

Dipole Perturbations of the Reissner-Nordström Solution: The Polar Case

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Abstract

The formalism developed by Chandrasekhar for the linear polar perturbations of the Reissner-Nordström solution is generalized to include the case of dipole ($l=1$) perturbations. Then, the perturbed metric coefficients and components of the Maxwell tensor are computed.

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The gravitational and electromagnetic perturbations of Schwarzschild and Reissner-Nordström black holes have been studied in detail [1, 2, 3, 4]. Even though these two exact solutions are spherically symmetric, there is an important difference in the character of their perturbations: In the Schwarzschild solution the gravitational and electromagnetic (linear) perturbations are uncoupled, while, in contradistinction, in the Reissner-Nordström solution, due to the background electric field, any electromagnetic perturbation causes a gravitational perturbation, and *vice versa*. This coupling of the electromagnetic and the gravitational perturbations complicates the study of the perturbations in Reissner-Nordström black holes considerably. Despite this complication, it turns out that even in the Reissner-Nordström black hole it is possible to decouple the perturbations (of each multipole order and for each parity) to two independent modes (each of which is made of an electromagnetic component and a gravitational component). This decoupling plays a crucial rôle in the study of perturbations in Reissner-Nordström black holes.

The decoupling of the perturbations of Reissner-Nordström into electromagnetic and metric perturbations was treated, for both polar and axial modes, in Ref. [2] and summarized in Ref. [3]. (The treatment for the Schwarzschild black hole is very similar, and is given in Refs. [1] and [3]). We shall see, however, that the formalism presented in Refs. [2, 3] is not valid in the case of dipole ($l = 1$) modes. For many applications, this difficulty is not very crucial, as one may be primarily interested in the dynamics of gravitational waves, for which there are no radiative modes with $l < 2$. However, it may be of interest to treat the propagation of dipole electromagnetic waves, especially in the Reissner-Nordström spacetime, because of the coupling of the gravitational and the electromagnetic fields. Thus, the late-time behavior of electromagnetic perturbations produced during the collapse decays like the $(2l + 2)$ inverse-power of external time [5], and is therefore dominated by the $l = 1$ mode. In addition, the $l = 1$ perturbations are especially important in the analysis of the (electromagnetic) effects of the blue-sheet at the Cauchy horizon of Reissner-Nordström black holes [6]. (Similar electromagnetic effects are to be expected at the inner horizon of the Kerr black hole, though we have not analyzed this case.)

for dipole polar modes. Then, we generalize the formalism to include polar perturbations of any l , including $l = 1$. The treatment of axial perturbations is different and we hope to treat them separately.

This Paper deals with perturbations of the Reissner-Nordström black hole. The perturbations of the Schwarzschild black hole are obtained from our formalism as a special case. Throughout this Paper we shall use the notation and convention of [3] unless when explicitly stated otherwise. As a rule, we shall not deviate from the notation of [3] except when necessary. When we do change the notation, it will be by adding ‘bars’ to the symbols of [3]. The ‘barred’ objects will be defined such that they are treated properly for dipole perturbations.

The outline of this Paper is as follows: In Section 2 we shall describe the definitions and notation. In Section 3 we give a full treatment for the general formalism of polar perturbations of the Reissner-Nordström solution. In Section 4 we shall decouple the fundamental equations for the perturbations for the dipole case, and in Section 5 we shall generalize the treatment for all polar modes. In Sections 6 and 7 we shall present the completion of the solution, and in Section 8 we shall discuss the formalism and give some concluding remarks.

2 DEFINITIONS AND NOTATION

Following Chandrasekhar [3], we write the line-element of an unperturbed Reissner-Nordström black hole in the form

$$\begin{aligned} ds^2 &= e^{2\nu} (dx^0)^2 - e^{2\psi} (dx^1)^2 - e^{2\mu_2} (dx^2)^2 - e^{2\mu_3} (dx^3)^2 \\ &= e^{2\nu} (dx^0)^2 - e^{2\mu_2} (dx^2)^2 - r^2 d\Omega^2, \end{aligned} \quad (1)$$

where the co-ordinates are

$$(x^0 \ x^1 \ x^2 \ x^3) = (t \ \phi \ r \ \theta), \quad (2)$$

$d\Omega^2$ is the unit two-sphere line-element, and the metric coefficients are $e^{2\nu} = e^{-2\mu_2} = (r^2 - 2Mr + Q_*^2)/r^2 \equiv \Delta/r^2$, M, Q_* being the mass and electric charge, respectively, of the Reissner-Nordström black hole, and r being the

circumference $2\pi r$. The general form of the line-element (1) is preserved under polar perturbations (sometimes called even-parity perturbations); On the other hand, axial perturbations (called also odd-parity perturbations), will lead in general to non-vanishing off-diagonal metric coefficients¹. Therefore, the form of the metric of a generally-perturbed Reissner-Nordström black hole will be much more complicated than the line-element (1). It has been shown [3], that a metric of sufficient generality is of the form

$$ds^2 = e^{2\nu} (dx^0)^2 - e^{2\psi} (dx^1 - \omega dx^0 - q_2 dx^2 - q_3 dx^3)^2 - e^{2\mu_2} (dx^2)^2 - e^{2\mu_3} (dx^3)^2. \quad (3)$$

