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The "Dual" Variables Of Yang-Mills Theory And Local Gauge Invariant Variables.

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Abstract

After adding auxiliary fields and integrating out the original variables, the Yang-Mills action can be expressed in terms of local gauge invariant variables. This method reproduces the known solution of the two dimensional $SU(N)$ theory. In more than two dimensions the action splits into a topological part and a part proportional to α_s. We demonstrate the procedure for $SU(2)$ in three dimensions where we reproduce a gravity-like theory. We discuss the four dimensional case as well. We use a cubic expression in the fields as a space-time metric to obtain a covariant Lagrangian. We also show how the four-dimensional $SU(2)$ theory can be expressed in terms of a local action with six degrees of freedom only.

Contents

1 Introduction

One of the fascinating properties of certain quantum field theories is the existence of a strong-weak duality of the coupling constant[1]. Recently, various novel dualities were discovered both in the context of supersymmetric gauge theories[5, 6] and topological field theories [2] as well as in string theories[7]. In certain theories, like the compactified boson in 2D or the abelian gauge theory in 4D, the duality transformations were performed by adding auxiliary fields and integrating out the original variables[3, 4, 5].

In this paper we explore this procedure in non-Abelian non-supersymmetric Yang-Mills theories. It is well known that the latter do not possess a strong-weak duality invariance. (In fact such a duality is usually meaningless when the coupling constant is running.) However, the exciting exact results derived recently[5] taught us that the study of the resulting action in terms of the auxiliary fields maybe very fruitful.

After the integration over the original variables, the resulting action can be expressed in terms of *local gauge invariant* variables. Having a gauge invariant description could be important for the large N limit of $SU(N)$ Yang-Mills theory. The correlation functions of proper gauge invariant variables vanish as $1/N^2$ and thus those variables can be considered classical (Obviously, not every gauge invariant variable has this property). This is the essence of the original master-field idea[8]. In the approach of Migdal and Makeenko[9, 10] the gauge invariant variables are the non-local Wilson loops. More recently, Gross and Gopakumar[11] suggested a different approach where the master field is local but not gauge invariant. This master field corresponded to the gauge field but as a non-commutative random variable.

In the present paper we obtain actions with local gauge invariant variables for the $SU(2)$ theory.

In two-dimensions we obtain, by the above procedure, the $SU(N)$ partition function on a torus. This corresponds to the well-known result[12][13] which is given as a sum over representations of $SU(N)$. In our case, each representation corresponds to a different configuration of the auxiliary field (whose value becomes quantized).

In more than two dimensions the action that we obtain splits into a topological field theory plus a term proportional to the coupling constant α_s. The topological field theory describes the flat gauge configurations at $\alpha_s = 0$. We demonstrate this by reproducing Lunev's result [19] for the three dimensional $SU(2)$ theory. It is expressed as a theory similar to 3D gravity (the topological part) plus a non-covariant coupling proportional to α_s.

The topological field theory of flat gauge connections was introduced in [14] in relation to 2D topological gravity. In [15] theories of flat gauge connections of different groups and for $D > 2$ where written down. The 2D cases were then discussed in [16][17]. In [17] the exact instanton expansion of the 2D partition function was obtained by expressing the action as a topological perturbation (proportional to α_s) to the flat gauge connection theory. The topological theory of flat gauge connections in 4D was recently discussed in[18] in relation to a twisting of the super-symmetric $N = 4$ Yang-Mills theory.

Continuing to 4D in gauge invariant variables, we obtain an expression for the $SU(2)$ action in terms of a (non-positive) metric $g_{\mu\nu}$ and a chiral spin-2 field. We then show that we can restrict to conformal metrics and thus obtain a description of $SU(2)$ in 4D in terms of a local action of six gauge invariant fields only.

We mention again the related results of Lunev[19][20][21]. In [19] the gravitational description of 3D $SU(2)$ was found. In [20] a gravity-like theory of 3D $SU(2)$ was found, but it seems different from our description. In a recent work [21] more relations between gravity and Yang-Mills theories are discussed.

The Hamiltonian approach to the gauge invariant description of Yang-Mills is very interesting [22][23][24]. In this approach the metric is spatial and the wave functional can be factorized[22].

The paper is organized as follows. In section (2) we describe the general framework. In section (3) we rederive the 2D partition function of $SU(N)$ on a torus. In section (4) we discuss $SU(2)$ in 3D and rederive Lunev's result[19]. Section (5) is devoted to $SU(2)$ in 4D. We show how $SU(2)$ can be expressed with only six fields – a chiral spin-2 field and a spin-0 field. Appendix (A) describes the topology of the auxiliary fields for $SU(N)$ in 2D. Appendix (B) describes an algebraic prescription to rewrite the action in gauge invariant variables (which we use in section(4)). Appendices (C-D) describes the detailed calculations for the 4D case.

2 Reformulation of the theory

We start with the Euclidean partition function in a D dimensional space with a metric $g_{\mu\nu}$

$$\begin{aligned} \mathcal{Z}^{(D)}_{(G)} = \int [\mathcal{D}A_\mu][\mathcal{D}F_{\mu\nu}][\mathcal{D}\widetilde{G}_{\mu\nu}] exp\{-\int \sqrt{g}\{\frac{1}{4\alpha_s} F_{\mu\nu,a}F^{\mu\nu,a} \\ + \quad i\widetilde{G}^{\mu\nu,a}(F^a_{\mu\nu} - \partial_\mu A^a_\nu + \partial_\nu A^a_\mu - t^{abc}A^b_\mu A^c_\nu)\} d^D x\} \end{aligned} \tag{1}$$

where $F^a_{\mu\nu}$ is treated as an independent field, $\widetilde{G}^{\mu\nu,a}$ is an anti-symmetric Lagrange multiplier and α_s is the strong coupling constant. The anti-symmetric structure constants are defined as usual

$$[T^a, T^b] = it^{abc}T^c \tag{2}$$

We proceed for simplicity on a flat toroidal space-time. Integrating over $[\mathcal{D}F_{\mu\nu}]$ we obtain

$$\mathcal{Z}^{(D)}_{(G)} = \int [\mathcal{D}A_\mu][\mathcal{D}\widetilde{G}_{\mu\nu}] e^{-\int d^D x \{\alpha_s \widetilde{G}^a_{\mu\nu}\widetilde{G}^a_{\mu\nu} + 2i\partial_\mu \widetilde{G}^a_{\mu\nu} A^a_\nu - i\widetilde{G}^a_{\mu\nu} t^{abc} A^b_\mu A^c_\nu)\}} \tag{3}$$

This action is quadratic in A^a_μ and the quadratic matrix

$$(\widetilde{\mathcal{G}})^{ab}_{\mu\nu} = t^{abc}\widetilde{G}^c_{\mu\nu} \tag{4}$$

is local in the new field variable $\widetilde{G}^c_{\mu\nu}$. For $D > 2$ and a *generic* field configuration $\widetilde{G}^c_{\mu\nu}$, it is also a non-singular matrix and we can define its inverse $(\widetilde{\mathcal{G}}^{-1})^{ab}_{\mu\nu}$:

$$(\widetilde{\mathcal{G}}^{-1})^{ab}_{\mu\nu}(\widetilde{\mathcal{G}})^{bc}_{\nu\tau} = \delta^{ac}\delta_{\mu\tau} \tag{5}$$

Integrating $[\mathcal{D}A_\mu]$ we obtain

$$\mathcal{Z}^{(D>2)}_{(G)} = \int [\mathcal{D}\widetilde{G}_{\mu\nu}] \prod_x (\det \widetilde{\mathcal{G}}(x))^{-1/2} e^{-\int d^D x \{\alpha_s \widetilde{G}^a_{\mu\nu}\widetilde{G}^a_{\mu\nu} + i\partial_\mu \widetilde{G}^a_{\mu\sigma}(\widetilde{\mathcal{G}}^{-1})^{ab}_{\sigma\tau}\partial_\nu \widetilde{G}^b_{\tau\nu}\}} \tag{6}$$

The term $\partial_\mu \widetilde{G}^a_{\mu\sigma}(\widetilde{\mathcal{G}}^{-1})^{ab}_{\sigma\tau}\partial_\nu \widetilde{G}^b_{\tau\nu}$ is invariant only up to a total derivative. After adding a total derivative we can write the Lagrangian as:

$$\begin{aligned}
\mathcal{L} &= \alpha_s \widetilde{G}^a_{\mu\nu}\widetilde{G}^a_{\mu\nu} + 2i\widetilde{G}^a_{\mu\nu}(\widetilde{\mathcal{G}}^{-1})^{ab}_{\nu\tau}\partial_\mu(\widetilde{\mathcal{G}})^{bc}_{\tau\sigma}(\widetilde{\mathcal{G}}^{-1})^{cd}_{\sigma\rho}\partial_\alpha \widetilde{G}^d_{\alpha\rho} \\
&- 2i\widetilde{G}^a_{\mu\nu}(\widetilde{\mathcal{G}}^{-1})^{ab}_{\nu\tau}\partial_\mu\partial_\sigma \widetilde{G}^b_{\sigma\tau} - i(\widetilde{\mathcal{G}}^{-1})^{cd}_{\nu\rho}\partial_\alpha \widetilde{G}^c_{\alpha\nu}\partial_\beta \widetilde{G}^d_{\beta\rho} \\
&= \alpha_s \widetilde{G}^a_{\mu\nu}\widetilde{G}^a_{\mu\nu} + i(\widetilde{\mathcal{G}}^{-1})^{cd}_{\nu\rho}\partial_\alpha \widetilde{G}^c_{\alpha\nu}\partial_\beta \widetilde{G}^d_{\beta\rho} \\
&- 2i\partial_\mu[\widetilde{G}^a_{\mu\nu}(\widetilde{\mathcal{G}}^{-1})^{ab}_{\nu\tau}\partial_\sigma \widetilde{G}^b_{\sigma\tau}]
\end{aligned} \tag{7}$$

To see that it is gauge invariant we note the following observation. If the *independent* variable $\widetilde{G}^a_{\mu\nu}$ were a field strength of some gauge field $\overline{A}^a_\mu$:

$$\widetilde{G}^a_{\mu\nu} = \partial_\mu \overline{A}^a_\nu - \partial_\nu \overline{A}^a_\mu - t^{abc}\overline{A}^b_\mu \overline{A}^c_\nu \tag{8}$$

and if it satisfied an equation of motion (in 4D we could also use the Bianchi identity and let $G^a_{\mu\nu}$ be the original field strength, but we want to keep the dimension D general):

$$D_\mu \widetilde{G}^a_{\mu\nu} \stackrel{def}{=} \partial_\mu \widetilde{G}^a_{\mu\nu} - t^{abc}\overline{A}^b_\mu \widetilde{G}^c_{\mu\nu} = 0 \tag{9}$$

we could solve for $\overline{A}^a_\mu$ and write

$$\overline{A}^a_\mu \stackrel{def}{=} (\widetilde{\mathcal{G}}^{-1})^{ab}_{\mu\nu}\partial_\tau \widetilde{G}^b_{\tau\nu} \tag{10}$$

