SPIN-$\frac{3}{2}$ POTENTIALS IN BACKGROUNDS WITH BOUNDARY

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This paper studies the two-spinor form of the Rarita-Schwinger potentials subject to local boundary conditions compatible with local supersymmetry. The massless Rarita-Schwinger field equations are studied in four-real-dimensional Riemannian backgrounds with boundary. Gauge transformations on the potentials are shown to be compatible with the field equations providing the background is Ricci-flat, in agreement with previous results in the literature. However, the preservation of boundary conditions under such gauge transformations leads to a restriction of the gauge freedom. The recent construction by Penrose of secondary potentials which supplement the Rarita-Schwinger potentials is then applied. The equations for the secondary potentials, jointly with the boundary conditions, imply that the background four-geometry is further restricted to be totally flat. The analysis of other gauge transformations confirms that, in the massless case, the only admissible class of Riemannian backgrounds with boundary is totally flat.

1. Introduction

Over the last few years, many efforts have been produced to study locally supersymmetric boundary conditions in perturbative quantum cosmology.[1−7] The aim of this paper is to perform a complete analysis of the corresponding *classical* elliptic boundary-value problems. Indeed, in Ref. 8 it was shown that one possible set of local boundary conditions for a massless field of spin s, involving field strengths $\phi_{A...L}$ and $\widetilde{\phi}_{A'...L'}$ and the Euclidean normal ${}_{e}n^{AA'}$ to the boundary:

$$2^{s} \; {}_{e}n^{AA'} ... {}_{e}n^{LL'} \; \phi_{A...L} = \pm \, \widetilde{\phi}^{A'...L'} \quad \text{at} \quad \partial M \; , \tag{1.1}$$

can only be imposed in flat Euclidean backgrounds with boundary. However, such boundary conditions (motivated by supergravity theories in anti-de Sitter space-time,[9−10] where (1.1) is essential to obtain a well-defined quantum theory after taking the covering space of anti-de Sitter) do not make it possible to relate bosonic and fermionic fields through the action of complementary projection operators at the boundary.[2] For this purpose, one has to impose another set of local and supersymmetric boundary conditions, first proposed in Ref. 1. These are in general *mixed*, and involve in particular Dirichlet conditions for the transverse modes of the vector potential of electromagnetism, a mixture of Dirichlet and Neumann conditions for scalar fields, and local boundary conditions for the spin-$\frac{1}{2}$ field and the spin-$\frac{3}{2}$ potential. Using two-component spinor notation for supergravity,[5,11−12] the spin-$\frac{3}{2}$ boundary conditions relevant for quantum cosmology take the form[4]

$$\sqrt{2} \; {}_{e}n_{A}^{\;\;A'} \; \psi^{A}_{\;\;i} = \pm \widetilde{\psi}^{A'}_{\;\;i} \quad \text{at} \quad \partial M \; . \tag{1.2}$$

With our notation, $\left(\psi^A_{\ i}, \widetilde{\psi}^{A'}_{\ i}\right)$ are the *independent* (i.e. not related by any conjugation) spatial components (hence $i = 1, 2, 3$) of the spinor-valued one-forms appearing in the action functional of Euclidean supergravity.[5,11]

In the light of the results outlined so far, a naturally occurring question is whether an analysis motivated by the one in Ref. 8 can be used to derive restrictions on the classical boundary-value problem corresponding to (1.2). Such a question is of crucial importance for at least two reasons:

(i) In the absence of boundaries, extended supergravity theories are naturally formulated on curved backgrounds with a cosmological constant.[5,9,10] Thus, if a local theory in terms of spin-$\frac{3}{2}$ potentials and in the presence of boundaries can only be studied in flat Euclidean four-space, this result would make it impossible to consider the most interesting supergravity models when a four-manifold with boundaries occurs.

(ii) One of the main problems of the twistor programme for general relativity lies in the impossibility to achieve a twistorial reconstruction of (complex) vacuum space-times which are not right-flat (i.e. such that the Ricci spinor $R_{AA'BB'}$ and the self-dual Weyl spinor $\widetilde{\psi}_{A'B'C'D'}$ vanish). To overcome this difficulty, Penrose has proposed a new definition of twistors as charges for massless spin-$\frac{3}{2}$ fields in Ricci-flat Riemannian manifolds (see references in the following sections). However, since gravitino potentials have been studied also in backgrounds which are not Ricci-flat,[9–10] one is led to ask whether the recent Penrose formalism can be applied to study a larger class of Riemannian four-manifolds with boundary.