Since the unperturbed Reissner-Nordström background is spherically symmetric, we can consider only axisymmetric modes of perturbations without any loss of generality². The line-element (3) involves seven functions, namely, $\nu, \psi, \mu_2, \mu_3, \omega, q_2$, and q_3 . Because the Einstein equations involve only six independent functions, not all seven functions can be determined arbitrarily, and there is one constraint on the metric coefficient. It has been shown [3], that this constraint is

$$(\omega_{,2} - q_{2,0})_{,3} - (\omega_{,3} - q_{3,0})_{,2} + (q_{2,3} - q_{3,2})_{,0}.$$

3 THE GENERAL FORMALISM

For completeness, we shall first present the linearized field-equations and the decoupling of the r, θ variables as given in [3]: The formalism for the treatment of the perturbations is made of the linearization of the coupled Einstein-Maxwell equations about the Reissner-Nordström solution. In particular,

¹Axial perturbations are characterized by the non-vanishing of the metric functions ω, q_2, q_3 (the non-vanishing of these metric-coefficients induce a dragging of the inertial-frame and impart a rotation to the black hole), while polar perturbations are those which alter the values of the metric functions ν, μ_2, μ_3 and ψ (which are in general non-zero for the unperturbed black hole).

²This is because all non-axisymmetric modes can be obtained from the axisymmetric modes, if the unperturbed spacetime is spherically symmetric [3].

equations [5]:

$$(\delta\psi + \delta\mu_3)_{,r} + \left(\frac{1}{r} - \nu_{,r}\right) (\delta\psi + \delta\mu_3) - \frac{2}{r} \delta\mu_2 = -\delta R_{(0)(2)} = 0, \quad (4)$$

$$\left[(\delta\psi + \delta\mu_2)_{,\theta} + (\delta\psi - \delta\mu_3) \cot \theta\right]_{,0} = -e^{\nu+\mu_3} \delta R_{(0)(3)} = -2Q_* \frac{e^\nu}{r} F_{(2)(3)}, \quad (5)$$

$$\begin{aligned} (\delta\psi + \delta\nu)_{r,\theta} + (\delta\psi - \delta\mu_3)_{,r} \cot \theta - \left(\frac{1}{r} - \nu_{,r}\right) \delta\nu_{,\theta} - \left(\frac{1}{r} + \nu_{,r}\right) \delta\mu_{2,\theta} \\ = -e^{\mu_2+\mu_3} \delta R_{(2)(3)} = -2Q_* \frac{e^{-\nu}}{r} F_{(0)(3)}, \end{aligned} \quad (6)$$

$$\begin{aligned} e^{2\nu} \left[\frac{2}{r} \delta\nu_{,r} + \left(\frac{1}{r} + \nu_{,r}\right) (\delta\psi + \delta\mu_3)_{,r} - 2 \left(\frac{1}{r^2} + 2\frac{\nu_{,r}}{r}\right) \delta\mu_2 \right] \\ + \frac{1}{r^2} [(\delta\psi + \delta\nu)_{,\theta,\theta} + (2\delta\psi + \delta\nu - \delta\mu_3)_{,\theta} \cot \theta + 2\delta\mu_3] \\ - e^{-2\nu} (\delta\psi + \delta\mu_3)_{,0,0} \\ = \delta G_{(2)(2)} = \delta R_{(2)(2)} = 2\frac{Q_*}{r^2} \delta F_{(0)(2)}, \end{aligned} \quad (7)$$

$$\begin{aligned} e^{2\nu} \left[\delta\psi_{,r,r} + 2 \left(\frac{1}{2} + \nu_{,r}\right) + \frac{1}{r} (\delta\psi + \delta\nu + \delta\mu_3 - \delta\mu_2)_{,r} \right. \\ - 2 \left(\frac{1}{2} + 2\nu_{,r}\right) \frac{1}{r} \delta\mu_2 \left. \right] + \frac{1}{r^2} [\delta\psi_{,\theta,\theta} + \delta\psi_{,\theta} \cot \theta + (\delta\psi + \delta\nu \\ - \delta\mu_3 + \delta\mu_2)_{,\theta} \cot \theta + 2\delta\mu_3] - e^{-2\nu} \delta\psi_{,0,0} \\ = -\delta R_{(1)(1)} = 2\frac{Q_*}{r^2} \delta F_{(0)(2)}, \end{aligned} \quad (8)$$

$$re^{-\nu} F_{(0)(3),0} = [re^\nu F_{(2)(3)}]_{,r}, \quad (9)$$

$$\delta F_{(0)(2),0} - \frac{Q_*}{r^2} (\delta\psi + \delta\mu_3)_{,0} + \frac{e^\nu}{r \sin \theta} [F_{(2)(3)} \sin \theta]_{,\theta} = 0, \quad (10)$$

$$\left[\delta F_{(0)(2)} - \frac{Q_*}{r^2} (\delta\nu + \delta\mu_2) \right]_{,\theta} + [re^\nu F_{(3)(0)}]_{,r} + re^{-\nu} F_{(2)(3),0} = 0, \quad (11)$$