Clearly $\overline{A}^a_\mu$ transforms as a gauge field when $\widetilde{G}^a_{\mu\nu}$ transforms in the adjoint representation. We can thus define

$$\overline{G}^a_{\mu\nu} = \partial_\mu \overline{A}^a_\nu - \partial_\nu \overline{A}^a_\mu - t^{abc}\overline{A}^b_\mu \overline{A}^c_\nu \tag{11}$$

which transforms like $\widetilde{G}^a_{\mu\nu}$ and we can write the Lagrangian as

$$\mathcal{L} = \alpha_s \widetilde{G}^a_{\mu\nu}\widetilde{G}^a_{\mu\nu} - i\overline{G}^a_{\mu\nu}\widetilde{G}^a_{\mu\nu} \tag{12}$$

which is gauge invariant. The terms in (7) that contain the inverse of $\widetilde{\mathcal{G}}$ must be accompanied with a prescription for the pole when $\widetilde{\mathcal{G}}$ becomes singular. For this purpose we have to add the gauge-breaking term $-\eta A^a_\mu A^a_\mu$ to the action before integrating, and then take $\eta \to 0$. This has the effect of replacing $\widetilde{\mathcal{G}}^{-1}$ with

$$(\widetilde{\mathcal{G}} + i\eta I)^{-1} = P\frac{\mathrm{adj}\widetilde{\mathcal{G}}}{\det \widetilde{\mathcal{G}}} - i\pi\delta(\det \widetilde{\mathcal{G}})(\mathrm{adj}\widetilde{\mathcal{G}})\mathrm{sgn}(\mathrm{tr}\{\mathrm{adj}\widetilde{\mathcal{G}}\}) \tag{13}$$

where $\mathrm{adj}\widetilde{\mathcal{G}} \stackrel{def}{=} (\det \widetilde{\mathcal{G}})\widetilde{\mathcal{G}}^{-1}$. The addition to $\overline{A}^a_\mu$ is

$$-i\pi\delta(\det \widetilde{\mathcal{G}})(\mathrm{adj}\widetilde{\mathcal{G}})^{ac}_{\mu\nu}\,\mathrm{sgn}(\mathrm{tr}\{\mathrm{adj}\widetilde{\mathcal{G}}\})\partial_\alpha \widetilde{G}^a_{\alpha\mu} \tag{14}$$

and it is seen that this addition does not change the gauge transformation properties of $\overline{A}^a_\mu$. When we plug it in (7) we obtain

$$\begin{aligned}\mathcal{L} = \alpha_s \widetilde{G}^a_{\mu\nu}\widetilde{G}^a_{\mu\nu} \; &+ \; i(\widetilde{\mathcal{G}}^{-1})^{cd}_{\nu\rho}\partial_\alpha \widetilde{G}^c_{\alpha\nu}\partial_\beta \widetilde{G}^d_{\beta\rho} - 2i\partial_\mu[\widetilde{G}^a_{\mu\nu}(\widetilde{\mathcal{G}}^{-1})^{ab}_{\nu\tau}\partial_\sigma \widetilde{G}^b_{\sigma\tau}] \\ &+ \; \pi\delta(\det(\widetilde{\mathcal{G}}))\,\mathrm{sgn}(\mathrm{tr}\{\mathrm{adj}\widetilde{\mathcal{G}}\})(\mathrm{adj}\widetilde{\mathcal{G}})^{cd}_{\nu\rho}\partial_\alpha \widetilde{G}^c_{\alpha\nu}\partial_\beta \widetilde{G}^d_{\beta\rho}\end{aligned} \tag{15}$$

It is enough to add the pole prescription just in the term $(\widetilde{\mathcal{G}}^{-1})^{cd}_{\nu\rho}\partial_\alpha \widetilde{G}^c_{\alpha\nu}\partial_\beta \widetilde{G}^d_{\beta\rho}$ and not in the total derivative. It is easy to check that the addition

$$\pi\delta(\det(\widetilde{\mathcal{G}}))\,\mathrm{sgn}(\mathrm{tr}\{\mathrm{adj}\widetilde{\mathcal{G}}\})(\mathrm{adj}\widetilde{\mathcal{G}})^{cd}_{\nu\rho}\partial_\alpha \widetilde{G}^c_{\alpha\nu}\partial_\beta \widetilde{G}^d_{\beta\rho} \tag{16}$$

Is gauge invariant by itself, because of the δ-function.

When the determinant $\det(\widetilde{\mathcal{G}})$ contains multiple roots (as will be the case for $SU(2)$ in $D = 3$ in section (4) where the determinant is a third power of an algebraic expression, and for $SU(2)$ in $D = 4$ in section (5) where the determinant will be quadratic) we will have to be more careful. We will deal with such a situation in chapter (4).

We have seen that the terms in (7) are local invariants. However it is an expression that contains derivatives, but doesn't contain any gauge field. Thus it must be possible to write it in terms of local *algebraic* invariants of $\widetilde{G}^a_{\mu\nu}$. We will demonstrate this in various cases in the next sections.

At $\alpha_s = 0$ we obtain a topological field theory that describes flat gauge connections[14, 17]. The Lagrangian $-i\overline{G}^a_{\mu\nu}\widetilde{G}^a_{\mu\nu}$ thus describes a topological field theory. Furthermore, all the manipulations above have been done in a flat space. It is easy to see that if the term $-\frac{1}{4\alpha_s}\int d^Dx\{F^a_{\mu\nu}F^a_{\mu\nu}\}$ were missing from (1) we keep general covariance, since we can write the Lagrange multiplier $\widetilde{G}^a_{\mu\nu}$ term in terms of forms only:

$$\int \mathrm{tr}\{G \wedge (F - dA - [A, A])\}d^Dx \tag{17}$$

where G is now a $(D-2)$ form. The $[\mathcal{D}A_\mu]$ measure depends on the space-time metric through its volume form. Since the integral is quadratic and we just substitute the solution of the equations of motion, we do not destroy covariance in the topological part of the action

$$\mathcal{L}_{\mathrm{topo}} = -i\overline{G}^a_{\mu\nu}G^a_{\mu\nu} \tag{18}$$

Thus, after we re-write $-i\overline{G}^a_{\mu\nu}G^a_{\mu\nu}$ in gauge invariant variables, we obtain a topological field theory that has general covariance. When $\alpha_s = 0$, only the term $\prod_x(\det\widetilde{\mathcal{G}}(x))^{-1/2}$ in (6) will break general covariance and behave as a density. We will indeed see in section (4) that for three dimensional SU(2) we reproduce Lunev's result [19] and obtain (a "close cousin" of) three dimensional gravity. Three dimensional gravity itself is known to be a topological field theory [25]. For four dimensional $SU(2)$ we obtain another topological field theory. Those theories describe flat gauge connections.

We will start with a discussion of the well-understood two dimensional Yang-Mills theory in this formalism.

3 Rederiving the 2D partition function for $SU(N)$ on a torus

Two dimensional Yang-Mills theory can be solved using several different methods see for instance [12, 13, 17, 11]. The partition function on a torus of area A, for a gauge group G is

$$\mathcal{Z}^{(2D)}_{(G)} = \sum_R e^{-\frac{1}{2}\alpha_s A C_2(R)} \tag{19}$$

where R runs over all representations of the group. It is important to mention that (19) is obtained after we sum over all G-fibre bundles in the functional integral. For a torus, the fibre-bundles are characterized by $\pi_1(G)$. Consider an $SU(N)$ group for G. The difference between $SU(N)/Z_N$ and $SU(N)$ in (19) manifests itself in the functional integral in whether we sum over all $N = \#\,\pi_1(SU(N)/Z_N)$ bundles or just the trivial bundle, since both algebras are the same. The representations R of $SU(N)$ are labeled by sets of increasing natural numbers $0 < l_1 < l_2 < \cdots < l_{N-1}$. The second Casimir is given by

$$C_2(l_1, l_2, \ldots, l_{N-1}) = \frac{1}{2}\sum_{i=1}^{N-1} l_i^2 - \frac{1}{2N}(\sum_{i=1}^{N-1} l_i)^2 \tag{20}$$

3.1 The $SU(2)$ partition function on a torus

Let us begin with $SU(2)$ for which

$$\begin{aligned} a &= 1,2,3 \\ t^{abc} &= \epsilon^{abc} \\ \widetilde{G}^a_{\mu\nu} &= \epsilon_{\mu\nu} g^a \end{aligned} \tag{21}$$

Choosing g_a locally at x to be in the positive $a = 3$ direction, i.e. $g^1 = g^2 = 0$ and $g^3 = g$, the matrix of the quadratic term in A^a_μ in (4) is then:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & -ig & 0 \\ 0 & 0 & 0 & ig & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & ig & 0 & 0 & 0 & 0 \\ -ig & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{22}$$

It has two zero eigenvectors which force two delta functions $\prod_{\nu=1,2}\delta(\partial_\mu \widetilde{G}^3_{\mu\nu})$, after an integration over A^3_μ. The remaining integral (over A^1_μ, A^2_μ) is Gaussian and gives

$$g^{-2}e^{-\frac{i}{g}\epsilon_{\sigma\tau}(\partial_\mu \widetilde{G}^1_{\mu\sigma}\partial_\nu \widetilde{G}^2_{\nu\tau}-\partial_\mu \widetilde{G}^2_{\mu\sigma}\partial_\nu \widetilde{G}^1_{\nu\tau})\Delta^2 x} \tag{23}$$

In the case that g_a is in a general iso-direction we write

$$g^a = g\hat{g}^a, \qquad \sum_{a=1}^{3}\hat{g}^a\hat{g}^a = 1, \qquad g \geq 0 \tag{24}$$

and obtain

$$\prod_\nu \delta(\hat{g}^a\partial_\mu G^a_{\mu\nu}) = \prod_\mu \delta(\partial_\mu g) \tag{25}$$

where we used $\hat{g}^a\partial_\mu\hat{g}^a = 0$. The product of delta-functions means that g is a constant field. However, there is an infinite constant involved because the arguments of the delta functions are related by $\epsilon^{\mu\nu}\partial_\mu\partial_\nu g = 0$. Since this infinite constant is independent of α_s or g we disregard it! We decompose the measure

$$[\mathcal{D}G_{\mu\nu}] = \prod_{x,a} dG^a_{\mu\nu}(x) = \prod_x (g(x)^2 dg(x)(d^2\hat{g}(x)) \tag{26}$$

and obtain

$$\mathcal{Z}^{(2D)}_{(SU(2))} = \int dg\{\int [\mathcal{D}\hat{g}]e^{-2\alpha_s A g^2 - ig\int d^2x\epsilon^{\mu\nu}\epsilon_{abc}\hat{g}^a\partial_\mu\hat{g}^b\partial_\nu\hat{g}^c}\} \tag{27}$$

where A is the area of the torus. Since $\hat{g}$ is a unit iso-vector, we have the identity

$$8\pi n = \int d^2x\epsilon^{\mu\nu}\epsilon_{abc}\hat{g}^a\partial_\mu\hat{g}^b\partial_\nu\hat{g}^c \tag{28}$$

where n is the integer topological number of the map

$$\hat{g} : \Sigma \to S^2$$

from our torus to S^2. More precisely n is the rank of the map

$$\hat{g}^* : H^2(S^2, \mathbf{Z}) \to H^2(\Sigma, \mathbf{Z})$$

The functional integral $[\mathcal{D}\hat{g}]$ decomposes into a sum over integer n. For each n we have a number v_n which is the "volume" of the subspace of maps $\hat{g}^a$ with topological number n. The v_n-s are universal constant numbers, *independent of* α_s. We assume that they are all equal, and we rescale them to one. We obtain

$$\mathcal{Z}^{(2D)}_{(SU(2))} = \int_0^\infty dg \sum_{n=-\infty}^{\infty} e^{-2\alpha_s A g^2 + 8gn\pi i} \tag{29}$$

where again the equality sign is up to a factor that is independent of α_s. Next we write

$$\begin{aligned}
\mathcal{Z}^{(2D)}_{(SU(2))} &= \frac{1}{2}\int_0^\infty dg \sum_{n=-\infty}^{\infty} (e^{-2\alpha_s A g^2 + 8gn\pi i} + e^{-2\alpha_s A g^2 - 8gn\pi i}) \\
&= \frac{1}{2}\int_{-\infty}^\infty dg \sum_{n=-\infty}^{\infty} e^{-2\alpha_s A g^2 + 8gn\pi i}
\end{aligned} \tag{30}$$

We finally obtain

$$\sqrt{\frac{\pi}{8\alpha_s A}} \sum_{n=-\infty}^{\infty} e^{-\frac{8\pi^2}{\alpha_s A} n^2} \tag{31}$$

which, after a Poisson resummation, differs from (19) by a constant $\frac{1}{2}$. This constant comes from the $g = 0$ contribution to the integral. If we had been more careful with the prescription to pass around the pole $g = 0$ we would have obtained the extra $\frac{1}{2}$. We will show this now.