For this purpose, we introduce in Sec. 2 the Rarita-Schwinger potentials with their gauge transformations in Riemannian background four-geometries. Section 3 derives compatibility conditions from the gauge transformations of Sec. 2, and from the boundary conditions (1.2). Section 4 is devoted to the secondary potentials which supplement the Rarita-Schwinger potentials in Ricci-flat backgrounds. Section 5 studies other sets of gauge transformations. Concluding remarks and open problems are presented in Sec. 6. Relevant details about the two-spinor form of Rarita-Schwinger equations are given in the appendix.

2. Rarita-Schwinger Potentials and their Gauge Transformations

For the reasons described in the introduction, we are here interested in the independent spatial components $\left(\psi^A_{\ i}, \widetilde{\psi}^{A'}_{\ i}\right)$ of the gravitino field in Riemannian backgrounds. In terms of the spatial components $e_{AB'i}$ of the tetrad, and of spinor fields, they can be expressed as[11,13–14]

$$\psi_{A\ i} = \Gamma^{C'}_{\ \ AB}\ e^B_{\ \ C'i}\ , \tag{2.1}$$

$$\widetilde{\psi}_{A'\ i} = \gamma^C_{\ \ A'B'}\ e_C^{\ \ B'}{}_i\ . \tag{2.2}$$

A first important difference with respect to the Dirac form of the potentials studied in Ref. 8 is that the spinor fields $\Gamma^{C'}_{\ \ AB}$ and $\gamma^C_{\ \ A'B'}$ are no longer symmetric in the second and third index.[14] From now on, they will be referred to as spin-$\frac{3}{2}$ potentials. They obey the differential equations (see appendix and cf. Refs. 13 and 14)

$$\epsilon^{B'C'}\ \nabla_{A(A'}\ \gamma^A_{\ \ B')C'} = -3\Lambda\ \widetilde{\alpha}_{A'}\ , \tag{2.3}$$

$$\nabla^{B'(B} \, \gamma^{A)}_{\ \ B'C'} = \Phi^{ABL'}_{\ \ \ \ C'} \, \widetilde{\alpha}_{L'} \ , \tag{2.4}$$

$$\epsilon^{BC} \, \nabla_{A'(A} \, \Gamma^{A'}_{\ \ B)C} = -3\Lambda \, \alpha_A \ , \tag{2.5}$$

$$\nabla^{B(B'} \, \Gamma^{A')}_{\ \ BC} = \widetilde{\Phi}^{A'B'L}_{\ \ \ \ C} \, \alpha_L \ , \tag{2.6}$$

where $\nabla_{AB'}$ is the spinor covariant derivative corresponding to the curved connection ∇ of the background, the spinors $\Phi^{AB}_{\ \ C'D'}$ and $\widetilde{\Phi}^{A'B'}_{\ \ CD}$ correspond to the trace-free part of the Ricci tensor, the scalar Λ corresponds to the scalar curvature $R = 24\Lambda$ of the background, and $\alpha_A, \widetilde{\alpha}_{A'}$ are a pair of independent spinor fields, corresponding to the Majorana field in the Lorentzian regime. Moreover, the potentials are subject to the gauge transformations (cf. Sec. 5)

$$\widehat{\gamma}^A_{\ \ B'C'} \equiv \gamma^A_{\ \ B'C'} + \nabla^A_{\ \ B'} \, \lambda_{C'} \ , \tag{2.7}$$

$$\widehat{\Gamma}^{A'}_{\ \ BC} \equiv \Gamma^{A'}_{\ \ BC} + \nabla^{A'}_{\ \ B} \, \nu_C \ . \tag{2.8}$$

A second important difference with respect to the Dirac potentials[8] is that the spinor fields ν_B and $\lambda_{B'}$ are no longer taken to be solutions of the Weyl equation. They should be freely specifiable (see Sec. 3).