are the Maxwell, Einstein, and Ricci tensors, respectively. The variables r and θ in Eqs. (4)–(11) can be separated by the Friedman substitutions [7]

$$\delta\nu = N(r)P_l(\cos\theta), \quad (12)$$

$$\delta\mu_2 = L(r)P_l(\cos\theta), \quad (13)$$

$$\delta\mu_3 = [T(r)P_l(\cos\theta) + V(r)P_{l,\theta,\theta}(\cos\theta)], \quad (14)$$

$$\delta\psi = [T(r)P_l(\cos\theta) + V(r)P_{l,\theta}(\cos\theta)\cot\theta], \quad (15)$$

$$\delta F_{(0)(2)} = \frac{r^2 e^{2\nu}}{2Q_*} B_{(0)(2)}(r)P_l(\cos\theta), \quad (16)$$

$$F_{(0)(3)} = \frac{r^2 e^\nu}{2Q_*} B_{(0)(3)}(r)P_{l,\theta}(\cos\theta), \quad (17)$$

and

$$F_{(2)(3)} = -i\sigma \frac{r^2 e^{-\nu}}{2Q_*} B_{(2)(3)}(r)P_{l,\theta}(\cos\theta). \quad (18)$$

$P_l(\cos\theta)$ are the Legendre functions of order l . We assume that the perturbations can be analyzed into their normal modes with a time dependence $e^{i\sigma t}$. This Fourier decomposition of the perturbations can be done without any loss of generality due to the linearized theory we assume³. Using these substitutions, we obtain the following equations for the radial functions defined by Eqs. (12)–(18):

$$\left[\frac{d}{dr} + \left(\frac{1}{r} - \nu_{,r} \right) \right] [2T - l(l+1)V] - \frac{2}{r}L = 0, \quad (19)$$

$$(T - V + L) = B_{(2)(3)}, \quad (20)$$

$$(T - V + N)_{,r} - \left(\frac{1}{r} - \nu_{,r} \right) N - \left(\frac{1}{r} + \nu_{,r} \right) L = B_{(0)(3)}, \quad (21)$$

$$\begin{aligned} & \frac{2}{r}N_{,r} + \left(\frac{1}{r} + \nu_{,r} \right) [2T - l(l+1)V] - \frac{2}{r} \left(\frac{1}{r} + 2\nu_{,r} \right) L \\ & - \frac{l(l+1)}{r^2} e^{-2\nu} N - \frac{(l-1)(l+2)}{r^2} e^{-2\nu} T + \sigma^2 e^{-4\nu} [2T - l(l+1)V] \\ & = B_{(0)(2)}, \end{aligned} \quad (22)$$

³We note, that these components are frequency-dependent. To obtain components independent of the frequency one should Fourier-transform from the frequency-plane to the temporal-plane.

$$r^4 e^{2\nu} B_{(0)(2)} = 2Q_*^2 [2T - l(l+1)V] - l(l+1)r^2 B_{(2)(3)}, \quad (24)$$

$$\left[r^2 e^{2\nu} B_{(0)(3)} \right]_{,r} + r^2 e^{2\nu} B_{(0)(2)} + \sigma^2 r^2 e^{-2\nu} B_{(2)(3)} = 2Q_*^2 \frac{N+L}{r^2}. \quad (25)$$

Note, that in Eq. (22) we changed the formalism of [3]. We shall now see the reasons for this change in the formalism, which makes the extension of the formalism to $l = 1$ necessary. In Ref. [3], a new radial function X is defined by

$$X = nV = \frac{1}{2}(l-1)(l+2)V.$$

For dipole radiation n , and consequently X , vanish. Hence, it is clear that the variable X – because it vanishes identically for dipole perturbations – cannot carry any information on the original variable V , which is to be calculated. As it is clear that the perturbative terms do not vanish identically (it is well known that there is in general a dipole electromagnetic mode, also in Minkowski spacetime), the formalism of [3] needs to be generalized to be valid for the treatment of dipole radiation too. Furthermore, in Ref. [3] physically-meaningful variables are divided by n or by μ , where $\mu^2 \equiv 2n$. This is clearly inappropriate⁴ for dipole radiation due to the unity value of l and consequently the identically-vanishing values of n and μ .

We re-write Eq. (20) as

$$2T - l(l+1)V = -2 \left[L + \frac{1}{2}(l-1)(l+2)V - B_{(2)(3)} \right], \quad (26)$$

which, after substitution in Eq. (19) yields

$$\begin{aligned} \left[L + \frac{1}{2}(l-1)(l+2)V - B_{(2)(3)} \right]_{,r} = & - \left(\frac{1}{r} - \nu_{,r} \right) \left[L + \frac{1}{2}(l-1)(l+2)V \right. \\ & \left. - B_{(2)(3)} \right] - \frac{1}{r} L. \end{aligned} \quad (27)$$

Combining Eqs. (20),(21), and (23) we obtain

$$(N - L)_{,r} = \left(\frac{1}{r} - \nu_{,r} \right) N + \left(\frac{1}{r} + \nu_{,r} \right) L + \frac{2}{r} B_{(2)(3)} \quad (28)$$

From Eqs. (22),(27), and (28) we find the following equations for the radial functions L, N , and V : (Note, that Eq. (31) is an equation for the variable

