closer to continuum perturbation theory.

I show, in the present work, that the classical Cayley parametrization of group elements has two things to recommend it over the exponential parametrization in common use:
I. This parametrization leads to gauge transformations for the lattice gauge potentials associated with links, that are 'closer' to continuum gauge transformations, than are the gauge transformations for the potentials obtained via the exponential map.
II. In this parametrization, it is natural to reorganize the lattice perturbation theory in a manner that eliminates certain tadpoles, and renormalizes the naïve lattice coupling

constant in the way posited by Lepage and Mackenzie. I calculate the background field effective action in this parametrization after a suitable resummation, and show that some care is needed in interpreting the resulting expansion parameter.

2. Gauge transformations

The Cayley parametrization of a unitary matrix is

$$U \equiv \frac{1 + igA/2}{1 - igA/2}, \qquad A \text{ Hermitian.}$$

This parametrization is valid so long as $\det(U + 1) \neq 0$, so we may use it in lattice perturbation theory, when we expect $U \approx 1$. So at the most simplistic level, one could arrive at a relation close to the Lepage-Mackenzie form by expanding

$$U_\mu = \frac{1 + igaA_\mu/2}{1 - igaA_\mu/2} = \frac{1}{1 - c^2} \left[1 + \Delta + \Delta^2 + \ldots\right] \left(1 + igaA_\mu + \Delta(1 - c^2)\right),$$

with $\Delta \equiv (c^2 - g^2 a^2 A_\mu^2/4)/(c^2 - 1)$. This expansion is now consistent with unitary link variables. The series expansion will converge the best when inserted in correlation functions if $c^2 = -\langle g^2 a^2 A_\mu^2 \rangle$. The higher terms in the series will then involve insertions of powers of $(c^2 - g^2 a^2 A_\mu^2/4)$, and self-contractions (tadpoles) will be cancelled. The higher terms are, of course, crucial for unitary link variables. This particular form of resumming tadpoles is likely not a good idea, since it is manifestly not gauge invariant. The same idea can, however, be implemented at the level of the lattice gauge theory action, as will be shown later in this paper.

Let us turn now to the question of gauge invariance of the gauge potential A defined by the Cayley parametrization. It will be useful to contrast the lattice gauge invariance of this parametrization with that of the usual exponential parametrization, so I first recall the standard result. (I have rescaled A to absorb g for this part of the discussion.) If $U_\mu(x) \equiv \exp(iaA_\mu(x))$, then gauge transformations map U_μ to $\tilde{U}_\mu(x) \equiv \eta(x)U_\mu(x)\eta^\dagger(x + e_\mu)$. Let $\eta(x) = 1 + i\varepsilon(x)$, then one has

$$\delta U_\mu(x)U_\mu^\dagger(x) = -iaD_\mu\varepsilon(x),$$

with the lattice covariant derivative defined by

$$D_\mu\varepsilon(x) \equiv \frac{1}{a}\left(U_\mu(x)\varepsilon(x + e_\mu)U_\mu^\dagger(x) - \varepsilon(x)\right).$$

We are interested in the change in A_μ, induced by such a change in U_μ, so we expand both sides in powers of A, to find

$$i\delta A_\mu(x) - \frac{a}{2}\{A_\mu, \delta A_\mu(x)\} - \frac{ia^2}{6}\left(\{\delta A_\mu(x), A_\mu(x)^2\} + A_\mu(x)\delta A_\mu(x)A_\mu(x)\right) + \ldots$$

$$= -i\left[\tilde{D}_\mu\varepsilon(x) + \frac{ia}{2}\{A_\mu(x), \Delta_\mu\varepsilon(x)\}B - \frac{a}{2}[A_\mu(x)^2, \varepsilon(x)] + \ldots\right],$$

where $\Delta_\mu \varepsilon(x) \equiv (\varepsilon(x + e_\mu) - \varepsilon(x))/a$, and

$$\tilde{D}_\mu \varepsilon(x) \equiv \frac{1}{a}(\varepsilon(x + e_\mu) - \varepsilon(x)) + \frac{i}{2}[A_\mu(x), \varepsilon(x + e_\mu) + \varepsilon(x)].$$