3.1.1 The problem near $g = 0$

From the formula

$$\begin{aligned}
\mathcal{Z}^{(\mathrm{wrong})}_{(SU(2))} &= 2\int_0^\infty dg \sum_{n=-\infty}^{\infty} e^{-2\alpha_s A g^2 + 8gn\pi i} \\
&= \frac{1}{2}\int_0^\infty e^{-2\alpha_s A g^2} \sum_{m=-\infty}^{\infty} \delta(g - \frac{m}{4}) dg \\
&= \sum_{m=1}^{\infty} e^{-\frac{\alpha_s A m^2}{8}} + \frac{1}{2}
\end{aligned} \tag{32}$$

Where as we should obtain

$$\sum_{j=0,\frac{1}{2},\ldots} e^{-\frac{\alpha_s A}{8}(2j+1)^2} \tag{33}$$

So (32) differs from the correct answer just by the contribution of $m = 0$ or, what is the same, from the $\delta(g)$ that appears in the sum over m. We claim that this $\delta(g)$ appears because of a wrong treatment of the $g \approx 0$ region.

At $g = 0$ the matrix $\widetilde{\mathcal{G}}$ at (22) becomes zero. For a regularization we add $-\eta A^a_\mu A^a_\mu$ to the original Lagrangian. We thus have

$$\mathcal{L} = 2\alpha_s g^a g^a - 2i\epsilon_{\mu\nu}\partial_\mu g^a A^a_\nu - (ig^a\epsilon^{abc}\epsilon_{\mu\nu} + \eta\delta^{bc}\delta_{\mu\nu})A^b_\mu A^c_\nu \qquad (34)$$

The inverse of the quadratic matrix

$$\mathbf{M}^{bc}_{\mu\nu} = ig^a\epsilon^{abc}\epsilon_{\mu\nu} + \eta\delta^{bc}\delta_{\mu\nu} \qquad (35)$$

is

$$(\mathbf{M}^{-1})^{ab}_{\mu\nu} = \frac{1}{\eta^2 + g^2}(\eta\delta^{ab}\delta_{\mu\nu} - ig^c\epsilon^{abc}\epsilon_{\mu\nu} + \eta^{-1}g^a g^b\delta_{\mu\nu}) \qquad (36)$$

Integrating out A^a_μ we obtain

$$+\frac{\eta}{\eta^2 + g^2}|\partial_\mu g^a|^2 + \frac{1}{4\eta(\eta^2 + g^2)}|\partial_\mu(g^2)|^2 - \frac{i}{\eta^2 + g^2}\epsilon_{\mu\nu}\epsilon^{abc}g^a\partial_\mu g^b\partial_\nu g^c \qquad (37)$$

For $g \gg \eta$ the second term produces the delta function that forces $g^2 =$ const , while the first term can be ignored and the last term produces the topological invariant. We see that for $g \gg \eta$ indeed all topological sectors appear with the same weight (since the coefficient of $|\partial_\mu g^a|^2$ in the first term is negligible). Thus for $g \neq 0$ we indeed get $\sum_{m\neq 0}\delta(g - \frac{m}{4})$. However as $g \approx \eta$ the first term $|\partial_\mu g^a|^2$ damps the fluctuations of $\hat{g}^a$. The result of this is that the sum $\sum_n e^{8\pi i g n}$ is finite, and we do not get the $\delta(g)$ term.

3.2 Generalization to $SU(N)$

Let us choose a Cartan subalgebra of the Lie algebra and denote its generators by T^i with $i = 1, \ldots, N-1$. The rest of the roots will be denoted by T^α. As in the $SU(2)$ case, where we parameterized $\frac{1}{2}\epsilon_{\mu\nu}\widetilde{G}^a_{\mu\nu}$ by g and $\hat{g}^a$ we now parameterize $\epsilon_{\mu\nu}\widetilde{G}^a_{\mu\nu}$ by

$$\widetilde{G}^a_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu}\sum_{i=1}^{N-1} H^i(\hat{g}T^i\hat{g}^{-1})^a \qquad (38)$$

where $\sum_i H^i T^i$ is an element of the Cartan subalgebra which is conjugate to $\widetilde{G}^a_{\mu\nu}$. For generic $\widetilde{G}^a_{\mu\nu}$, the H^i-s are unique up to the Weyl group S_N. $\hat{g}$ is the (generically) unique element from the coset

$$\hat{g} \in SU(N)/U(1)^{N-1} \tag{39}$$

where $U(1)^{N-1}$ represents the Cartan torus in $SU(N)$ that corresponds to our choice of Cartan subalgebra. $\hat{g}$ is represented by $\hat{g} \in SU(N)$ such that $\hat{g} \equiv \hat{g}h$ for $h \in U(1)^{N-1}$. The matrix $\widetilde{\mathcal{G}}(x)$ has now $2(N-1)$ zero modes, two for each Cartan direction T^i. The integration over A^i_μ in the zero mode direction will, as before, produce a constancy condition:

$$\partial_\mu H^i = 0, \qquad i = 1, \ldots, N-1 \tag{40}$$

Choosing locally at x, $\hat{g}(x) = 1$ we find that the remaining terms decouple for each positive root α:

$$iA^\alpha_\nu \epsilon_{\mu\nu} \partial_\nu G^{-\alpha} + iA^{-\alpha}_\nu \epsilon_{\mu\nu} \partial_\nu G^\alpha - i\epsilon_{\mu\nu} (\sum_i \alpha(i) H^i) A^\alpha_\nu A^{-\alpha}_\mu \tag{41}$$

where

$$G^a \stackrel{def}{=} \frac{1}{2} \epsilon_{\mu\nu} \widetilde{G}^a_{\mu\nu} \tag{42}$$

After the integration we have in the exponent

$$\begin{aligned} & \sum_\alpha \frac{1}{\sum_i \alpha(i) H^i} \epsilon_{\mu\nu} \partial_\nu G^\alpha \partial_\mu G^{-\alpha} \\ = & \sum_\alpha \frac{1}{\sum_i \alpha(i) H^i} \epsilon_{\mu\nu} \sum_{i,j} H^i H^j \partial_\nu \mathrm{tr}\{\hat{g} T^i \hat{g}^{-1} T^\alpha\} \partial_\mu \mathrm{tr}\{\hat{g} T^i \hat{g}^{-1} T^{-\alpha}\} \end{aligned} \tag{43}$$

and in front of the exponent we have

$$\frac{1}{\prod_\alpha (\sum_i \alpha(i) H^i)^2} \tag{44}$$

which cancels with the Jacobian for passing from $\prod_a dG^a$ to $\prod_i dH^i d^{(N^2-N)} \hat{g}$. In fact, it is the well known square of the Van-Der-Monde of H^i in an appropriate basis for the Cartan subalgebra.

At x, since $\hat{g}(x) = 1$ we have

$$\begin{aligned}\sum_i H^i \partial_\mu \mathrm{tr}\{\hat{g} T^i \hat{g}^{-1} T^\alpha\} &= -\sum_i H^i \mathrm{tr}\{T^i [T^\alpha, \hat{g}^{-1}\partial_\mu \hat{g}]\} \\ = \sum_i H^i \mathrm{tr}\{[T^i, T^\alpha]\hat{g}^{-1}\partial_\mu \hat{g}\} &= (\sum_i \alpha(i) H^i) \mathrm{tr}\{T^\alpha \hat{g}^{-1}\partial_\mu \hat{g}\}\end{aligned}$$

The exponent (43) reads, at x,

$$\sum_\alpha (\sum_i \alpha(i) H^i) \epsilon_{\mu\nu} \mathrm{tr}\{T^{-\alpha} \hat{g}^{-1}\partial_\nu \hat{g}\} \mathrm{tr}\{T^\alpha \hat{g}^{-1}\partial_\mu \hat{g}\} \tag{45}$$

Since

$$\hat{g}^{-1}\partial_\mu \hat{g} = \sum_i T^i \mathrm{tr}\{T^i \hat{g}^{-1}\partial_\mu \hat{g}\} + \sum_\alpha T^{-\alpha} \mathrm{tr}\{T^\alpha \hat{g}^{-1}\partial_\mu \hat{g}\} + \sum_\alpha T^\alpha \mathrm{tr}\{T^{-\alpha} \hat{g}^{-1}\partial_\mu \hat{g}\} \tag{46}$$

we can write the exponent as

$$i\epsilon_{\mu\nu} \sum_i H^i \mathrm{tr}\{T^i [\hat{g}^{-1}\partial_\mu \hat{g}, \hat{g}^{-1}\partial_\nu \hat{g}]\} \tag{47}$$

But the expressions

$$\frac{1}{2\pi} \int \epsilon_{\mu\nu} \mathrm{tr}\{T^i [\hat{g}^{-1}\partial_\mu \hat{g}, \hat{g}^{-1}\partial_\nu \hat{g}]\} d^2x \tag{48}$$

are integers. In fact (see Appendix (A)) they are the pullbacks by $\hat{g}$ of a basis of the second integer cohomology group $H^2(SU(N)/U(1)^{N-1}, Z)$ back to the torus. So we write

$$n_i = \frac{1}{2\pi} \int \epsilon_{\mu\nu} \mathrm{tr}\{T^i [\hat{g}^{-1}\partial_\mu \hat{g}, \hat{g}^{-1}\partial_\nu \hat{g}]\} d^2x \tag{49}$$

and $n_1, \ldots, n_{N-1}$ characterize the topological sector of the map $\hat{g}$. We are left with:

$$\sum_{\{n_i\}} \int \frac{1}{N!} \prod_i dH^i e^{-2\alpha_s \sum_{i,j} H^i H^j \mathrm{tr}\{T^i T^j\} - 8\pi i \sum_i H^i n_i} \tag{50}$$

Using (see Appendix (A))

$$\mathrm{tr}\{T^i T^j\} = \delta^{ij} - \frac{1}{N} \tag{51}$$

and substituting

$$\sum_{n_i} e^{-8\pi i H^i n_i} = \frac{1}{4}\sum_{l_i} \delta(H^i - \frac{1}{4}l_i), \tag{52}$$

we obtain

$$\frac{1}{4^{N-1}N!}\sum_{\{l_i\}} e^{-\frac{1}{2}\alpha_s(\sum_{i=1}^{N-1} l_i^2 - \frac{1}{N}(\sum_{i=1}^{N-1} l_i)^2)} \tag{53}$$

where the $N!$ is the size of the Weyl group. Again the remaining terms in (19-20) come from carefully taking care of the points where the matrix $\widetilde{\mathcal{G}}$ is more singular than usual. These are points where $\sum_i \alpha(i)H^i = 0$ for some root α. In our case, these are points where $H^i = H^j$ for some $i \neq j$, or $H^i = 0$ for some i. Those points must be excluded from the δ-function as in the $SU(2)$ case. Defining

$$D(m_1, \ldots, m_N) \stackrel{def}{=} \frac{1}{N}\sum_{i<j}(m_i - m_j)^2 \tag{54}$$

We can write

$$\sum_{i=1}^{N-1} l_i^2 - \frac{1}{N}(\sum_{i=1}^{N-1} l_i)^2 = D(0, l_1, \ldots, l_{N-1}) \tag{55}$$

Noting that $D(m_1, \ldots, m_N)$ is invariant under permutations and that for any integer m:

$$D(m_1, \ldots, m_N) = D(m_1 - m, \ldots, m_N - m) \tag{56}$$

We see that we can write (19-20) as

$$\frac{1}{N!}\sum_{\substack{l_1, \ldots, l_{N-1} \\ l_i \neq 0, l_i \neq l_j}} e^{-\frac{1}{2}\alpha_s A D(0, l_1, \ldots, l_N)} \tag{57}$$

Which is the same as (53) with the above restrictions on H^i-s.