⁴See, e.g., Eqs. (180)–(181) of Chapter 5 of [3].

vanishes identically, and therefore is not to be treated.)

$$N_{,r} = aN + bL + c \left[\frac{1}{2}(l-1)(l+2)V - B_{(2)(3)} \right], \quad (29)$$

$$L_{,r} = \left(a - \frac{1}{r} + \nu_{,r} \right) N + \left(b - \frac{1}{r} - \nu_{,r} \right) L + c \left[\frac{1}{2}(l-1)(l+2)V - B_{(2)(3)} \right] - \frac{2}{r} B_{(2)(3)}, \quad (30)$$

$$\begin{aligned} \frac{1}{2}(l-1)(l+2)V_{,r} = & - \left(a - \frac{1}{r} + \nu_{,r} \right) N - \left(b + \frac{1}{r} - 2\nu_{,r} \right) L \\ & - \left(c + \frac{1}{r} - \nu_{,r} \right) \left[\frac{1}{2}(l-1)(l+2)V - B_{(2)(3)} \right] \\ & + B_{(0)(3)}, \end{aligned} \quad (31)$$

where

$$a = \frac{1 + (l-1)(l+2)/2}{r} e^{-2\nu}, \quad (32)$$

$$\begin{aligned} b = & -\frac{1}{r} - \left[\frac{(l-1)(l+2)}{2r} - \frac{M}{r^2} \right] e^{-2\nu} \\ & + \left[\frac{M^2}{r^3} + \sigma^2 r - \frac{Q_*^2}{r^3} (1 + 2e^{2\nu}) \right] e^{-4\nu}, \end{aligned} \quad (33)$$

$$c = -\frac{1}{r} + \frac{1}{r} e^{-2\nu} + \left[\frac{M^2}{r^3} + \sigma^2 r - \frac{Q_*^2}{r^3} (1 + 2e^{2\nu}) \right] e^{-4\nu}. \quad (34)$$

It is important to notice, that for dipole radiation Eq. (31) becomes an algebraic equation rather than a differential equation. (We shall see this in detail when we explicitly discuss the dipole mode.) Eqs. (23),(25),(29),(30), and (31) can be reduced to a pair of second-order equations (and thus allow for a special solution [3]). We now define the following functions: (Notice the difference between these functions and the functions defined in Ref. [3].)

$$\bar{H}_2^{(+)} = rV - \frac{r^2}{\varpi} \left[L + \frac{1}{2}(l-1)(l+2)V - B_{(2)(3)} \right], \quad (35)$$

$$\bar{H}_1^{(+)} = -\frac{1}{Q_*} \left\{ r^2 B_{(2)(3)} + 2Q_*^2 \frac{r}{\varpi} \left[L + \frac{(l-1)(l+2)}{2} V - B_{(2)(3)} \right] \right\}, \quad (36)$$

where $\varpi = (l-1)(l+2)r/2 + 3M - 2Q_*^2/r$. The newly-defined functions satisfy the following coupled equations:

$$\Lambda^2 \bar{H}_2^{(+)} = \frac{\Delta}{r^5} \left\{ \bar{U} \bar{H}_2^{(+)} + \bar{W} \left[-3M \bar{H}_2^{(+)} + 2Q_* \bar{H}_1^{(+)} \right] \right\}, \quad (37)$$

$$\Lambda^2 \bar{H}_1^{(+)} = \frac{\Delta}{r^5} \left\{ \bar{U} \bar{H}_1^{(+)} + \bar{W} \left[2Q_* (l-1)(l+2) \bar{H}_2^{(+)} + 3M \bar{H}_1^{(+)} \right] \right\}, \quad (38)$$

$$\begin{aligned} \bar{U} &= [(l-1)(l+2)r + 3M] \bar{W} + [\varpi - (l-1)(l+2)r - M] \\ &\quad - \frac{(l-1)(l+2)\Delta}{\varpi}, \end{aligned} \quad (39)$$

$$\bar{W} = \frac{\Delta}{r\varpi^2} [(l-1)(l+2)r + 3M] + \frac{(l-1)(l+2)r + M}{\varpi}, \quad (40)$$

and $\Lambda^2 \equiv d^2/dr_*^2 + \sigma^2$, r_* being the Regge-Wheeler ‘tortoise’ co-ordinate defined by $(\Delta/r^2)d/dr = d/dr_*$.

4 DECOUPLING OF THE EQUATIONS – DIPOLE CASE

The decoupling of the equations for the radial functions $\bar{H}_1^{(+)}, \bar{H}_2^{(+)}$ is easier when one first decouples them for the special case $l = 1$, and then uses this case for the determination of parameters for the decoupling of the general equations. In the next section we shall decouple the equations for any l . for $l = 1$, Eqs. (37) and (38) assume the form

$$\Lambda^2 \bar{H}_1^{(+)} = \frac{\Delta}{r^5} (\bar{U} + 3M\bar{W}) \bar{H}_1^{(+)}, \quad (41)$$

$$\Lambda^2 \bar{H}_2^{(+)} = \frac{\Delta}{r^5} [(\varpi - M) \bar{H}_2^{(+)} + 2Q_* \bar{W} \bar{H}_1^{(+)}]. \quad (42)$$