3. Compatibility Conditions

Our task is now to derive compatibility conditions, by requiring that the field equations (2.3)-(2.6) should also be satisfied by the gauge-transformed potentials appearing on the left-hand side of Eqs. (2.7)-(2.8). For this purpose, after defining the operators

$$\square_{AB} \equiv \nabla_{M'(A} \, \nabla_{B)}^{\ \ M'} \ , \tag{3.1}$$

$$\Box_{A'B'} \equiv \nabla_{F(A'} \nabla_{B')}^{\ \ F} \ , \tag{3.2}$$

we need the standard identity[15–17] $\Omega_{[AB]} = \frac{1}{2}\epsilon_{AB} \ \Omega_{C}^{\ \ C}$ and the spinor Ricci identities[8]

$$\Box_{AB} \ \nu_C = \psi_{ABCD} \ \nu^D - 2\Lambda \ \nu_{(A} \ \epsilon_{B)C} \ , \tag{3.3}$$

$$\Box_{A'B'} \lambda_{C'} = \widetilde{\psi}_{A'B'C'D'} \ \lambda^{D'} - 2\Lambda \ \lambda_{(A'} \ \epsilon_{B')C'} \ , \tag{3.4}$$

$$\Box^{AB} \ \lambda_{B'} = \Phi^{AB}_{\ \ \ \ M'B'} \ \lambda^{M'} \ , \tag{3.5}$$

$$\Box^{A'B'} \ \nu_B = \widetilde{\Phi}^{A'B'}_{\ \ \ \ \ MB} \ \nu^M \ . \tag{3.6}$$

Of course, $\widetilde{\psi}_{A'B'C'D'}$ and ψ_{ABCD} are the self-dual and anti-self-dual Weyl spinors respectively.

Thus, on using the Eqs. (2.3)-(2.8) and (3.1)-(3.6), the basic rules of two-spinor calculus[15–17] lead to the compatibility equations

$$3\Lambda \ \lambda_{A'} = 0 \ , \tag{3.7}$$

$$\Phi^{AB \ \ \ \ C'}_{\ \ \ \ M'} \ \lambda^{M'} = 0 \ , \tag{3.8}$$

$$3\Lambda \ \nu_A = 0 \ , \tag{3.9}$$

$$\widetilde{\Phi}^{A'B' \ \ \ C}_{\ \ \ \ \ \ M} \ \nu^M = 0 \ . \tag{3.10}$$

Non-trivial solutions of (3.7)-(3.10) only exist if the scalar curvature and the trace-free part of the Ricci tensor vanish. Hence the gauge transformations (2.7)-(2.8) lead to spinor fields ν_A and $\lambda_{A'}$ which are freely specifiable *inside* Ricci-flat backgrounds, while the boundary conditions (1.2) are preserved under the action of (2.7)-(2.8) providing the following conditions hold at the boundary:

$$\sqrt{2} \; {}_{e}n_{A}^{\;\;A'} \left(\nabla^{AC'} \; \nu^{B}\right) e_{BC'i} = \pm \left(\nabla^{CA'} \lambda^{B'}\right) e_{CB'i} \quad \text{at} \quad \partial M \; . \tag{3.11}$$

4. Secondary Potentials in Ricci-Flat Backgrounds

As shown by Penrose in Ref. 18, in a Ricci-flat manifold the Rarita-Schwinger potentials may be supplemented by secondary potentials. Here we use such a construction in its local form. For this purpose, we introduce secondary potentials for the γ-potentials by requiring that locally (see Ref. 18)

$$\gamma_{A'B'}^{\;\;\;\;\;\;C} \equiv \nabla_{BB'} \; \rho_{A'}^{\;\;\;CB} \; . \tag{4.1}$$

Of course, special attention should be payed to the index ordering in (4.1), since the primary and secondary potentials are not symmetric (cf. Ref. 8). On inserting (4.1) into (2.3), a repeated use of symmetrizations and anti-symmetrizations leads to the equation (hereafter $\Box \equiv \nabla_{CF'} \nabla^{CF'}$)

$$\begin{aligned} &\epsilon_{FL} \; \nabla_{AA'} \; \nabla^{B'(F} \; \rho_{B'}^{\;\;\;A)L} + \frac{1}{2} \nabla^{A}_{\;\;A'} \; \nabla^{B'M} \; \rho_{B'(AM)} \\ &+ \Box_{AM} \; \rho_{A'}^{\;\;\;(AM)} + \frac{3}{8} \Box \rho_{A'} = 0 \; , \end{aligned} \tag{4.2}$$

where, following Ref. 18, we have defined

$$\rho_{A'} \equiv \rho_{A'C}^{\;\;\;\;\;\;C} \; , \tag{4.3}$$

and we bear in mind that our background has to be Ricci-flat. Thus, if the following equation holds (cf. Ref. 18):