4 Three dimensional $SU(2)$

Proceeding to three dimensions, we wish to rewrite the YM Lagrangian in terms of only gauge invariant variables.

We define the " magnetic" field B_i^a by

$$\widetilde{G}_{ij}^a = \epsilon_{ijk} B_k^a \tag{58}$$

The final variables will be bilinear in the field strength. We further define the symmetric semi-positive matrix

$$T_{ij} \stackrel{def}{=} B_i^a B_j^a \tag{59}$$

We mention in advance a puzzle that might arise about our plan. It is known[26, 27] that in three dimensions the invariant bilinears T_{ij} are not enough to completely describe a gauge configuration A_i^a that yields B_i^a. Namely, there can be two different configurations of A_i^a with identical T_{ij} but different $B_i^a (D_j B_k)^a$ where D_j is the covariant derivative (although this is not the generic situation) [27]. In our case, however, it is important to remember that the T_{ij}-s are not bilinear in the original field strength but in the *auxiliary* fields, thus there is no conflict.

4.1 The action

In general the inverse matrix $\widetilde{\mathcal{G}}^{-1}$ can be expressed as a rational function in $\widetilde{G}_{\mu\nu}^a$ with a denominator that is $\det(\widetilde{\mathcal{G}})$ of degree $D(N^2-1)$. For three-dimensional $SU(2)$ we can reduce the degree from 9 to 3. This is related to the fact that $\det(\widetilde{\mathcal{G}})$ is a third power of a cubic polynomial. Defining

$$\Delta \stackrel{def}{=} \frac{1}{6} \epsilon^{abc} \epsilon_{ijk} B_i^a B_j^b B_k^c \tag{60}$$

we have

$$\Delta^2 = \det(T_{ij}) \tag{61}$$

we get

$$(\widetilde{\mathcal{G}}^{-1})_{ij}^{ab} = \frac{1}{2\Delta} (B_i^a B_j^b - 2 B_j^a B_i^b) \tag{62}$$

Plugging this in (7) we obtain

$$\begin{aligned} \epsilon_{ijp} B_p^a \overline{G}_{ij}^a &= \frac{1}{\Delta} \epsilon_{klm} \epsilon_{pji} T_{pk} \partial_i \partial_m T_{jl} + \frac{1}{\Delta} T_{kp} \epsilon_{klm} \epsilon_{pij} \partial_m T_{lj} \partial_i \log \Delta \\ &+ \frac{1}{4\Delta} D_{ij} T_{st} (\epsilon_{slp} \epsilon_{tmk} - \epsilon_{spk} \epsilon_{tlm}) \partial_p T_{ki} \partial_m T_{lj} + \frac{3}{4\Delta} \epsilon_{klm} \epsilon_{pij} \partial_p T_{ki} \partial_m T_{lj} \end{aligned} \tag{63}$$

where

$$\begin{aligned} D_{ij} &= \frac{1}{2\Delta^2}\epsilon_{jst}\epsilon_{ilm}T_{sl}T_{mt} \\ T_{ij}D_{jk} &= \delta_{ik} \end{aligned}$$

After integration by parts the topological part of the action takes the form

$$\begin{aligned} S &= i\int\{\frac{1}{4\Delta}D_{ij}T_{st}(\epsilon_{slp}\epsilon_{tmk} - \epsilon_{spk}\epsilon_{tlm})\partial_p T_{ki}\partial_m T_{lj} \\ &- \frac{1}{4\Delta}\epsilon_{klm}\epsilon_{pij}\partial_p T_{ki}\partial_m T_{lj}\} \end{aligned} \tag{64}$$

This reproduces the result of Lunev[19] since it is easy to check that if we take $g_{ij} = T_{ij}$ to be the metric, then we can write

$$\mathcal{L} = i\sqrt{g}R \tag{65}$$

where R is the curvature built out of the metric. We see that we get a topological theory (3D gravity) as expected.

There are still two more things to be taken care of: The first has to do with the fact that Δ can have either $\pm$ sign, where as $\sqrt{g}$ is defined to be positive. The second and related problem, is the prescription of passing around zeroes of Δ.

4.2 Going round $\Delta = 0$

It remains to calculate the analog of (16). Since $\det(\widetilde{\mathcal{G}})$ is a cubic power, (16) has to be modified. We can write

$$\begin{aligned} (\widetilde{\mathcal{G}}^{-1})^{ab}_{ij} &= \frac{1}{\Delta}K^{ab}_{ij} \\ K^{ab}_{ij} &\stackrel{def}{=} \frac{1}{2}(B^a_i B^b_j - 2B^a_j B^b_i) \end{aligned}$$

Now we can write the addition due to the pole prescription as

$$\delta\mathcal{L}_{PP} = \pi\delta(\Delta)\,\mathrm{sgn}(\mathrm{tr}\{\mathrm{adj}K\})K^{ab}_{ij}\epsilon_{ipk}\partial_p B^a_k\epsilon_{jql}\partial_q B^b_l \tag{66}$$

We can put

$$\mathrm{sgn}(\mathrm{tr}\{\mathrm{adj}K\}) = -sgn(B^a_i B^a_i) = -1 \tag{67}$$

and obtain

$$\delta\mathcal{L}_{PP} = -\pi\delta(\Delta)K^{ab}_{ij}\epsilon_{ipk}\partial_p B^a_k \epsilon_{jql}\partial_q B^b_l \tag{68}$$

which is invariant at points x where $\Delta(x) = 0$. At such a point there is a direction $\hat{n}_i$ such that

$$B^a_i \hat{n}_i = 0, \qquad a = 1, 2, 3 \tag{69}$$

We can write the invariant projection matrix on this direction as

$$P_{ij} \stackrel{def}{=} \hat{n}_i \hat{n}_j = \frac{T_{ik}T_{kj} - T_{kk}T_{ij}}{\frac{1}{2}(T_{kk}T_{ll} - T_{kl}T_{lk})} + \delta_{ij} \tag{70}$$

which can easily be checked by diagonalizing T_{ij}. Next, choose for simplicity $\hat{n}$ in the $\hat{x}_3$ direction, i.e. $\hat{n}_i = \delta_{i3}$. We denote by $\hat{i}, \hat{j}, \hat{k}, \ldots$ indices that run only on $1, 2$. We further denote

$$\epsilon_{\hat{i}\hat{j}} \stackrel{def}{=} \epsilon_{\hat{i}\hat{j}3} \tag{71}$$

and use

$$\begin{aligned} K^{ab}_{\hat{i}\hat{j}}\epsilon_{\hat{j}\hat{l}}\partial_3 B^b_{\hat{l}} &= -\frac{1}{2}\epsilon_{\hat{j}\hat{l}} B^a_{\hat{j}} \partial_3 T_{\hat{l}\hat{i}} \qquad \text{at } x = 0 \\ \partial_i T_{j3} &= B^a_j \partial_i B^a_3 \qquad \text{at } x = 0 \end{aligned}$$

We find

$$\begin{aligned} K^{ab}_{ij}\epsilon_{ipk}\partial_p B^a_k \epsilon_{jql}\partial_q B^b_l &= -\frac{1}{4}\epsilon_{\hat{j}\hat{l}}\epsilon_{\hat{i}\hat{k}}\partial_3 T_{\hat{j}\hat{k}}\partial_3 T_{\hat{l}\hat{i}} \\ +\epsilon_{\hat{j}\hat{l}}\epsilon_{\hat{i}\hat{k}}\partial_{\hat{k}} T_{\hat{j}3}\partial_3 T_{\hat{l}\hat{i}} &+ \frac{1}{2}\epsilon_{\hat{j}\hat{l}}\epsilon_{\hat{i}\hat{k}}\partial_{\hat{k}} T_{\hat{i}3}\partial_{\hat{l}} T_{\hat{j}3} - \epsilon_{\hat{j}\hat{l}}\epsilon_{\hat{i}\hat{k}}\partial_{\hat{k}} T_{\hat{j}3}\partial_{\hat{l}} T_{\hat{i}3} \end{aligned}$$

Using the invariant projection operator we can write this as

$$\begin{aligned} \delta\mathcal{L}_{PP} &= -\pi\delta(\Delta)P_{pq}P_{st}\epsilon_{jlp}\epsilon_{ikq}(\partial_k T_{js}\partial_t T_{li} \\ &- \frac{1}{4}\partial_s T_{jk}\partial_t T_{li} + \frac{1}{2}\partial_k T_{is}\partial_l T_{jt} - \partial_k T_{js}\partial_l T_{it}) \\ P_{ij} \stackrel{def}{=} \hat{n}_i\hat{n}_j &= \frac{T_{ik}T_{kj} - T_{kk}T_{ij}}{\frac{1}{2}(T_{kk}T_{ll} - T_{kl}T_{lk})} + \delta_{ij} \end{aligned} \tag{72}$$

4.3 The full functional integral

The local change of variables from B_i^a to T_{ij} is of course accompanied by a Jacobian which joins with the factor

$$\frac{1}{\sqrt{\det(\widetilde{\mathcal{G}})}} = \Delta^{-3/2} \tag{73}$$

The Jacobian of the transformation is

$$J(T) \stackrel{def}{=} \int \prod_{i \le j} \delta(T_{ij} - B_i^a B_j^a) \prod dB_i^a = \text{const } \times \det(T)^{-1/2} = \text{const } \times |\Delta|^{-1} \tag{74}$$

The functional integral now looks like

$$\sum_{sgn(\Delta(x))} \int [\mathcal{D}T_{ij}] \prod_x (\frac{1}{\sqrt{\Delta^3}}) exp\{-\int (\frac{iS(T_{ij})}{\Delta} - \alpha_s \text{tr}\{T\})\} \tag{75}$$

where the generally covariant integration measure is

$$[\mathcal{D}T_{ij}] \stackrel{def}{=} \prod_x (\frac{1}{\sqrt{T}} \prod_{i \le j} dT_{ij}) = \prod_x (|\Delta|^{-1} \prod_{i \le j} dT_{ij}) \tag{76}$$

There are several problems with this action because of the sum over the sign of each Δ. The entire space is divided into regions separated by surfaces where $\Delta = 0$. In each region, we have to sum over the *global* sign of Δ in that region. In the vicinity of the surfaces of zero Δ we have to put in the addition (72). The problems arise because of two reasons. First, when we choose the negative sign for Δ, the term $\frac{1}{\sqrt{\Delta^3}}$ contains an extra i. Since we have to multiply those terms for every x, there is a phase ambiguity. Furthermore, at the boundaries $\Delta = 0$ and $\frac{1}{\sqrt{\Delta^3}}$ gives an infinite contribution.