It is important to notice, that Eq. (41) is already decoupled. We shall find it convenient to define new radial functions $\bar{Z}_1^{(+)}, \bar{Z}_2^{(+)}$ by

$$\bar{H}_1^{(+)} = \alpha \bar{Z}_1^{(+)} + \beta \bar{Z}_2^{(+)}, \quad (43)$$

$$\bar{H}_2^{(+)} = \gamma \bar{Z}_1^{(+)} + \delta \bar{Z}_2^{(+)}. \quad (44)$$

Because Eq. (41) is decoupled, we find that for $l = 1$, $\beta = 0$. Substituting Eqs. (43) and (44) in Eqs. (41) and (42), we find that

$$\begin{aligned} \alpha \Lambda^2 \bar{Z}_1^{(+)} &= \alpha \frac{\Delta}{r^5} (\bar{U} + 3M\bar{W}) \bar{Z}_1^{(+)}, \quad (45) \\ \gamma \Lambda^2 \bar{Z}_1^{(+)} + \delta \Lambda^2 \bar{Z}_2^{(+)} &= \frac{\Delta}{r^5} [\gamma(\varpi - M) + 2\alpha Q_* \bar{W}] \bar{Z}_1^{(+)} \\ &\quad + \frac{\Delta}{r^5} \delta(\varpi - M) \bar{Z}_2^{(+)}. \quad (46) \end{aligned}$$

equations, we find that

$$\begin{aligned}\alpha\delta\Lambda^2\bar{Z}_2^{(+)} &= \alpha\frac{\Delta}{r^5}\left[\gamma(\varpi - M) + 2\alpha Q_*\bar{W} - \gamma(\bar{U} + 3M\bar{W})\right]\bar{Z}_1^{(+)} \\ &+ \alpha\delta\frac{\Delta}{r^5}(\varpi - M)\bar{Z}_2^{(+)}.\end{aligned}\quad (47)$$

In order that Eq. (47) indeed be decoupled, the decoupling parameters α and γ must be such that

$$\gamma(\varpi - M) + 2\alpha Q_*\bar{W} - \gamma(\bar{U} + 3M\bar{W}) = 0. \quad (48)$$

We thus obtain that

$$\alpha = \frac{3M\bar{W} + \bar{U} + M - \varpi}{2Q_*\bar{W}}\gamma, \quad (49)$$

or, substituting Eq. (39) for \bar{U} ,

$$\alpha = 3\frac{M}{Q_*}\gamma. \quad (50)$$

We still have the freedom to fix one of the parameters α or γ . Choosing $\alpha = 1/(6M)$ [and, consequently, $\gamma = Q_*/(18M^2)$], we obtain for the decoupled equations (in the $l = 1$ case):

$$\Lambda^2\bar{Z}_1^{(+)} = \frac{\Delta}{r^5}\left(2M - 2\frac{Q_*^2}{r} + 6M\bar{W}\right)\bar{Z}_1^{(+)}, \quad (51)$$

$$\Lambda^2\bar{Z}_2^{(+)} = \frac{\Delta}{r^5}\left(2M - 2\frac{Q_*^2}{r}\right)\bar{Z}_2^{(+)}. \quad (52)$$

We notice, that δ remains free to be fixed arbitrarily.

5 DECOUPLING OF THE EQUATIONS – GENERAL CASE

In this section, we shall decouple Eqs. (37) and (38) for any l . We again use Eqs. (43) and (44), but in this case, of course, β will in general not vanish identically. We thus find that

$$\alpha\Lambda^2\bar{Z}_1^{(+)} + \beta\Lambda^2\bar{Z}_2^{(+)} = \frac{\Delta}{r^5}\left[\alpha\bar{U} + 2\gamma Q_*(l-1)(l+2)\bar{W} + 3\alpha M\bar{W}\right]\bar{Z}_1^{(+)}$$

$$\begin{aligned}
\gamma\Lambda^2\bar{Z}_1^{(+)} + \delta\Lambda^2\bar{Z}_2^{(+)} &= \frac{\Delta}{r^5}(\gamma\bar{U} - 3\gamma M\bar{W} + 2\alpha Q_*\bar{W})\bar{Z}_1^{(+)} \\
&+ \frac{\Delta}{r^5}(\delta\bar{U} - 3\delta M\bar{W} + 2\beta Q_*\bar{W})\bar{Z}_2^{(+)}.
\end{aligned} \tag{54}$$

We now multiply Eq. (53) by γ and Eq. (54) by α . Subtracting the equations we find that

$$\begin{aligned}
(\beta\gamma - \alpha\delta)\Lambda^2\bar{Z}_2^{(+)} &= \frac{\Delta}{r^5} \left[2\gamma^2 Q_*(l-1)(l+2)\bar{W} + 6\alpha\gamma M\bar{W} \right. \\
&- 2\alpha^2 Q_*\bar{W} \left. \right] \bar{Z}_1^{(+)} + \frac{\Delta}{r^5} \left[\beta\gamma\bar{U} + 2\gamma\delta Q_*(l-1)(l+2)\bar{W} + 3\beta\gamma M\bar{W} \right. \\
&- \left. \alpha\delta\bar{U} + 3\alpha\delta M\bar{W} - 2\alpha\beta Q_*\bar{W} \right] \bar{Z}_2^{(+)}.
\end{aligned} \tag{55}$$