$$\nabla^{B'(F} \rho_{B'}^{\ \ A)L} = 0 , \tag{4.4}$$

one finds

$$\nabla^{B'M} \rho_{B'(AM)} = \frac{3}{2} \nabla_A^{\ \ F'} \rho_{F'} , \tag{4.5}$$

and hence Eq. (4.2) may be cast in the form

$$\square_{AM} \rho_{A'}^{\ \ (AM)} = 0 . \tag{4.6}$$

A very useful identity resulting from Eq. (4.9.13) of Ref. 19 enables one to show that

$$\square_{AM} \rho_{A'}^{\ \ (AM)} = -\Phi_{AMA'}^{\ \ \ \ \ \ L'} \rho_{L'}^{\ \ (AM)} . \tag{4.7}$$

Hence Eq. (4.6) reduces to an identity by virtue of Ricci-flatness. Moreover, we have to insert (4.1) into the field equation (2.4) for γ-potentials. By virtue of (4.4) and of the identities (cf. Ref. 19)

$$\square^{BM} \rho_{B'\ \ M}^{\ \ A} = -\psi^{ABLM} \rho_{(LM)B'} - \Phi^{BM}_{\ \ \ \ B'}{}^{D'} \rho^A_{\ \ MD'} + 4\Lambda \rho^{(AB)}_{\ \ \ \ \ \ B'} , \tag{4.8}$$

$$\square^{B'F'} \rho_{B'}^{\ \ (AB)} = 3\Lambda \rho^{(AB)F'} + \widetilde{\Phi}^{B'F'}_{\ \ \ \ \ \ L}{}^{A} \rho^{(LB)}_{\ \ \ \ \ \ B'} + \widetilde{\Phi}^{B'F'B}_{\ \ \ \ \ \ \ \ L} \rho^{(AL)}_{\ \ \ \ \ \ B'} , \tag{4.9}$$

this leads to the equation

$$\psi^{ABLM} \rho_{(LM)C'} = 0 , \tag{4.10}$$

where we have used again the Ricci-flatness condition.

Of course, secondary potentials supplementing Γ-potentials may also be constructed locally. On defining

$$\Gamma_{AB}^{\ \ \ \ C'} \equiv \nabla_{B'B} \, \theta_A^{\ \ C'B'} \ , \tag{4.11}$$

$$\theta_A \equiv \theta_{AC'}^{\ \ \ \ \ C'} \ , \tag{4.12}$$

and requiring that[18]

$$\nabla^{B(F'} \, \theta_B^{\ \ A')L'} = 0 \ , \tag{4.13}$$

one finds

$$\nabla^{BM'} \, \theta_{B(A'M')} = \frac{3}{2} \nabla_{A'}^{\ \ \ F} \, \theta_F \ , \tag{4.14}$$

and a similar calculation yields an identity and the equation

$$\widetilde{\psi}^{A'B'L'M'} \, \theta_{(L'M')C} = 0 \ . \tag{4.15}$$

Note that Eqs. (4.10) and (4.15) relate explicitly the secondary potentials to the curvature of the background. This inconsistency is avoided if one of the following conditions holds:

(i) The whole conformal curvature of the background vanishes.

(ii) ψ^{ABLM} and $\theta_{(L'M')C}$, or $\widetilde{\psi}^{A'B'L'M'}$ and $\rho_{(LM)C'}$, vanish.

(iii) The symmetric parts of the secondary potentials vanish.

In the first case one finds that the only admissible background is again flat Euclidean four-space with boundary, as in Ref. 8. By contrast, in the other cases, left-flat, right-flat or Ricci-flat backgrounds are still admissible, providing the secondary potentials take the form

$$\rho_{A'}^{\ \ CB} = \epsilon^{CB} \, \widetilde{\alpha}_{A'} \ , \tag{4.16}$$

$$\theta_A^{\ \ C'B'} = \epsilon^{C'B'} \, \alpha_A \ , \tag{4.17}$$

where α_A and $\widetilde{\alpha}_{A'}$ solve the Weyl equations

$$\nabla^{AA'} \, \alpha_A = 0 \ , \tag{4.18}$$

$$\nabla^{AA'} \, \widetilde{\alpha}_{A'} = 0 \ . \tag{4.19}$$

Eqs. (4.16)-(4.19) ensure also the validity of Eqs. (4.4), (4.13), and (A.6)-(A.7) of the appendix.