It follows then that the gauge transformations of the gauge potential on the lattice, defined by the exponential parametrization, differ from the continuum form of the gauge transformations at $O(a)$. Explicitly, if $\delta A_\mu = -\tilde{D}_\mu \varepsilon(x) + a\varpi$, then

$$\varpi = -\frac{1}{4}\{A_\mu, [A_\mu, \varepsilon(x + e_\mu) + \varepsilon(x)]\} + \frac{1}{2}[A_\mu(x)^2, \varepsilon(x)] + O(a).$$

The same calculation for the Cayley parametrization is much simpler, and leads in a straightforward manner to

$$\delta A_\mu(x) = -\tilde{D}_\mu \varepsilon(x) - \frac{a^2}{4} A_\mu(x)\Delta_\mu \varepsilon(x) A_\mu(x),$$

an exact result as opposed to the infinite series one obtains in the exponential parametrization. Thus the Cayley parametrization is closer to the continuum in the sense that gauge transformations of the Cayley gauge potential differ *only* at $O(a^2)$, from $-\tilde{D}_\mu \varepsilon(x)$, whereas gauge transformation for the exponential parametrization differ at $O(a)$, with contributions at all higher powers of a. Of course, $-\tilde{D}_\mu \varepsilon(x)$ itself differs from the continuum form at $O(a)$.

One may wonder if it is possible to improve the lattice definition of gauge potential even further, to make the deviation from the continuum even higher order in a. It is a simple matter to show that parametrizations of the form $U_\mu = 1 + f(iaA)/1 + f(-iaA)$, with f a real polynomial, do not accomplish this.

3. Tadpole resummation in the action

Consider now using the Cayley parametrization in the Wilson action,

$$S_W = \frac{1}{g^2} \sum_{\text{plaquettes}} \Re\text{tr}\,[1 - U_\mu(x)U_\nu(x + e_\mu)U_\mu(x + e_\nu)U_\nu(x)] \equiv \frac{1}{g^2} \sum_{\text{plaquettes}} \Re\text{tr}\,[1 - W_{\mu\nu}].$$

A particularly interesting question to address is the following: The renormalization group scale parameter for weak coupling lattice perturbation theory, Λ_L, is very small[6,3] compared to the continuum value, computed with *e.g.* Pauli-Villars,

$$\frac{\Lambda_L}{\Lambda_{PV}} \approx 0.02, \qquad \text{for SU}(\infty).$$

The fact that this ratio is small due to tadpole diagrams was noted in the original calculation of Dashen and Gross[3]—this is one of the motivations for tadpole renormalization according to Lepage and Mackenzie. Therefore, a good test of the efficacy of Cayley's

parametrization in lattice perturbation theory is to compute the same background field one-loop effective action taking into account a resummation of tadpoles.

In the background field approach, we parametrize the link variables as

$$U_\mu(x) \equiv C(\alpha_\mu) U_\mu^0(x),$$

where $C(x) \equiv (1 + iax/2)/(1 - iax/2)$, α are the fluctuations in the gauge potential, and U^0 is the background configuration of the link variables, supposed to be weak (*i.e.* close to the identity) and varying only over long distances. The lattice covariant derivatives D_μ^0 are now defined with parallel transport by U^0. It is a standard exercise to show that

$$\mathrm{tr} W_{\mu\nu} = \mathrm{tr} C(X_{\mu\nu}) W_{\mu\nu}^0, \mathrm{with} \qquad W_{\mu\nu}^0 \equiv U_\mu^0(x) U_\nu^0(x + e_\mu) U_\mu^0(x + e_\nu) U_\nu^0(x),$$