We now require that

$$2\gamma^2 Q_*(l-1)(l+2) + 6\alpha\gamma M - 2\alpha^2 Q_* = 0. \tag{56}$$

The solution of this constraint is

$$\alpha = \frac{\gamma \left[3M \pm \sqrt{9M^2 + 4Q_*^2(l-1)(l+2)} \right]}{2Q_*}. \tag{57}$$

To obtain the result of the previous section for the $l = 1$ mode, we choose the positive root. We now define

$$q_1 = 3M + \sqrt{9M^2 + 4Q_*^2(l-1)(l+2)}, \tag{58}$$

and find that

$$\alpha = \frac{q_1}{2Q_*}\gamma. \tag{59}$$

For the $l = 1$ case we find that $q_1 = 6M$, and we thus indeed recover our previous result for the $l = 1$ case [Eq. (50)]. To obtain a corresponding connection between β and δ we multiply Eq. (53) by δ and Eq. (54) by β . Subtracting the equations we find that

$$\begin{aligned}
(\beta\gamma - \alpha\delta)\Lambda^2\bar{Z}_1^{(+)} &= \frac{\Delta}{r^5} \left[\beta\gamma\bar{U} - 3\beta\gamma M\bar{W} + 2\alpha\beta Q_*\bar{W} - \alpha\delta\bar{U} \right. \\
&- \left. 2\gamma\delta(l-1)(l+2)Q_*\bar{W} - 3\alpha\delta M\bar{W} \right] \bar{Z}_1^{(+)} + \frac{\Delta}{r^5} \left[-6\beta\delta M\bar{W} \right. \\
&+ \left. 2\beta^2 Q_*\bar{W} - 2\delta^2(l-1)(l+2)Q_*\bar{W} \right] \bar{Z}_2^{(+)}.
\end{aligned} \tag{60}$$

$$2\beta^2 Q_* - 2\delta^2(l-1)(l+2)Q_* - 6\beta\delta M = 0. \quad (61)$$

The solution of Eq. (61) is:

$$\beta = \frac{\delta \left[3M \pm \sqrt{9M^2 + 4(l-1)(l+2)Q_*^2} \right]}{2Q_*}. \quad (62)$$

Because for $l = 1$ we have $\beta = 0$, we choose the negative root, and define

$$q_2 = 3M - \sqrt{9M^2 + 4Q_*^2(l-1)(l+2)}. \quad (63)$$

We thus find that

$$\beta = \frac{q_2}{2Q_*} \delta. \quad (64)$$

We now fix $\delta = 1/q_1$, and consequently $\beta = q_2/(2q_1Q_*)$. Thus, we found the four parameters of Eqs. (43) and (44), and completed the decoupling of the equations.

6 THE DECOUPLED EQUATIONS

In the previous section we found that Eqs. (43) and (44) can be explicitly written as

$$\bar{H}_1^{(+)} = \frac{1}{q_1} \bar{Z}_1^{(+)} + \frac{q_2}{2q_1 Q_*} \bar{Z}_2^{(+)}, \quad (65)$$

$$\bar{H}_2^{(+)} = \frac{Q_*}{3Mq_1} \bar{Z}_1^{(+)} + \frac{1}{q_1} \bar{Z}_2^{(+)}. \quad (66)$$

Substituting Eqs. (65) and (66) in Eqs. (37) and (38) we find that the differential equations satisfied by $\bar{Z}_1^{(+)}$, $\bar{Z}_2^{(+)}$ are

$$\Lambda^2 \bar{Z}_1^{(+)} = \frac{\Delta}{r^5} \left\{ \bar{U} + \frac{9M^2 \bar{W}}{\sqrt{9M^2 + 4Q_*^2(l-1)(l+2)}} + \frac{[q_1 q_2 - 4Q_*^2(l-1)(l+2)] \bar{W}}{q_2 - q_1} \right\} \bar{Z}_1^{(+)}, \quad (67)$$

$$\Lambda^2 \bar{Z}_2^{(+)} = \frac{\Delta}{r^5} \left\{ \bar{U} - \frac{9M^2 \bar{W}}{\sqrt{9M^2 + 4Q_*^2(l-1)(l+2)}} - \frac{[q_1 q_2 - 4Q_*^2(l-1)(l+2)] \bar{W}}{q_2 - q_1} \right\} \bar{Z}_2^{(+)}. \quad (68)$$

$$\Lambda^2 \bar{Z}_i^{(+)} = \bar{V}_i^{(+)} \bar{Z}_i^{(+)}, \quad (69)$$

where $i = 1, 2$ and

$$\bar{V}_{1,2}^{(+)} = \frac{\Delta}{r^5} \left[\bar{U} \pm \frac{1}{2} (q_1 - q_2) \bar{W} \right]. \quad (70)$$