However, if one requires the preservation of Eqs. (4.4) and (4.13) under the following gauge transformations for secondary potentials (the order of the indices AL, $A'L'$ is of crucial importance):

$$\widehat{\rho}_{B'}^{\ \ AL} \equiv \rho_{B'}^{\ \ AL} + \nabla_{B'}^{\ \ A} \, \mu^L \ , \tag{4.20}$$

$$\widehat{\theta}_B^{\ \ A'L'} \equiv \theta_B^{\ \ A'L'} + \nabla_B^{\ \ A'} \, \sigma^{L'} \ , \tag{4.21}$$

one finds compatibility conditions in Ricci-flat backgrounds of the form

$$\psi_{AFLD} \, \mu^D = 0 \ , \tag{4.22}$$

$$\widetilde{\psi}_{A'F'L'D'} \, \sigma^{D'} = 0 \ . \tag{4.23}$$

Thus, to ensure *unrestricted* gauge freedom (except at the boundary) for the secondary potentials, one is forced to work with flat Euclidean backgrounds. The boundary conditions

(1.2) play a role in this respect, since they make it necessary to consider both ψ_i^A and $\widetilde{\psi}_i^{A'}$, and hence both ${\rho_{B'}}^{AL}$ and ${\theta_B}^{A'L'}$. Otherwise, one might use (4.22) to set to zero the anti-self-dual Weyl spinor only, *or* (4.23) to set to zero the self-dual Weyl spinor only, so that self-dual (left-flat) or anti-self-dual (right-flat) Riemannian backgrounds with boundary would survive.

5. Other Gauge Transformations

In the massless case, flat Euclidean backgrounds with boundary are really the only possible choice for spin-$\frac{3}{2}$ potentials with a gauge freedom. To prove this, we have also investigated an alternative set of gauge transformations for primary potentials, written in the form (cf. (2.7)-(2.8))

$$\widehat{\gamma}^A{}_{B'C'} \equiv \gamma^A{}_{B'C'} + \nabla^A{}_{C'} \, \lambda_{B'} \; , \tag{5.1}$$

$$\widehat{\Gamma}^{A'}{}_{BC} \equiv \Gamma^{A'}{}_{BC} + \nabla^{A'}{}_{C} \, \nu_{B} \; . \tag{5.2}$$

These gauge transformations *do not* correspond to the usual formulation of the Rarita-Schwinger system, but we will see that they can be interpreted in terms of familiar physical concepts.

On imposing that the field equations (2.3)-(2.6) should be preserved under the action of (5.1)-(5.2), and setting to zero the trace-free part of the Ricci spinor (since it is inconsistent to have gauge fields $\lambda_{B'}$ and ν_B which depend explicitly on the curvature of the background) one finds compatibility conditions in the form of differential equations, i.e. (cf. Ref. 20)

$$\Box \lambda_{B'} = -2\Lambda \, \lambda_{B'} \; , \tag{5.3}$$

$$\nabla^{(A(B'} \, \nabla^{C')B)} \lambda_{B'} = 0 \; , \tag{5.4}$$

$$\Box \nu_{B} = -2\Lambda \, \nu_{B} \; , \tag{5.5}$$

$$\nabla^{(A'(B} \, \nabla^{C)B')} \, \nu_{B} = 0 \; . \tag{5.6}$$

In a flat Riemannian four-manifold with flat connection D, covariant derivatives commute and $\Lambda = 0$. Hence it is possible to express $\lambda_{B'}$ and ν_B as solutions of the Weyl equations

$$D^{AB'} \, \lambda_{B'} = 0 \; , \tag{5.7}$$

$$D^{BA'} \, \nu_{B} = 0 \; , \tag{5.8}$$

which agree with the flat-space version of (5.3)-(5.6). The boundary conditions (1.2) are then preserved under the action of (5.1)-(5.2) if ν_B and $\lambda_{B'}$ obey the boundary conditions (cf. (3.11))

$$\sqrt{2} \, {}_{e}n_{A}^{\;\;A'} \Bigl(D^{BC'} \, \nu^{A} \Bigr) e_{BC'i} = \pm \Bigl(D^{CB'} \, \lambda^{A'} \Bigr) e_{CB'i} \quad \text{at} \quad \partial M \; . \tag{5.9}$$