and

$$\frac{X_{\mu\nu}}{a} \equiv D_\mu^0 \alpha_\nu - D_\nu^0 \alpha_\mu + \frac{i}{2} \left(2[\alpha_\mu, \alpha_\nu] - a[D_\nu^0 \alpha_\mu, \alpha_\mu] + a[D_\mu^0 \alpha_\nu, \alpha_\nu] - a^2 [D_\nu^0 \alpha_\mu, D_\mu^0 \alpha_\nu] \right),$$

which we divide into two parts, $X_{\mu\nu}/a \equiv 2(E_{\mu\nu} + iB_{\mu\nu})$. We write $2W_{\mu\nu}^0 \equiv G_{\mu\nu} + iH_{\mu\nu}$, with $G = 2 + O(F^2)$, $H = O(F)$, and note that $X^2 = 4a^2(E^2 - B^2 + i\{E, B\})$. Further,

$$4\Re \mathrm{tr} W_{\mu\nu} = \mathrm{tr} \left[\frac{1}{1 + a^2 X_{\mu\nu}^2/4} \left(\left(1 - i \frac{X_{\mu\nu} a}{2} \right)^2 (G_{\mu\nu} - iH_{\mu\nu}) \right. \right.$$

$$\left. \left. + \left(1 + i \frac{X_{\mu\nu} a}{2} \right)^2 (G_{\mu\nu} + iH_{\mu\nu}) \right) \right],$$

so for a one-loop computation we need only

$$2\Re \mathrm{tr} W_{\mu\nu} = \mathrm{tr} \left[\frac{1}{1 + a^2 E_{\mu\nu}^2} \left((1 - a^2 E_{\mu\nu}^2) G_{\mu\nu} - 4ia B_{\mu\nu} H_{\mu\nu} \right) \right].$$

Note that at this order in the weak-coupling background field calculation we do not need to consider the lattice functional Haar measure.

Expanding S_W about U^0, we have

$$S_W = \frac{1}{g^2} \sum_{\mathrm{plaquettes}} \Re \mathrm{tr} \left(1 - W_{\mu\nu}^0 \right) + \Re \mathrm{tr} \left(W_{\mu\nu}^0 - W_{\mu\nu} \right),$$

which gives

$$S_W = \frac{1}{g^2} \sum_{\mathrm{plaquettes}} \Re \mathrm{tr} \left(1 - W_{\mu\nu}^0 \right) + \mathrm{tr} \left[\frac{a^2 E_{\mu\nu}^2}{1 + a^2 E_{\mu\nu}^2} G + i \frac{a B_{\mu\nu} H_{\mu\nu}}{1 + a^2 E_{\mu\nu}^2} \right].$$

Define $\mathcal{E}_{\mu\nu} = (a^2 E_{\mu\nu}^2 - C)/(1 + C)$. with C a number that we shall fix momentarily. In terms of $\mathcal{E}_{\mu\nu}$,

$$S_W = \frac{1}{g^2}\left(\frac{1-C}{1+C}\right)\sum_{\text{plaquettes}}\Re\text{tr}\left(1 - W_{\mu\nu}^0\right)$$

$$+ \frac{1}{g^2(1+C)}\sum_{\text{plaquettes}}\text{tr}\left[\frac{2a^2 E_{\mu\nu}^2}{1+\mathcal{E}_{\mu\nu}} + \frac{\mathcal{E}_{\mu\nu}}{1+\mathcal{E}_{\mu\nu}}(G_{\mu\nu} - 2) - i\frac{aB_{\mu\nu}H_{\mu\nu}}{1+\mathcal{E}_{\mu\nu}}\right].$$

Observe that the coupling constant g^2 in front of the classical action has been renormalized to $g^2(1+C)/(1-C)$.