We will now propose a way out, though we do not know if it is really well defined. The problems really started because we changed the order of the integration in (1), which is not absolutely converging. The pole prescription (13) makes the integral absolutely converging but is not enough to settle the phase ambiguity in (75). We propose to put the problematic term $\frac{1}{\sqrt{\Delta^3}}$ back into the exponential where it came from, by adding an *invariant* field ϕ and write

$$\sum_{sgn(\Delta(x))} \int [\mathcal{D}\phi] \int [\mathcal{D}T_{ij}] exp\{-\int (\frac{iS(T_{ij})}{\Delta} - \pi\delta(\Delta)P(T_{ij}) - \alpha_s \text{tr}\{T\} - i\Delta^3\phi^2)\} \tag{77}$$

where we added the pole prescription term $\pi\delta(\Delta)P(T_{ij})$ from (72). The integration over ϕ should be performed at the end, and can be done with a regulator $-\eta\phi^2$.

5 Four dimensional $SU(2)$

In four dimensions, our invariant variables will again be bilinear expressions in the field strength. However, as there are $D(D-1)/2 = 6$ field strengths, there are 21 bilinear variables. Under $O(4)$ they decompose into real representations as

$$21 = \underline{\mathbf{1}} \oplus \underline{\mathbf{1}} \oplus \underline{\mathbf{9}} \oplus \underline{\mathbf{10}} \tag{78}$$

(the two $\underline{\mathbf{1}}$-s are $G^a_{\mu\nu}G^a_{\mu\nu}$ and $\widetilde{G}^a_{\mu\nu}G^a_{\mu\nu}$, the $\underline{\mathbf{9}}$ is the traceless symmetric tensor $G^a_{\mu\tau}G^a_{\tau\nu}-$ (trace), and $\underline{\mathbf{10}} = \underline{\mathbf{5}} \oplus \underline{\mathbf{5}}^*$ where ($\underline{\mathbf{5}}^*$) $\underline{\mathbf{5}}$ is the product of two (anti-)self-dual parts (minus trace). The number of degrees of freedom minus gauge degrees is $3*6-3=15$ so we have to use an overcomplete set of variables if we wish to use only bilinear variables and keep $O(4)$ invariance.

Although the representation $\underline{\mathbf{5}}$ describes self-dual tensors, there is a way to use it without the anti-self-dual parts. We will see shortly that a combination that is cubic in the field strength describes a (non-positive) metric which makes the field strengths $G^a_{\mu\nu}$ self-dual!

5.1 Algebraic identities

Our aim is again to write (7) in terms of gauge invariant variables. We start with some interesting algebraic properties of our $SU(2)$ variables. The algebraic facts in this subsection, can all be checked with a tedious but straightforward calculation.

Recalling that

$$G^a_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\sigma\tau}\widetilde{G}^a_{\sigma\tau} \tag{79}$$

The equation for the inverse is:

$$\begin{aligned}(\widetilde{\mathcal{G}}^{-1})^{ab}_{\mu\nu} &= \frac{1}{\Delta}[\frac{1}{8}(G^a_{\alpha\beta}\widetilde{G}^b_{\gamma\delta} - G^b_{\alpha\beta}\widetilde{G}^a_{\gamma\delta})(\epsilon^{def}G^d_{\mu\nu}\widetilde{G}^e_{\alpha\beta}G^f_{\gamma\delta}) \\ &- \frac{1}{6}G^a_{\gamma\delta}\widetilde{G}^b_{\gamma\delta}(\epsilon^{def}G^d_{\mu\alpha}\widetilde{G}^e_{\alpha\beta}G^f_{\beta\nu})]\end{aligned} \tag{80}$$

$$\Delta = \frac{1}{48}(\epsilon^{abc}G^a_{\alpha\beta}G^b_{\gamma\delta}G^c_{\sigma\tau})(\epsilon^{def}\widetilde{G}^d_{\alpha\beta}\widetilde{G}^e_{\gamma\delta}\widetilde{G}^f_{\sigma\tau}) \tag{81}$$

The determinant is

$$\det(\widetilde{\mathcal{G}}) = \frac{\Delta^2}{4} \tag{82}$$

We further define a symmetric tensor

$$g_{\mu\nu} = \frac{1}{3\Delta^{1/3}}\epsilon^{def}G^d_{\mu\alpha}\widetilde{G}^e_{\alpha\beta}G^f_{\beta\nu} \tag{83}$$

$$g^{\mu\nu} = \frac{2}{3\Delta^{2/3}}\epsilon^{def}\widetilde{G}^d_{\mu\alpha}G^e_{\alpha\beta}\widetilde{G}^f_{\beta\nu} \tag{84}$$

As the notation suggests $g_{\mu\nu}$ is a symmetric tensor (10 components) and $g^{\mu\nu}$ is its inverse. We have

$$\det_{4\times4}(g_{\mu\nu}) = \frac{1}{4}\Delta^{2/3} \tag{85}$$

The Δ scalings in (83-84) have been chosen so that $g_{\mu\nu}$ will be a covariant tensor. The metric (83) has the important property that it makes the field strengths *self-dual*.

$$\frac{1}{2\sqrt{g}}\epsilon^{\alpha\beta\tau\sigma}g_{\mu\alpha}g_{\nu\beta}G^a_{\tau\sigma} = G^a_{\mu\nu} \tag{86}$$

Actually, the distinction between self-dual and anti-self-dual here is just what sign we take in $\sqrt{g} = \pm\frac{1}{2}\Delta^{-2/3}$. Since the metric $g_{\mu\nu}$ is not necessarily positive definite, we can choose either sign, and we need to sum over a *global* sign, just like the summation over signs in the 3D case in (75).

5.2 Gauge invariant variables

Up to now we have not paid much attention to the distinction between covariant and contra-variant tensors, because we were working in flat Euclidean space-time. Now that we have chosen the metric (83) we will make this distinction. The original field $G^a_{\mu\nu}$ is by definition covariant. We now define

$$F^a_{\mu\nu} \stackrel{def}{=} G^a_{\mu\nu} = g_{\mu\alpha}g_{\nu\beta}\widetilde{G}^{\alpha\beta,a}$$

$$F^{\mu\nu,a} \stackrel{def}{=} g^{\mu\alpha}g^{\nu\beta}G^a_{\alpha\beta} = \widetilde{G}^{\alpha\beta,a}$$

From now on, raising and lowering of indices are with respect to (83-84).

We can write (80) as:

$$(\widetilde{\mathcal{G}}^{-1})^{ab}_{\mu\nu} = -\frac{1}{2} g^{\alpha\beta} F^a_{\mu\alpha} F^b_{\beta\nu} - \frac{1}{4} F^{\alpha\beta,a} F^b_{\alpha\beta}\, g_{\mu\nu} \tag{87}$$

The 10 fields of the metric $g_{\mu\nu}$ plus the 5 fields of the traceless, $g_{\mu\nu}$-self-dual tensor

$$W_{\mu\nu\sigma\tau} = F^a_{\mu\nu} F^a_{\sigma\tau} - \frac{1}{24\sqrt{g}} \epsilon^{\alpha\beta\gamma\delta} F^a_{\alpha\beta} F^a_{\gamma\delta} (g_{\mu\sigma} g_{\nu\tau} - g_{\mu\tau} g_{\nu\sigma} + \sqrt{g}\epsilon_{\mu\nu\sigma\tau}) \tag{88}$$

form 15 independent variables, which is exactly the number of degrees of freedom that are left after eliminating the gauge degrees. We note, however, that the metric (83) is not necessarily positive definite.

The topological Lagrangian (7) can be written in terms of the "metric" and the field $W_{\mu\nu\sigma\tau}$. The resulting expression is rather cumbersome, and is described in appendix (C). We define

$$\Phi \stackrel{def}{=} \frac{1}{24\sqrt{g}} \epsilon^{\mu\nu\sigma\tau} G^a_{\mu\nu} G^a_{\sigma\tau} \tag{89}$$

Φ is a scalar field which, given that metric, can be written in terms of the chiral spin-2 field W. We show this in Appendix (D).

The non-covariant α_s-dependent term is

$$\begin{aligned} & \alpha_s G^a_{\mu\nu} G^a_{\mu\nu} = \alpha_s \delta^{\mu\sigma} \delta^{\nu\tau} \widetilde{G}^a_{\mu\nu} \widetilde{G}^a_{\sigma\tau} \\ = \; & \alpha_s \delta^{\mu\sigma} \delta^{\nu\tau} \{ W_{\mu\nu\sigma\tau} + (g_{\mu\sigma} g_{\nu\tau} - g_{\mu\tau} g_{\nu\sigma}) \Phi \} \end{aligned} \tag{90}$$

where we used (88) and (89).

5.3 Reduction to conformal metrics

So far we have a description with 10 + 5 fields. However, we can in fact, restrict to conformal metrics. What happens if in (1) we restrict $\widetilde{G}^a_{\mu\nu}$ to be self-dual? This corresponds to an original Yang-Mills action

$$\frac{1}{4\alpha_s} \int F^a_{\mu\nu} F^a_{\mu\nu} d^4x - \frac{1}{4\alpha_s} \int F^a_{\mu\nu} \widetilde{F}^a_{\mu\nu} d^4x \tag{91}$$

Note that θ angle is imaginary. When we restrict to a trivial $SU(2)$-bundle, the instanton number is zero and we may drop the θ-term. For non-trivial bundles, we have to be more careful in the $[\mathcal{D}A_\mu]$ integration in (1) that produces (7) .We will elaborate on those matters in a later work[30]. Note that when the $G^a_{\mu\nu}$-s are self-dual $\sqrt{g} \neq 0$ everywhere, the instanton number must be trivial since the three fields $G^a_{14}, G^a_{24}, G^a_{34}$ are linearly independent and establish a *frame* of the associated rank-3 vector bundle.

When $\widetilde{G}^a_{\mu\nu}$ is self dual, the metric (83) is conformal.

$$g_{\mu\nu} = \psi \delta_{\mu\nu} \tag{92}$$

The action can thus be expressed in terms of the six fields: the spin-2 self-dual $W_{\mu\nu\sigma\tau}$ (5 fields) and the spin-0 ψ. The resulting action can be derived from the formula in appendix (C). This will be described in more detail in a future publication[30].

6 Discussion

We have explored the description of Yang-Mills theory in terms of the auxiliary "dual" variables. We have seen that in two dimensions this formalism reproduces the known solution[12, 13] of the torus partition function. The conjugacy class of the dual field becomes constant (over space) and quantized. The sum over representations of $SU(N)$ in [12, 13] corresponds to a sum over the quantized values of the conjugacy class of the auxiliary field. This may be compared to an instanton expansion[17] which gives the Poisson resummed partition function.