7 THE COMPLETION OF THE SOLUTION

As five differential equations of the first order are reduced to a pair of second-order equations, it is clear that there is a special solution. This special solution is [3]:

$$N^{(0)} = r^{-2} e^\nu \left[M - \frac{r}{\Delta} (M^2 - Q_*^2 + \sigma^2 r^4) - 2 \frac{Q_*^2}{r} \right] \quad (71)$$

$$L^{(0)} = r^{-3} e^\nu (3Mr - 4Q_*^2) \quad (72)$$

$$V^{(0)} = e^\nu r^{-1} \quad (73)$$

$$B_{(2)(3)}^{(0)} = -2Q_*^2 r^{-3} e^\nu \quad (74)$$

$$B_{(0)(3)}^{(0)} = 2Q_*^2 r^{-6} e^{-\nu} (2Q_*^2 + r^2 - 3Mr). \quad (75)$$

As in Ref. [3], the completion of the solution is given by:

$$\begin{aligned} N &= N^{(0)} \Phi + (l-1)(l+2) \frac{e^{2\nu}}{\varpi} \bar{H}_2^{(+)} - \frac{e^{2\nu}}{\varpi} \left[\frac{1}{2} (l-1)(l+2) r \bar{H}_2^{(+)} \right. \\ &+ \left. Q_* \bar{H}_1^{(+)} \right]_{,r} + \frac{1}{r\varpi^2} \left\{ e^{2\nu} [\varpi - (l-1)(l+2)r - 3M] \right. \\ &- \left. \left[\frac{1}{2} (l-1)(l+2) + 1 \right] \varpi \right\} \left[\frac{1}{2} (l-1)(l+2) r \bar{H}_2^{(+)} + Q_* \bar{H}_1^{(+)} \right] \quad (76) \end{aligned}$$

$$L = L^{(0)} \Phi - \frac{1}{r^2} \left[\frac{1}{2} (l-1)(l+2) r \bar{H}_2^{(+)} + Q_* \bar{H}_1^{(+)} \right] \quad (77)$$

$$V = V^{(0)} \Phi + \frac{1}{r} \bar{H}_2^{(+)} \quad (78)$$

$$B_{(2)(3)} = B_{(2)(3)}^{(0)} \Phi - \frac{Q_*}{r^2} \bar{H}_1^{(+)} \quad (79)$$

$$\begin{aligned} B_{(0)(3)} &= B_{(0)(3)}^{(0)} \Phi - \frac{Q_*}{r^2} \bar{H}_{1,r}^{(+)} \\ &- 2 \frac{Q_*^2}{r^2 \varpi} \left[\frac{1}{2} (l-1)(l+2) r \bar{H}_2^{(+)} + Q_* \bar{H}_1^{(+)} \right] \quad (80) \end{aligned}$$

$$T = B_{(2)(3)} + V - L \quad (81)$$

$$B_{(0)(2)} = r^{-4} e^{-2\nu} \left\{ 2Q_*^2 [2T - l(l+1)V] - l(l+1)r^2 B_{(2)(3)} \right\}, \quad (82)$$

$$\Phi = \int \left[\frac{1}{2}(l-1)(l+2)r\bar{H}_2^{(+)} + Q_*\bar{H}_1^{(+)} \right] \frac{e^{-\nu}}{\varpi r} dr. \quad (83)$$

8 DISCUSSION

The formalism presented here is adequate for the treatment of polar modes of any l , including $l = 1$. Now, we shall see in detail the perturbation formalism for dipole polar perturbations. We can simply substitute a value of unity for l , and obtain the equations for the dipole mode. We observe, that Eq. (31) becomes an algebraic equation rather than a differential equation. This results from the non-radiative character of the dipole gravitational mode. Hence, dynamics is obtained from just one differential equation (of the second order) and not by a pair of second-order differential equations. The expression for Eq. (31) in the case of dipole perturbations then reads

$$\begin{aligned} & - \left(a - \frac{1}{r} + \nu_{,r} \right) N - \left(b + \frac{1}{r} - 2\nu_{,r} \right) L \\ & + \left(c + \frac{1}{r} - \nu_{,r} \right) B_{(2)(3)} + B_{(0)(3)} = 0, \end{aligned} \quad (84)$$

with a, b and c defined by Eqs. (32)–(34). Now, we re-write Eqs. (35) and (36) as

$$\bar{H}_2^{(+)}(l=1) = rV - \frac{r^2}{\varpi}(L - B_{(2)(3)}), \quad (85)$$

and

$$\bar{H}_1^{(+)}(l=1) = -\frac{1}{Q_*} \left\{ r^2 B_{(2)(3)} + 2Q_*^2 \frac{r}{\varpi} [L - B_{(2)(3)}] \right\}, \quad (86)$$

where $\varpi(l=1) = 3M - 2Q_*^2/r$. With these definitions, the differential equation for $\bar{H}_1^{(+)}(l=1)$ is already decoupled from the equation for $\bar{H}_2^{(+)}(l=1)$, and it reads

$$\Lambda^2 \bar{H}_1^{(+)}(l=1) = \frac{\Delta}{r^5} (\tilde{U} + 3M\tilde{W}) \bar{H}_1^{(+)}(l=1), \quad (87)$$

where $\tilde{U} = 3M\tilde{W} + (\varpi - M)$ and $\tilde{W} = 3M\Delta/(r\varpi^2) + M/\varpi$. It turns out, that all the physically meaningful quantities are fully determined by $\bar{H}_1^{(+)}$.