We now wish to fix C. To this end, rescale the fluctuation fields by a factor of $g\sqrt{1+C}$. The part of S_W governing fluctuations becomes

$$S(\alpha) = \sum_{\text{plaquettes}}\text{tr}\left[\frac{2a^2 E_{\mu\nu}^2}{1+g^2\widehat{\mathcal{E}}_{\mu\nu}} + \frac{\widehat{\mathcal{E}}_{\mu\nu}}{1+g^2\widehat{\mathcal{E}}_{\mu\nu}}\frac{1}{1+C}(G_{\mu\nu} - 2) - ia\frac{B_{\mu\nu}H_{\mu\nu}}{1+g^2\widehat{\mathcal{E}}_{\mu\nu}}\right],$$

with

$$\widehat{\mathcal{E}}_{\mu\nu} = a^2 E_{\mu\nu}^2 - \frac{C}{g^2(1+C)}.$$

It follows therefore that at this order, we should choose

$$C = \frac{g^2\langle a^2 E_{\mu\nu}^2\rangle}{1 - g^2\langle a^2 E_{\mu\nu}^2\rangle},$$

which implies that

$$\frac{1-C}{1+C} = 1 - 2g^2\langle a^2 E_{\mu\nu}^2\rangle.$$

The gauge fixing terms are as usual, with the gauge parameter dependent on C for Feynman gauge, but this is not a problem.

It should be noted that there are two types of contributions to the one-loop renormalization of the lattice coupling constant, those that have continuum analogues and those that are specific to lattice gauge theory. Ideally, one would want to incorporate all the lattice specific renormalizations into a resummed action—the resummation proposed above does *not* include all the lattice specific renormalizations, it only includes the tadpole diagram that provides the bulk of the renormalization. Within the present calculation, it does not seem possible to resum the other type of lattice specific renormalizations, essentially because they are not tadpole diagrams. It may be possible to eliminate such contributions if one starts with an action different from the Wilson action.

When $\beta = 6$, for SU(3), this tells us that the coupling constant in front of the classical action is

$$\tilde{g}^2 = \frac{4}{3}g^2,$$

at this order in perturbation theory. This should be compared with the non-perturbative value found numerically by Lepage and Mackenzie,

$$\tilde{g}^2 \approx 1.7 g^2.$$

It would appear then that the use of the Cayley parametrization combined with our resummation of the perturbation expansion is a decent approximation to the numerical renormalization, since this value of β corresponds to $g^2 = 1$. Note that we have ignored terms of order $g^2 \alpha^2$ from the measure[6] for example, and we have not included any Symanzik-type improvements in the action—nevertheless, we still obtained a lowest order result that is about half of the numerical non-perturbative value.

Two more comments are in order. Firstly, perturbation theory is still organized in powers of g^2 since higher terms in the fluctuation field appear in the combination $g^2 \widehat{\mathcal{E}}$. Of course, if one calculated at higher order, one would also need to change the definition of C to be consistent. Secondly, the tadpole contribution to Λ_L/Λ_{PV} does not appear since $G_{\mu\nu} - 2$ couples to $\widehat{\mathcal{E}}$, and $\langle\widehat{\mathcal{E}}\rangle = 0$. The one-loop renormalization of the tree action is still non-trivial since we have only resummed some of the graphs contributing to the determinant. In particular, C should be adjusted order by order in the perturbative expansion in order to eliminate the largest contribution from diagrams specific to lattice gauge theory.

4. Conclusions

It is plausible, on the basis of the simple calculation presented here, that combining the Cayley parametrization with Symanzik-type improvements will lead to a systematic lattice perturbation theory that is much closer to the continuum and to the non-perturbative numerical simulations. It seems unlikely that lattice perturbation theory, even if it is reformulated as described above, will become a practical tool for calculations in gauge theories. The interest in understanding the structure of lattice perturbation theory is in being able to match numerical results with continuum results. As I hope is evident from the preceding discussion, there is freedom in the definition of the lattice perturbation expansion that can be used to render such a comparison more transparent. Deviations from perturbation theory can then be understood as either lattice artifacts, or genuine non-perturbative effects. The numerical phenomenology of genuine non-perturbative physics would be useful in constructing analytical approximations to gauge theory dynamics. While such systematic study does not have the theoretical allure of direct attempts at phenomenological models of low-energy dynamics in gauge theories, keep in mind the words of Simone Weil—'We must prefer real hell to an imaginary paradise.'

References

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