In the three dimensional $SU(2)$ gauge theory, we showed that the action can be expressed as a sum of 3D gravity[19] plus a non-covariant coupling. We have seen that there is an extra non-covariant contribution from the $(\det \widetilde{\mathcal{G}})^{-1/2}$ in (6) which arose from the Gaussian integration.

The generalization to 4D is interesting because it separates the action again into a topological generally covariant action (which describes the pure gauge configurations) plus an action proportional to α_s. We have expressed it in terms of a complicated, albeit local, Lagrangian that contains a (non-positive) 4D metric and a spin-2 self-dual tensor. It appears to be different from Lunev's gauge invariant formulation of 4D $SU(2)$ [20]. Restricting to zero instanton number, the metric becomes conformal. Thus, the sector of the

$SU(2)$ theory with a trivial gauge bundle can be transformed into a theory with a local Lagrangian with 6 degrees of freedom – five from the spin-2 self-dual tensor and an additional degree of freedom from the conformal metric. We intend to elaborate on this description in a later paper [30].

It is an interesting algebraic problem to express the Lagrangian (7) for general $SU(N)$ in terms of local gauge invariant variables in such a way that is suitable for a large N expansion.

It is also interesting to relate the Lagrangian (7) to the Hamiltonian approach of [22, 23].

Finally, for the 3D theory, in a recent work Das and Wadia[29] have generalized Polyakov's result[28] and have shown how "dressed" monopoles generate confinement. It is interesting to extract from the Lagrangian (7) that part of the action that corresponds to just integrating over the collective coordinates of the monopoles in $[\mathcal{D}A_\mu]$ in (3) and compare to the full Lagrangian.

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Appendix A: On the cohomology $H^2(SU(N)/U(1)^{N-1})$

From the fibre bundle

$$\begin{array}{ccc} U(1)^{N-1} & \longrightarrow & SU(N) \\ & & \downarrow \\ & & SU(N)/U(1)^{N-1} \end{array} \tag{93}$$

We obtain, by standard spectral sequence arguments, a basis for $H^2(SU(N)/U(1)^{N-1}) = \mathbf{Z}^{N-1}$. Let the $(N-1)$ $U(1)$-s correspond to a Cartan torus, which is generated by $T^i, i = 1, \ldots, N-1$ in the Cartan subalgebra. We choose

$$T^i = \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & & \vdots \\ 0 & \cdots & 1_{(i,i)} & \cdots & 0 \\ \vdots & & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix} - \frac{1}{N}\mathbf{I} \tag{94}$$

so that

$$\mathrm{tr}\{T^i T^j\} = \delta^{ij} - \frac{1}{N} \tag{95}$$

We construct the i-th generator of H^2 as the Chern class of the projection

$$\frac{SU(N)}{W_i \otimes \mathbf{Z}_N} \xrightarrow{\pi_i} \frac{SU(N)}{U(1)^{N-1}} \tag{96}$$

where W_i is generated by the e^{iT}-s with T is the Cartan subalgebra such that $\mathrm{tr}\{TT^i\} = 0$. To write the Chern class explicitly let

$$A^{(i)} = \mathrm{tr}\{T^i g^{-1} dg\}, \qquad g \equiv gh \qquad \text{for } h \in W_i \otimes \mathbf{Z}_N \tag{97}$$

$A^{(i)}$ is a one-form field that is well defined for $g \equiv gh$ when

$$h \in W_i \otimes \mathbf{Z}_N \tag{98}$$

and transforms as a $U(1)_i$ gauge field for $h \in U(1)_i$. The two-form $dA^{(i)}$ is the desired Chern class

$$C^{(i)} = dA^{(i)} = \mathrm{tr}\{T^i (g^{-1}dg) \wedge (g^{-1}dg)\} \tag{99}$$

which is well defined for $g \equiv gh$ with $h \in U(1)^{N-1}$. Standard spectral sequence arguments show that the $C^{(i)}$ span H^2 and that $H^1 = 0$. Since every map from a two dimensional CW-complex to another CW-complex can be homotopically pushed to the two-skeleton of the target complex, we see that the maps f from the torus to $SU(N)/U(1)^{N-1}$ are characterized by the $N-1$ integer classes of $f^* C^{(i)}$ on the torus.

Appendix B: On $SU(2)$ gauge invariant expressions

We are given an expression of the form

$$R = S^{a,b}_{ij,\mu\nu}(F)\partial_\mu F^a_i \partial_\nu F^b_j + W^a_{i,\mu\nu}(F)\partial_\mu\partial_\nu F^a_i \tag{100}$$

where $S^{a,b}_{ij,\mu\nu}(F)$ and $W^a_{i,\mu\nu}(F)$ are local algebraic expressions in F^a_i with $a = 1, 2, 3$ an $SU(2)$ index and $i = 1, \ldots, K$ (with $K = D(D-1)/2$ for our purposes). Supposing that R is gauge invariant under local gauge transformations which transform the F^a_i as

$$\delta_\eta F^a_i = \epsilon^{abc}\eta^b F^c_i \tag{101}$$

Our goal is to write (100) in a manifestly invariant form, i.e.

$$R = \sum_{(\alpha)} \varpi_{(\alpha),\mu\nu}\partial_\mu\varsigma_{(\alpha)}\partial_\nu\varphi_{(\alpha)} + \sum_{(\alpha)} \varrho_{(\alpha),\mu\nu}\partial_\mu\partial_\nu\vartheta_{(\alpha)} \tag{102}$$

where $\varpi_{(\alpha),\mu\nu}$,$\varrho_{(\alpha)(\beta),\mu\nu}$, $\vartheta_{(\alpha)}$ and $\varsigma_{(\alpha)}$ are gauge invariant algebraic expressions.

By substituting (101) in (100) it is clear that for each $\mu\nu$

$$W^a_{i,\mu\nu}(F)\frac{d}{dt}F^a_i \tag{103}$$

(where now we treat $F^a_i(t)$ as depending on a single parameter t) is also gauge invariant. So our first task will be to write (103) as

$$\sum_{(\alpha)} \widetilde{\varrho}_{(\alpha),\mu\nu}\frac{d}{dt}\widetilde{\vartheta}_{(\alpha)} \tag{104}$$

We then subtract from R

$$\sum_{(\alpha)} \widetilde{\varrho}_{(\alpha),\mu\nu}\partial_\mu\partial_\nu\widetilde{\vartheta}_{(\alpha)} \tag{105}$$

The resulting expression does not contain second derivatives and is still gauge invariant. Thus the problem reduces to writing invariantly the two separate expressions:

$$W(t) \stackrel{def}{=} W^a_i(F)\frac{d}{dt}F^a_i \tag{106}$$

$$S(x^\mu) \stackrel{def}{=} S^{a,b}_{ij,\mu\nu}(F)\partial_\mu F^a_i\partial_\nu F^b_j \tag{107}$$

We will separate the discussion to $K = 3$ and $K > 3$.

Three field strengths: $K = 3$

In this case the matrix F_i^a is a 3×3 matrix and is generically invertible. Denote the inverse by R_i^a

$$
\begin{aligned}
R_i^a &= \frac{1}{2\Delta}\epsilon^{abc}\epsilon_{ijk}F_j^b F_k^c \\
\Delta &= \frac{1}{6}\epsilon^{abc}\epsilon_{ijk}F_i^a F_j^b F_k^c = \det(F) \\
F_i^a R_j^a &= \delta_{ij} \\
F_i^a F_i^b &= \delta^{ab}
\end{aligned}
$$

We can now write

$$W_i^a = (W_i^b R_k^b)F_k^a \stackrel{def}{=} W_{ik}F_k^a \tag{108}$$

$$S_{ij,\mu\nu}^{a,b} = (S_{ij,\mu\nu}^{c,d}R_k^c R_l^d)F_k^a F_l^b \stackrel{def}{=} S_{ijkl,\mu\nu}F_k^a F_l^b \tag{109}$$

The newly defined quantities $W_{ik}, S_{ijkl,\mu\nu}$ are algebraic gauge invariants. Now the gauge invariant expressions (106-107) read

$$W(t) = W_{ik}(F)F_k^a \frac{d}{dt}F_i^a \tag{110}$$

$$S(x^\mu) = S_{ijkl,\mu\nu}(F)F_k^a F_l^b \partial_\mu F_i^a \partial_\nu F_j^b \tag{111}$$

$$S_{ijkl,\mu\nu} = S_{jilk,\nu\mu} \tag{112}$$

Now we use gauge invariance (and the symmetry of S) to obtain

$$W_{ik}\epsilon^{abc}F_k^a F_i^b = 0 \tag{113}$$

$$S_{ijkl,\mu\nu}\epsilon^{abc}F_k^a F_i^b = 0 \tag{114}$$

where in the second equation we used the fact that F_l^b is (generically) a nonsingular matrix, and also that $\partial_\nu F_j^b$ is generic. Finally, since

$$\epsilon^{abc}F_k^a F_i^b F_j^c = \epsilon_{ijk}\Delta \tag{115}$$

and generically $\Delta \neq 0$ we obtain

$$
\begin{aligned}
W_{ik} &= W_{ki} \\
S_{ijkl,\mu\nu} &= S_{kjil,\mu\nu}
\end{aligned}
$$

Using

$$T_{ij} \stackrel{def}{=} F_i^a F_k^a \tag{116}$$

we can write

$$\begin{aligned} W(t) &= \frac{1}{2} W_{ik}(F) \frac{d}{dt}(F_k^a F_i^a) = \frac{1}{2} W_{ik}(F) \frac{d}{dt} T_{ik} && (117) \\ S(x^\mu) &= \frac{1}{4} S_{ijkl,\mu\nu}(F) \partial_\mu T_{ik} \partial_\nu T_{lj} && (118) \end{aligned}$$

Which is explicitly invariant.

More than three field strengths: $K > 3$

We will limit ourselves to $W(t)$. The other invariant S is manipulated similarly. Define the 3×3 matrix

$$M^{ab} \stackrel{def}{=} F_i^a F_i^b \tag{119}$$

It is generically non-singular. The inverse of a 3×3 matrix $\mathbf{M}$ is given, by the Cayley-Hamilton theorem:

$$\begin{aligned} \mathbf{M}^{-1} &= \frac{1}{\Lambda}(\mathbf{M}^2 - \mathrm{tr}\{\mathbf{M}\}\mathbf{M} + \frac{1}{2}(\mathrm{tr}\{\mathbf{M}\}^2 - \mathrm{tr}\{\mathbf{M}^2\})\mathbf{I}) && (120) \\ \Lambda &= \det(\mathbf{M}) = \frac{1}{3}\mathrm{tr}\{\mathbf{M}^3\} - \frac{1}{2}\mathrm{tr}\{\mathbf{M}\}\mathrm{tr}\{\mathbf{M}^2\} + \frac{1}{6}\mathrm{tr}\{\mathbf{M}\}^3 && (121) \end{aligned}$$

and using this we can write

$$W_i^a = W_i^b (\mathbf{M}^{-1})^{bc} \mathbf{M}^{ca} = (W_i^b (\mathbf{M}^{-1})^{bc} F_k^c) F_k^a \tag{122}$$

So defining the gauge invariant algebraic expression

$$W_{ik} \stackrel{def}{=} W_i^b (\mathbf{M}^{-1})^{bc} F_k^c \tag{123}$$

we get

$$W(t) = W_{ik} F_k^a \frac{d}{dt} F_i^a \tag{124}$$

Since $W(t)$ is supposed to be gauge invariant we have

$$W_{ik} \epsilon^{abc} F_i^a F_k^b = 0 \tag{125}$$

for $c = 1, 2, 3$. We can decompose W_{ik} into symmetric and anti-symmetric parts

$$\begin{aligned} W_{ik}^{(A)} &\stackrel{def}{=} \frac{1}{2}(W_{ik} - W_{ki}) \\ W_{ik}^{(S)} &\stackrel{def}{=} \frac{1}{2}(W_{ik} + W_{ki}) \end{aligned}$$