In the curved case, on defining

$$\phi^{A} \equiv \nabla^{AA'} \, \lambda_{A'} \; , \tag{5.10}$$

$$\widetilde{\phi}^{A'} \equiv \nabla^{AA'} \, \nu_{A} \; , \tag{5.11}$$

Eqs. (5.4) and (5.6) imply that these spinor fields solve the equations (cf. Ref. 20)

$$\nabla_{C'}^{\ \ (A} \phi^{B)} = 0 , \tag{5.12}$$

$$\nabla_{C}^{\ \ (A'} \widetilde{\phi}^{B')} = 0 . \tag{5.13}$$

Moreover, Eqs. (5.3), (5.5) and the spinor Ricci identities imply that

$$\nabla_{AB'} \phi^{A} = 2\Lambda \lambda_{B'} , \tag{5.14}$$

$$\nabla_{BA'} \widetilde{\phi}^{A'} = 2\Lambda \nu_{B} . \tag{5.15}$$

Remarkably, the Eqs. (5.12)-(5.13) are the twistor equations[15,20] in Riemannian four-geometries. The consistency conditions for the existence of non-trivial solutions of such equations in curved four-manifolds are given by[15]

$$\psi_{ABCD} = 0 , \tag{5.16}$$

and

$$\widetilde{\psi}_{A'B'C'D'} = 0 , \tag{5.17}$$

respectively, unless one regards ϕ^{B} as a four-fold principal spinor[15] of ψ_{ABCD}, and $\widetilde{\phi}^{B'}$ as a four-fold principal spinor of $\widetilde{\psi}_{A'B'C'D'}$.

Further consistency conditions for our problem are derived by acting with covariant differentiation on the twistor equation, i.e.

$$\nabla_{A'}^{\ \ C} \nabla^{AA'} \phi^{B} + \nabla_{A'}^{\ \ C} \nabla^{BA'} \phi^{A} = 0 . \tag{5.18}$$

While the complete symmetrization in ABC yields (5.16), the use of (5.18), jointly with the spinor Ricci identities of Sec. 3, yields

$$\Box \phi^B = 2\Lambda \, \phi^B \ , \tag{5.19}$$

and an analogous equation is found for $\widetilde{\phi}^{B'}$. Thus, since Eq. (5.12) implies

$$\nabla_{C'}{}^{A} \, \phi^B = \epsilon^{AB} \, \pi_{C'} \ , \tag{5.20}$$

we may obtain from (5.20) the equation

$$\nabla^{BA'} \, \pi_{A'} = 2\Lambda \, \phi^B \ , \tag{5.21}$$

by virtue of the spinor Ricci identities and of (5.19). On the other hand, in the light of (5.20), Eq. (5.14) leads to

$$\nabla_{AB'} \, \phi^A = 2\pi_{B'} = 2\Lambda \, \lambda_{B'} \ . \tag{5.22}$$

Hence $\pi_{A'} = \Lambda \, \lambda_{A'}$, and the definition (5.10) yields

$$\nabla^{BA'} \, \pi_{A'} = \Lambda \, \phi^B \ . \tag{5.23}$$

By comparison of (5.21) and (5.23), one gets the equation $\Lambda \, \phi^B = 0$. If $\Lambda \neq 0$, this implies that ϕ^B, $\pi_{B'}$ and $\lambda_{B'}$ have to vanish, and there is no gauge freedom fou our model. This inconsistency is avoided if and only if $\Lambda = 0$, and the corresponding background is forced to be totally flat, since we have already set to zero the trace-free part of the Ricci spinor and the whole conformal curvature. The same argument applies to $\widetilde{\phi}^{B'}$ and the gauge field ν_B. The present analysis corrects the statements made in Sec. 8.8 of Ref. 20, where it was not realized that, in our massless model, a non-vanishing cosmological constant is incompatible with a gauge freedom for the spin-$\frac{3}{2}$ potential. More precisely, if one sets $\Lambda = 0$ from the beginning in (5.3) and (5.5), the system (5.3)-(5.6) admits solutions of the Weyl equation in Ricci-flat manifolds. These backgrounds are further restricted to be totally flat on considering the Eqs. (4.10) and (4.15) for an arbitrary form of the secondary potentials. As already pointed out at the end of Sec. 4, the boundary conditions (1.2) play a role, since otherwise one might focus on right-flat or left-flat Riemannian backgrounds with boundary.