$$N = N^{(0)}\Phi - \frac{e^{2\nu}}{\varpi} Q_* \bar{H}_{1,r}^{(+)} + \frac{Q_*}{r\varpi^2} \left[e^{2\nu} (\varpi - 3M) - \varpi \right] \bar{H}_1^{(+)} \quad (88)$$

$$L = L^{(0)}\Phi - \frac{Q_*}{r^2} \bar{H}_1^{(+)} \quad (89)$$

$$B_{(2)(3)} = B_{(2)(3)}^{(0)}\Phi - \frac{Q_*}{r^2} \bar{H}_1^{(+)} \quad (90)$$

$$B_{(0)(3)} = B_{(0)(3)}^{(0)}\Phi - \frac{Q_*}{r^2} \bar{H}_{1,r}^{(+)} - 2 \frac{Q_*^3}{r^2 \varpi} \bar{H}_1^{(+)} \quad (91)$$

$$B_{(0)(2)} = r^{-4} e^{-2\nu} \left\{ 4Q_*^2 [B_{(2)(3)} - L] - 2r^2 B_{(2)(3)} \right\}, \quad (92)$$

where we use Eqs. (47)–(51) for the definitions of the special functions used in the above equations. The function $\Phi(l=1)$ is

$$\Phi(l=1) = Q_* \int \bar{H}_1^{(+)} \frac{e^{-\nu}}{\varpi r} dr. \quad (93)$$

Using these radial functions, the metric perturbations [through Eqs. (12)–(15)] are given by

$$\delta\nu(l=1) = N(r) \cos \theta \quad (94)$$

$$\delta\mu_2(l=1) = L(r) \cos \theta \quad (95)$$

$$\begin{aligned} \delta\mu_3(l=1) &= \delta\psi \\ &= [B_{(2)(3)} - L] \cos \theta, \end{aligned} \quad (96)$$

and the perturbations of the tetrad components of the Maxwell tensor [Eqs. (16)–(18)] are

$$\delta F_{(0)(2)}(l=1) = \frac{r^2 e^{2\nu}}{2Q_*} B_{(0)(2)}(r) \cos \theta \quad (97)$$

$$F_{(0)(3)}(l=1) = \frac{r e^\nu}{2Q_*} B_{(0)(3)}(r) \sin \theta \quad (98)$$

$$F_{(2)(3)}(l=1) = i\sigma \frac{r e^{-\nu}}{2Q_*} B_{(2)(3)}(r) \sin \theta. \quad (99)$$

To obtain the perturbations for the Schwarzschild solution we cannot just set Q_* equal to zero in Eqs. (69) and (70), because we divided by Q_* in several places during the development of the formalism. However, Eqs. (37) and (38) are already decoupled for the Schwarzschild black hole. This is such because in the Schwarzschild spacetime the electromagnetic and gravitational fields are not coupled as in the Reissner-Nordström spacetime. Hence, one needs not decouple the equations.

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References

- [1] S. Chandrasekhar, Proc. Roy. Soc. Lond. Ser. A. **343**, 289-298 (1975); S. Chandrasekhar and S. Detweiler, *ibid.* **344**, 441-452 (1975).
- [2] S. Chandrasekhar, Proc. Roy. Soc. Lond. Ser. A. **365**, 453-465 (1979); S. Chandrasekhar and B.C. Xanthopoulos, Proc. Roy. Soc. Lond. Ser. A. **367**, 1-14 (1979); B.C. Xanthopoulos, *ibid.* **378**, 73-88 (1981); V. Moncrief, Phys. Rev. D **9**, 2707-2709 (1974); **10**, 1057-1059 (1974); **12**, 1526-1537 (1975); F.J. Zerilli, *ibid.* **9**, 860-868 (1974).
- [3] S. Chandrasekhar, *The Mathematical Theory of Black Holes* (Oxford University Press, Oxford, 1983).
- [4] See also: I.D. Novikov and V.P. Frolov, *Physics of Black Holes* (Kluwer, Dordrecht, 1989); K.S. Thorne, R.H. Price and D.A. Macdonald, *Black Holes: The Membrane Paradigm* (Yale University press, New Haven, 1986).
- [5] R.H. Price, Phys. Rev. D. **5**, 2419-2438 (1972); **5**, 2439-2454 (1972); J. Bičák, Gen. Rel. Grav. **3**, 331-349 (1972); N.R. Sibgatullin and G.A. Alekseev, Zh. Eksp. Teor. Fiz. **67**, 1233-1249 (1974) [Sov. Phys. - JETP **40**, 613-620 (1975)]; C. Gundlach, R.H. Price and J. Pullin, Phys. Rev. D. **49**, 883-889 (1994); **49**, 890-899 (1994).
- [6] L.M. Burko and A. Ori, Phys. Rev. Lett. **74**, 1064-1066 (1995); L.M. Burko and A. Ori, "On Crossing the Cauchy Horizon of a Black Hole," Technion preprint TECHNION-PH-95-14, to be published.
- [7] J.L. Friedman, Proc. Roy. Soc. Lond. Ser. A. **335**, 163-190 (1973).