When we plug the symmetric part into $W(t)$ we get the gauge invariant expression

$$W^{(S)}(t) = \frac{1}{2} W_{ik}^{(S)}(F) \frac{d}{dt}(F_k^a F_i^a) = \frac{1}{2} W_{ik}(F) \frac{d}{dt} T_{ik} \tag{126}$$

So from now on we will assume that W_{ik} is anti-symmetric, $W_{ik} = -W_{ki}$. Defining

$$R_{ik}^c \stackrel{def}{=} \epsilon^{abc} F_i^a F_k^b \tag{127}$$

which is antisymmetric in ik. (125) expresses the fact that the $K(K-1)/2$ vector W_{ij} is orthogonal to the three vectors R_{ik}^c. We need a projection operator on the space that is orthogonal to R_{ik}^c. Such a projection operator is given as follows. Suppose W_{ij} is anti-symmetric but not necessarily satisfying (125), then

$$\begin{aligned} V_{ij} &= W_{ij} - 3\mu^{-1} W_{kl} K_{klm} K_{ijm} && (128) \\ \mu &= K_{klm} K_{klm} && (129) \end{aligned}$$

where

$$\begin{aligned} T_{12} &\stackrel{def}{=} \widetilde{G}_1^a \widetilde{G}_2^a && (130) \\ K_{123} &\stackrel{def}{=} \epsilon^{def} \widetilde{G}_1^d \widetilde{G}_2^e \widetilde{G}_3^f && (131) \end{aligned}$$

If W_{ij} satisfies (125) then $V_{ij} = W_{ij}$. Furthermore, for any W_{ij}, not necessarily satisfying (125) we have $V_{ik} \epsilon^{abc} F_i^a F_k^b = 0$. Thus it has to be that for any W_{ij} the expression $V_{ik} F_k^a \frac{d}{dt} F_i^a$ can be written in a manifestly invariant way. This is indeed so. Define

$$\begin{aligned} U_{ij} &\stackrel{def}{=} T_{ik} T_{kj} = (\mathbf{T}^2)_{ij} \\ G_{ij} &\stackrel{def}{=} T_{ik} T_{kl} T_{lj} = (\mathbf{T}^3)_{ij} \end{aligned}$$

$$
\begin{aligned}
T &\stackrel{def}{=} T_{ii} = \mathrm{tr}\{\mathbf{T}\} \\
U &\stackrel{def}{=} U_{ii} = \mathrm{tr}\{\mathbf{T}^2\} \\
G &\stackrel{def}{=} G_{ii} = \mathrm{tr}\{\mathbf{T}^3\} \\
\mu &\stackrel{def}{=} T^3 - 3UT + 2G = 6 \det M^{ab}
\end{aligned}
$$

$$
P_{ij} \stackrel{def}{=} 6\mu^{-1}(G_{ij} - TU_{ij} + \frac{1}{2}(T^2 - U)T_{ij}) \tag{132}
$$

P_{ij} is a projection operator and satisfies

$$
P_{ij}F^a_j = F^a_j \tag{133}
$$

Now we can write

$$
\begin{aligned}
V_{ij}F^a_j \frac{d}{dt}F^a_i &= (W_{ij} - 3\mu^{-1}W_{kl}K_{klm}K_{ijm})F^a_i \frac{d}{dt}F^a_j \\
= -2P_{ik}W_{ij}\frac{d}{dt}T_{kj} &+ 6\mu^{-1}(T_{lm}K_{mik} - \frac{1}{2}TK_{lik})W_{ij}\frac{d}{dt}K_{klj}
\end{aligned} \tag{134}
$$

which is written in terms of local invariant objects.

Appendix C: Expressing $\mathcal{L}$ in terms of $W_{\mu\nu\sigma\tau}$ and $g_{\mu\nu}$

In appendix (B) we described in general how to write $SU(N)$ Lagrangians like (7) in gauge invariant variables. We saw that the $D = 3$ case is simpler because the field strength $G^a_{\mu\nu}$ can be thought of as a 3×3 matrix (three values of a, and three values of $\mu\nu$). In higher dimensions, $D > 3$ this is not the case and we had more complications. However, for $D = 4$, we have seen in section (5), that when passing to the special metric (83), the field strength becomes self-dual. Thus, given the metric, there are only three linearly independent $\mu\nu$-s, and $F^a_{\mu\nu}$ is effectively a 3×3 matrix.

We shall now describe in detail how the invariant Lagrangian is obtained. We start with the topological part of (7):

$$
\begin{aligned}
\mathcal{L}_{topo} &= 2\widetilde{G}^a_{\mu\nu}(\widetilde{\mathcal{G}}^{-1})^{ab}_{\nu\tau}\partial_\mu(\widetilde{\mathcal{G}})^{bc}_{\tau\sigma}(\widetilde{\mathcal{G}}^{-1})^{cd}_{\sigma\rho}\partial_\alpha\widetilde{G}^d_{\alpha\rho} \\
&- 2\widetilde{G}^a_{\mu\nu}(\widetilde{\mathcal{G}}^{-1})^{ab}_{\nu\tau}\partial_\mu\partial_\sigma\widetilde{G}^b_{\sigma\tau} - (\widetilde{\mathcal{G}}^{-1})^{cd}_{\nu\rho}\partial_\alpha\widetilde{G}^c_{\alpha\nu}\partial_\beta\widetilde{G}^d_{\beta\rho}
\end{aligned}
$$

Since the topological part is independent of the metric we pick our special induced metric (83) and write

$$
\begin{aligned}
\mathcal{L}_{topo} &= 2F^{\mu\nu,a}(\widetilde{\mathcal{G}}^{-1})^{ab}_{\nu\tau}\epsilon^{bce}\nabla_\mu F^{\tau\sigma,e}(\widetilde{\mathcal{G}}^{-1})^{cd}_{\sigma\rho}\nabla_\alpha F^{\alpha\rho,d} \\
&- 2F^{\mu\nu,a}(\widetilde{\mathcal{G}}^{-1})^{ab}_{\nu\tau}\nabla_\mu\nabla_\sigma F^{\sigma\tau,b} - (\widetilde{\mathcal{G}}^{-1})^{cd}_{\nu\rho}\nabla_\alpha F^{\alpha\nu,c}\nabla_\beta F^{\beta\rho,d}
\end{aligned}
$$

where ∇_μ is the covariant derivative with respect to that metric. We substitute our covariant expression for $(\widetilde{\mathcal{G}}^{-1})^{ab}_{\mu\nu}$ and obtain

$$
\begin{aligned}
\mathcal{L}_{topo} &= \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 \\
\mathcal{L}_1 &= \frac{1}{2}F_\nu{}^{\gamma,c}F^d_{\gamma\rho}\nabla_\alpha F^{\alpha\nu,c}\nabla_\beta F^{\beta\rho,d} + \frac{1}{4}F^{\alpha\beta,c}F^d_{\alpha\beta}g_{\nu\rho}\nabla_\alpha F^{\alpha\nu,c}\nabla_\beta F^{\beta\rho,d} \\
\mathcal{L}_2 &= F_\nu{}^{\gamma,a}F^b_{\gamma\tau}F^{\mu\nu,a}\nabla_\mu\nabla_\rho F^{\rho\tau,b} + \frac{1}{2}F^\mu{}_\tau{}^a F^{\alpha\beta,a}F^b_{\alpha\beta}\nabla_\mu\nabla_\sigma F^{\sigma\tau,b} \\
\mathcal{L}_3 &= \frac{1}{2}F^{\mu\nu,a}F_\nu{}^{\gamma,a}F^b_{\gamma\tau}\epsilon^{bce}\nabla_\mu F^{\tau\sigma,e}F_\sigma{}^{\gamma,c}F^d_{\gamma\rho}\nabla_\alpha F^{\alpha\rho,d} \\
&+ \frac{1}{4}F^{\mu\nu,a}F_\nu{}^{\gamma,a}F^b_{\gamma\tau}\epsilon^{bce}\nabla_\mu F^\tau{}_\sigma{}^e F^{\gamma\delta,c}F^d_{\gamma\delta}\nabla_\alpha F^{\alpha\sigma,d} \\
&+ \frac{1}{4}F^\mu{}_\tau{}^a F^{\alpha\beta,a}F^b_{\alpha\beta}\epsilon^{bce}\nabla_\mu F^{\tau\sigma,e}F_\sigma{}^{\gamma,c}F^d_{\gamma\rho}\nabla_\alpha F^{\alpha\rho,d} \\
&+ \frac{1}{8}F^\mu{}_\tau{}^a F^{\alpha\beta,a}F^b_{\alpha\beta}\epsilon^{bce}\nabla_\mu F^\tau{}_\sigma{}^e F^{\gamma\delta,c}F^d_{\gamma\delta}\nabla_\alpha F^{\alpha\sigma,d}
\end{aligned}
$$

It is tedious though straight-forward to check the identity

$$
F_\mu{}^{\gamma,a}F^a_{\gamma\nu} = -3\Phi g_{\mu\nu}
$$

We use it to express $\mathcal{L}_2$ as

$$
\begin{aligned}
\mathcal{L}_2 &= \mathcal{L}_2' + \frac{3}{2}\Phi\nabla^\mu\nabla_\mu\Phi - R\Phi^2 + \frac{1}{16}R(W^2) \\
&- \frac{1}{4}\Phi R_{\mu\nu\sigma\tau}W^{\mu\nu\sigma\tau} - \frac{1}{8}R_{\mu\nu\sigma\tau}W^{\sigma\tau}{}_{\alpha\beta}W^{\alpha\beta\sigma\tau} \\
&+ \frac{1}{4}W^{\alpha\beta\mu}{}_\tau\nabla_\mu\nabla_\sigma W_{\alpha\beta\sigma\tau} + \frac{1}{4}W^{\alpha\beta\sigma}{}_\tau\nabla^\mu\nabla_\sigma W_{\alpha\beta\mu\tau} - \frac{1}{4}W^{\alpha\beta\gamma\delta}\nabla^\mu\nabla_\mu W_{\alpha\beta\gamma\delta} \\
\mathcal{L}_2' &= -\frac{1}{2}\Phi\nabla_\mu F^\mu{}_\tau{}^b\nabla_\sigma F^{\sigma\tau,b} - \frac{1}{2}\Phi\nabla_\sigma F^\mu{}_\tau{}^b\nabla_\mu F^{\sigma\tau,b} \\
&- \frac{1}{2}W^{\alpha\beta\mu}{}_\tau\nabla_\mu F^b_{\alpha\beta}\nabla_\sigma F^{\sigma\tau,b} - \frac{1}{2}W^{\alpha\beta\mu}{}_\tau\nabla_\sigma F^b_{\alpha\beta}\nabla_\mu F^{\sigma\tau,b} \\
&+ \frac{1}{2}W^{\alpha\beta\gamma\delta}\nabla^\mu F^b_{\alpha\beta}\nabla_\mu F^b_{\gamma\delta}
\end{aligned}
$$

where R is the curvature:

$$\begin{aligned} R_{\alpha\beta\gamma\delta} &= g_{\delta\mu}(\partial_\alpha\Gamma^\mu_{\beta\gamma} - \partial_\beta\Gamma^\mu_{\alpha\gamma} + \Gamma^\mu_{\alpha\sigma}\Gamma^\sigma_{\beta\gamma} - \Gamma^\mu_{\beta\sigma}\Gamma^\sigma_{\alpha\gamma}) \\ R_{\alpha\beta} &= g^{\gamma\delta}R_{\alpha\gamma\beta\delta} \\ R &= g^{\alpha\beta}R_{\alpha\beta} \end{aligned}$$