Yet other gauge transformations can be studied (e.g. the ones involving gauge fields $\lambda_{B'}$ and ν_B which solve the twistor equations), but they are all incompatible with a non-vanishing cosmological constant in the massless case.

6. Concluding Remarks and Open Problems

The consideration of boundary conditions is essential to obtain a well-defined formulation of physical theories in quantum cosmology.[5,21,22] In particular, one-loop quantum cosmology[3-7] makes it necessary to study spin-$\frac{3}{2}$ potentials about four-dimensional Riemannian backgrounds with boundary. The corresponding classical analysis has been performed in our paper in the massless case, to supersede the analysis appearing in Refs. 8 and 23. Our results are as follows.

First, the gauge transformations (2.7)-(2.8) are compatible with the massless Rarita-Schwinger equations providing the background four-geometry is Ricci-flat. However, the presence of a boundary restricts the gauge freedom, since the boundary conditions (1.2) are preserved under the action of (2.7)-(2.8) only if the boundary conditions (3.11) hold.

Second, the Penrose construction of secondary potentials in Ricci-flat four-manifolds shows that the admissible backgrounds may be further restricted to be totally flat, or left-flat, or right-flat, unless these secondary potentials take the special form (4.16)-(4.17). Hence the secondary potentials supplementing the Rarita-Schwinger potentials have a very clear physical meaning in Ricci-flat four-geometries with boundary: they are related to the spinor fields $\left(\alpha_A, \widetilde{\alpha}_{A'}\right)$ corresponding to the Majorana field in the Lorentzian version of (2.3)-(2.6). [One should bear in mind that, in real Riemannian four-manifolds, the only admissible spinor conjugation is Euclidean conjugation, which is anti-involutory on spinor fields with an odd number of indices.[5,20,24] Hence no Majorana field can be defined in real Riemannian four-geometries]

Third, to ensure unrestricted gauge freedom for the secondary potentials, one is forced to work with flat Euclidean backgrounds, when the boundary conditions (1.2) are imposed. Thus, the very restrictive results obtained in Refs. 8 and 23 for massless Dirac potentials with the boundary conditions (1.1) are indeed confirmed also for massless Rarita-Schwinger potentials subject to the supersymmetric boundary conditions (1.2). Interestingly, a formalism originally motivated by twistor theory[15,18,20,23−26] has been applied to classical boundary-value problems relevant for one-loop quantum cosmology.

Fourth, the gauge transformations (5.1)-(5.2) with non-trivial gauge fields are compatible with the field equations (2.3)-(2.6) if and only if the background is totally flat.

The corresponding gauge fields solve the Weyl equations (5.7)-(5.8), subject to the boundary conditions (5.9). Indeed, it is well-known that the Rarita-Schwinger description of a massless spin-$\frac{3}{2}$ field is equivalent to the Dirac description in a special choice of gauge.[18] In such a gauge, the spinor fields $\lambda_{B'}$ and ν_B solve the Weyl equations, and this is exactly what we find in Sec. 5 on choosing the gauge transformations (5.1)-(5.2).

A non-vanishing cosmological constant can be consistently studied when a *massive* spin-$\frac{3}{2}$ potential is studied.[27] For this purpose, one has to replace the spinor covariant derivative $\nabla_{AA'}$ in the field equations (2.3)-(2.6) by a new spinor covariant derivative $S_{AA'}$ which reduces to $\nabla_{AA'}$ when $\Lambda = 0$. In the language of γ-matrices, one has (cf. Ref. 27)

$$S_\mu \equiv \nabla_\mu + f(\Lambda)\gamma_\mu \ , \tag{6.1}$$

where $f(\Lambda)$ vanishes at $\Lambda = 0$, and γ_μ are the γ-matrices. We are currently investigating the reformulation of Secs. 2-5 in terms of the definition (6.1). In particular, it appears interesting to understand, by using two-spinor formalism, whether twistors can generate the gauge freedom for a class of *massive* spin-$\frac{3}{2}$ potentials in conformally flat Einstein four-geometries with boundary. Moreover, other interesting problems are found to arise:

(i) Can one relate Eqs. (4.4) and (4.13) to the theory of integrability conditions relevant for massless fields in curved backgrounds (see Ref. 18 and our appendix) ? What happens when such equations do not hold ?