Now we repeat the arguments of appendix (B) for the $D = 3$ case to argue that if an expression of the form

$$\Upsilon_{\{\mu\nu,\alpha\beta,\kappa\},\{\sigma\tau,\gamma\delta,\rho\}}F^{\mu\nu,a}\nabla^\kappa F^{\alpha\beta,a}F^{\sigma\tau,b}\nabla^\rho F^{\gamma\delta,b} \tag{135}$$

is gauge invariant, it can be written as

$$\frac{1}{4}\Upsilon_{\{\mu\nu,\alpha\beta,\kappa\},\{\sigma\tau,\gamma\delta,\rho\}}\nabla^\kappa V^{\mu\nu\alpha\beta}\nabla^\rho V^{\sigma\tau\gamma\delta} \tag{136}$$

where

$$\begin{aligned} h_{\mu\nu\sigma\tau} &\stackrel{def}{=} g_{\mu\sigma}g_{\nu\tau} - g_{\mu\tau}g_{\nu\sigma} + \sqrt{g}\epsilon_{\mu\nu\sigma\tau} \\ \Phi &= \frac{1}{24\sqrt{g}}\epsilon^{\mu\nu\sigma\tau}F^a_{\mu\nu}F^a_{\sigma\tau} \\ V_{\mu\nu\sigma\tau} &\stackrel{def}{=} W_{\mu\nu\sigma\tau} + \Phi h_{\mu\nu\sigma\tau} \end{aligned}$$

In order to express our Lagrangian in a suitable form we need two more identities that can all be induced from the 3×3 nature of the matrices involved (see e.g. appendix (D)). For $\mathcal{L}_3$ we need

$$\epsilon^{abc}F^b_{\mu\tau}F^c_{\sigma\gamma} = \Psi_{\mu\tau,\sigma\gamma,\alpha\beta}F^{\alpha\beta,a} \tag{137}$$

with

$$\begin{aligned} &\Psi_{\mu\tau,\sigma\gamma,\alpha\beta} = \\ &\frac{1}{16}(W^2)_{\alpha\beta\tau\gamma}g_{\mu\sigma} - \frac{1}{16}(W^2)_{\alpha\beta\tau\sigma}g_{\mu\gamma} - \frac{1}{16}(W^2)_{\alpha\beta\mu\gamma}g_{\tau\sigma} + \frac{1}{16}(W^2)_{\alpha\beta\mu\sigma}g_{\tau\gamma} \\ - &\frac{1}{4}\Phi W_{\alpha\beta\tau\gamma}g_{\mu\sigma} + \frac{1}{4}\Phi W_{\alpha\beta\tau\sigma}g_{\mu\gamma} + \frac{1}{4}\Phi W_{\alpha\beta\mu\gamma}g_{\tau\sigma} - \frac{1}{4}\Phi W_{\alpha\beta\mu\sigma}g_{\tau\gamma} \\ + &(\frac{1}{4}\Phi^2 - \frac{1}{128}(W^2))(h_{\alpha\beta\tau\gamma}g_{\mu\sigma} - h_{\alpha\beta\tau\sigma}g_{\mu\gamma} - h_{\alpha\beta\mu\gamma}g_{\tau\sigma} + h_{\alpha\beta\mu\sigma}g_{\tau\gamma}) \end{aligned}$$

$$
\begin{aligned}
(W^2)_{\mu\nu\sigma\tau} &\stackrel{def}{=} W_{\mu\nu}{}^{\alpha\beta} W_{\alpha\beta\sigma\tau} \\
(W^2) &\stackrel{def}{=} W^{\mu\nu\sigma\tau} W_{\mu\nu\sigma\tau}
\end{aligned}
$$

For $\mathcal{L}_2$ we need

$$
\delta^{ab} = \vartheta_{\mu\nu\sigma\tau} F^{\mu\nu,a} F^{\sigma\tau,b} \tag{138}
$$

$$
\vartheta_{\mu\nu\sigma\tau} \stackrel{def}{=} \frac{1}{64}(W^2)_{\mu\nu\sigma\tau} - \frac{1}{16}\Phi W_{\mu\nu\sigma\tau} + \frac{1}{16}(\Phi^2 - \frac{1}{32}(W^2)) h_{\mu\nu\sigma\tau} \tag{139}
$$

Putting everything together we get the expression fo the Lagrangian:

$$
\begin{aligned}
& \frac{3}{8}\Phi\nabla^\mu\nabla_\mu\Phi - R\Phi^2 + \frac{1}{16}R(W^2) - \frac{1}{4}\Phi R_{\mu\nu\sigma\tau}W^{\mu\nu\sigma\tau} - \frac{1}{8}R_{\mu\nu\sigma\tau}W^{\sigma\tau}{}_{\alpha\beta}W^{\alpha\beta\sigma\tau} \\
+ & \frac{1}{4}W^{\alpha\beta\mu}{}_{\tau}\nabla_\mu\nabla_\sigma W_{\alpha\beta\sigma\tau} + \frac{1}{4}W^{\alpha\beta\sigma}{}_{\tau}\nabla^\mu\nabla_\sigma W_{\alpha\beta\mu\tau} - \frac{1}{4}W^{\alpha\beta\gamma\delta}\nabla^\mu\nabla_\mu W_{\alpha\beta\gamma\delta} \\
+ & \frac{1}{16}\nabla_\alpha V^{\alpha\nu\gamma\delta}\nabla^\beta V_{\beta\nu\gamma\delta} - \frac{1}{8}\Phi g_{\gamma\delta}\vartheta_{\mu\nu\sigma\tau}\nabla_\alpha V^{\alpha\gamma\mu\nu}\nabla_\beta V^{\beta\delta\sigma\tau} \\
- & \frac{1}{8}\Phi g_{\gamma\delta}\vartheta_{\mu\nu\sigma\tau}\nabla_\beta V^{\alpha\gamma\mu\nu}\nabla_\alpha V^{\beta\delta\sigma\tau} - \frac{1}{8}W^{\alpha\beta\mu}{}_{\tau}\vartheta_{\gamma\delta\nu\sigma}\nabla_\mu V^{\gamma\delta}{}_{\alpha\beta}\nabla_\rho V^{\rho\tau\nu\sigma} \\
- & \frac{1}{8}W^{\alpha\beta\mu}{}_{\tau}\vartheta_{\gamma\delta\nu\sigma}\nabla_\rho V^{\gamma\delta}{}_{\alpha\beta}\nabla_\mu V^{\rho\tau\nu\sigma} + \frac{1}{8}W^{\alpha\beta\gamma\delta}\vartheta_{\mu\nu\sigma\tau}\nabla^\mu V^{\mu\nu}{}_{\alpha\beta}\nabla_\mu V^{\sigma\tau}{}_{\gamma\delta} \\
- & \frac{3}{8}\Phi\Psi_{\mu\tau,\sigma\alpha,\beta\delta}\nabla^\mu V^{\beta\delta\tau\sigma}\nabla^\alpha\Phi - \frac{1}{16}\Phi\Psi_{\mu\tau,\sigma\gamma,\beta\delta}\nabla^\mu V^{\beta\delta\tau\rho}\nabla_\alpha V^{\sigma\gamma\alpha}{}_{\rho} \\
+ & \frac{3}{16}W^{\alpha\beta\mu}{}_{\tau}\Psi_{\alpha\beta,\sigma\rho,\gamma\delta}\nabla_\mu V^{\gamma\delta\tau\sigma}\nabla^\rho\Phi + \frac{1}{32}W^{\alpha\beta\mu}{}_{\tau}\Psi_{\alpha\beta,\sigma\rho,\gamma\delta}\nabla_\mu V^{\gamma\delta\tau\nu}\nabla_\kappa V^{\sigma\rho\kappa}{}_{\nu}
\end{aligned}
$$

with

Appendix D: Expressing Φ in terms of $W_{\mu\nu\sigma\tau}$

$W_{\mu\nu\sigma\tau}$ and Φ are defined as

$$
\begin{aligned}
W_{\mu\nu\sigma\tau} &= F^a_{\mu\nu}F^a_{\sigma\tau} - \frac{1}{24\sqrt{g}}\epsilon^{\alpha\beta\gamma\delta}F^a_{\alpha\beta}F^a_{\gamma\delta}(g_{\mu\sigma}g_{\nu\tau} - g_{\mu\tau}g_{\nu\sigma} + \sqrt{g}\epsilon_{\mu\nu\sigma\tau}) \\
\Phi &= \frac{1}{24\sqrt{g}}\epsilon^{\mu\nu\sigma\tau}F^a_{\mu\nu}F^a_{\sigma\tau}
\end{aligned}
$$

Define the 3×3 matrix

$$
\mathbf{M}^{ab} \stackrel{def}{=} \frac{1}{8\sqrt{g}}\epsilon^{\mu\nu\sigma\tau}F^a_{\mu\nu}F^b_{\sigma\tau} \tag{140}
$$

It is easy to see that

$$\det \mathbf{M} = \frac{\Delta}{8\sqrt{g}^3} = 1 \tag{141}$$

Let $\lambda_1, \lambda_2, \lambda_3$ be the three eigen-values of $\mathbf{M}$. Then

$$\begin{aligned} \Phi &= \frac{1}{3}(\lambda_1 + \lambda_2 + \lambda_3) \\ 1 &= \lambda_1 \lambda_2 \lambda_3 \end{aligned}$$

thus we need two more relations among $\lambda_1, \lambda_2, \lambda_3$ to determine Φ. Using the self-duality of W we have

$$\frac{1}{16} W^{\mu\nu\sigma\tau} W_{\mu\nu\sigma\tau} = (\frac{1}{4\sqrt{g}})^2 \epsilon^{\mu\nu\alpha\beta} \epsilon^{\sigma\tau\gamma\delta} W_{\mu\nu\sigma\tau} W_{\alpha\beta\gamma\delta} = \mathrm{tr}\{\mathbf{M}^2\} - 3\Phi^2 \tag{142}$$

From a similar equation for $\mathrm{tr}\{\mathbf{M}^3\}$

$$\mathrm{tr}\{\mathbf{M}^3\} = \frac{1}{64} W^{\mu\nu\sigma\tau} W_{\mu\nu}{}^{\alpha\beta} W_{\alpha\beta\sigma\tau} + \frac{3}{16} \Phi W^{\mu\nu\sigma\tau} W_{\mu\nu\sigma\tau} + 3\Phi^2 \tag{143}$$

we find that Φ is the solution of the cubic equation

$$\Phi^3 - A\Phi - B = 0 \tag{144}$$

with

$$\begin{aligned} A &= \frac{1}{32} W^{\mu\nu\sigma\tau} W_{\mu\nu\sigma\tau} \\ B &= 1 - \frac{1}{192} W^{\mu\nu\sigma\tau} W_{\mu\nu}{}^{\alpha\beta} W_{\alpha\beta\sigma\tau} \end{aligned}$$

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