(ii) Is there an underlying global theory of Rarita-Schwinger potentials ? In the affirmative case, what are the key features of the global theory ?

(iii) Can one reconstruct the Riemannian four-geometry from the twistor space in Ricci-flat or conformally flat backgrounds with boundary, or from whatever is going to replace twistor space ?

Thus, the results and problems presented in our paper seem to add evidence in favour of a deep link existing between twistor geometry, quantum cosmology and modern field theory.

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Appendix

Following Ref. 13, one can locally express the Γ-potentials of (2.1) as (cf. (4.11))

$$\Gamma^{A}{}_{BB'} \equiv \nabla_{BB'} \, \alpha^{A} \; . \tag{A.1}$$

Thus, acting with $\nabla_{CC'}$ on both sides of (A.1), symmetrizing over $C'B'$ and using the spinor Ricci identity (3.6), one finds

$$\nabla_{C(C'} \, \Gamma^{AC}{}_{B')} = \widetilde{\Phi}_{B'C'L}{}^{A} \, \alpha^{L} \; . \tag{A.2}$$

Moreover, acting with $\nabla_{C}{}^{C'}$ on both sides of (A.1), putting $B' = C'$ (with contraction over this index), and using the spinor Ricci identity (3.3) leads to

$$\epsilon^{AB} \, \nabla_{(C}{}^{C'} \, \Gamma_{|A|B)C'} = -3\Lambda \, \alpha_{C} \; . \tag{A.3}$$

Eqs. (A.1)-(A.3) rely on the conventions in Ref. 13. However, to achieve agreement with the conventions in Ref. 18 and in our paper, the Eqs. (2.3)-(2.6) are obtained by defining (for the effect of torsion terms, see comments following Eq. (21) in Ref. 13)

$$\Gamma_{B}{}^{A}{}_{B'} \equiv \nabla_{BB'} \, \alpha^{A} \; , \tag{A.4}$$

$$\gamma_{A'}{}^{B'}{}_{C} \equiv \nabla_{CA'} \, \widetilde{\alpha}^{B'} \; . \tag{A.5}$$

On requiring that (A.5) and (4.1) should agree, one finds by comparison that

$$\nabla_{BB'} \, \rho_{A'}{}^{(CB)} = 2\nabla^{C}{}_{[A'} \, \widetilde{\alpha}_{B']} \; , \tag{A.6}$$

which is obviously satisfied if ${\rho_{A'}}^{(CB)}=0$ and $\widetilde{\alpha}_{B'}$ obeys the Weyl equation (4.19). Similarly, by comparison of (A.4) and (4.11) one finds

$$\nabla_{B'B}\,{\theta_{A}}^{(C'B')}=2{\nabla^{C'}}_{[A}\,\alpha_{B]}\ , \tag{A.7}$$

which is satisfied if Eqs. (4.17)-(4.18) hold.

In the original approach by Penrose,[18] one describes Rarita-Schwinger potentials in flat space-time in terms of a rank-three vector bundle with local coordinates $\Bigr(\eta_{A},\zeta\Bigr)$, and an operator $\Omega_{AA'}$ whose action is defined by

$$\Omega_{AA'}(\eta_{B},\zeta)\equiv\Bigr(\mathcal{D}_{AA'}\eta_{B},\mathcal{D}_{AA'}\zeta-\eta^{C}\rho_{A'AC}\Bigr)\ , \tag{A.8}$$

$\mathcal{D}$ being the flat Levi-Civita connection of Minkowski space-time. The gauge transformations are then

$$\Bigr(\widehat{\eta}_{B},\widehat{\zeta}\Bigr)\equiv\Bigr(\eta_{B},\zeta+\eta_{A}\xi^{A}\Bigr)\ , \tag{A.9}$$

$$\widehat{\rho}_{A'AB}\equiv\rho_{A'AB}+\mathcal{D}_{AA'}\xi_{B}\ . \tag{A.10}$$

For the operator defined in (A.8), the integrability condition on β-planes[20] turns out to be

$$\mathcal{D}^{A'(A}\,{\rho_{A'}}^{B)C}=0\ . \tag{A.11}$$

It now remains to be seen whether, at least in Ricci-flat backgrounds, an operator can be defined (cf. (A.8)) whose integrability condition on β-surfaces[20] is indeed given by Eq. (4.4) (cf. Eq. (A.11)).

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