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Anyon trajectories and the systematics of the three-anyon spectrum

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Abstract

We develop the concept of trajectories in anyon spectra, i.e., the continuous dependence of energy levels on the kinetic angular momentum. It provides a more economical and unified description, since each trajectory contains an infinite number of points corresponding to the same statistics. For a system of non-interacting anyons in a harmonic potential, each trajectory consists of two infinite straight line segments, in general connected by a nonlinear piece. We give the systematics of the three-anyon trajectories. The trajectories in general cross each other at the bosonic/fermionic points. We use the (semi-empirical) rule that all such crossings are true crossings, i.e. the order of the trajectories with respect to energy is opposite to the left and to the right of a crossing.

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1 The concept of trajectories

Anyons [1, 2] are two-dimensional particles whose wave function ψ obeys the interchange conditions

$$P_{mn}\psi = \exp(i\pi\nu)\psi, \tag{1}$$

where P_{mn} denotes continuous anticlockwise interchange of particles m and n, such that no other particles are encircled, and ν is the statistics parameter, which may be any real number. For a given Hamiltonian, not explicitly dependent on ν and with a discrete spectrum for any ν, the energy eigenvalues $E_k(\nu)$ and (usually) the corresponding eigenfunctions $\psi_k(\nu)$ will be continuous functions of ν. A continuous function $E_k(\nu)$, for a fixed index k and $\nu \in \langle -\infty, \infty\rangle$, is what we will refer to as an (anyon) trajectory. For an earlier discussion of the ν dependence of the states, see [3].

Since the exponent in (1) is the only place where ν appears in this formulation, the spectrum and the set of energy eigenstates are obviously periodic in ν with period 2; thus, all the information is contained in the sets $\{E_k(\nu)\}$ and $\{\psi_k(\nu)\}$ for $\nu \in [0, 2\rangle$. However, individual trajectories are not generally periodic in ν; one way to see this is to observe that when ν increases continuously, the kinetic angular momentum L changes according to the formula

$$L(\nu) = L(0) + \frac{1}{2}N(N-1)\nu, \tag{2}$$

with $L(0)$ an integer[5], so that changing ν by 2 will not bring us back to the same state. It is assumed that the Hamiltonian is rotationally invariant, so that L is a good quantum number. Thus, $E_k(\nu + 2\ell)$, where $\ell = 0, \pm 1, \pm 2, \ldots$, will be the energy of some state with statistics parameter ν, but the index of that state will be different from k:

$$E_k(\nu + 2\ell) = E_{k'}(\nu); \tag{3}$$

where k' depends on k and ℓ: $k' = k'(k, \ell)$. At any given ν, the trajectories k and k' look different, having different angular momenta $L_k(\nu)$ and $L_{k'}(\nu) = L_k(\nu + 2\ell) = L_k(\nu) + N(N-1)\ell$, but when viewed on the interval $\langle -\infty, \infty\rangle$, they are seen to be copies of one and the same trajectory, shifted by 2ℓ along the ν axis.

To represent the trajectories in a way which is not redundant, it is convenient to take L as a parameter instead of ν. L is directly related to ν by equation (2), it is an observable (gauge invariant) quantity, and the dependence $E(L)$ is the same for all the trajectories that are shifted copies of one another. We arrive at the viewpoint that the states of an N-anyon problem may be obtained from the set of trajectories, i.e., functions $E(L)$ for $L \in \langle -\infty, \infty\rangle$. This provides a more economical and unified way of decribing the N-anyon spectra. For each trajectory there is an integer value $L(0)$ such that the statistics is bosonic at angular momenta $L = L(0) + N(N-1)\ell$. We may

[5] This formula can be proved by noting that a 2π rotation of the whole system multiplies the wave function by a phase factor, which equals, on the one hand, $\exp[2i\pi L(\nu)]$, and on the other hand, $\exp[N(N-1)i\pi\nu]$, since each of the $N(N-1)/2$ pairs of anyons is interchanged twice.

always choose $-N(N-1)/2 < L(0) \leq N(N-1)/2$. The value of $L(0)$ groups the set of trajectories into $N(N-1)$ classes. Only trajectories from the same class can cross, i.e. have the same E and L at the same statistics.

A part of a trajectory between any two neighboring bosonic points corresponds to what is usually referred to as one anyon state on the interval $\nu \in [0, 2\rangle$. It is useful to have a geometric picture of this: If one plots a trajectory for $-\infty < L < \infty$ and then wraps the plot around a cylinder of circumference $N(N-1)$, then points which correspond to ν differing by an even number (that is, to the same statistics) will fall on the same vertical line. On the surface of the cylinder one will see the set of all N-anyon states corresponding to the trajectory. Thus, an infinite number of pieces of trajectories (or states in the usual terminology) on the interval $\nu \in [0, 2\rangle$ are shown to make up one single trajectory on the interval $\nu \in \langle -\infty, \infty \rangle$. In other words, certain anyon (and in particular, boson or fermion) states that are usually considered entirely different, are in fact parts of one and the same continuous pattern. This is reminiscent of the method of organizing the spectra of particle physics into Regge trajectories. In fact, by applying the concept of Regge trajectories to a two-dimensional, two-particle system, one obtains the same grouping of bosonic/fermionic states.

It follows that if some trajectory $E(L)$ is analytic, then it is in principle sufficient to find only one of its pieces. The rest is uniquely determined by analytic continuation. This means that it is sufficient to find one anyonic state in some range of ν and then an infinite number of others are obtained "automatically" by analytic continuation and periodicity. However, trajectories are not always analytic. With our assumption that the Hamiltonian does not explicitly depend on ν (and involves only non-singular interactions) both energies and wave functions will depend analytically on ν, except possibly at bosonic points. The source of non-analyticity is that the relative angular momentum of one pair of particles becomes zero, which can only happen at bosonic points. When there is non-analytic behavior at some point with degeneracy, one must determine how each trajectory continues through this point by investigating the corresponding wave functions. They should change continuously across these points.

2 Two- and three-anyon trajectories

We are going now to apply this reasoning to the problems of two and three anyons in a rotation symmetric harmonic potential. We scale the variables so that $\hbar = 1$, the mass $m = 1$ and the angular frequency $\omega = 1$. Two more preliminary remarks are in order. First, the potential is parity invariant; hence, if ψ is an eigenfunction of the Hamiltonian and satisfies (1), then its complex conjugate $\bar{\psi}$ is an eigenfunction of the Hamiltonian with the same energy and satisfies (1) with $-\nu$ instead of ν. In particular, this implies that all information about the states is in fact contained in the interval $\nu \in [0, 1]$.

Second, there is the tower structure of the spectrum [4, 5]. It has been observed that, for any number of anyons, all the states come in towers, the angular momentum being the same for all members of a tower and the energies being $E(\nu)$, $E(\nu)+2$, $E(\nu)+4, \ldots$, where $E(\nu)$ is the energy of the lowest, "bottom" state. This sequence of levels is due to radial excitations. The radial coordinate r is defined by $r^2 = \sum_i (x_i - X)^2 + (y_i - Y)^2$, with (X, Y) the center-of-mass coordinates. Thus, it is sufficient to find only the bottom states (consequently, bottom trajectories); from now on, we will always mean these, unless otherwise specified.

Now we will demonstrate that in the two-anyon problem there is only *one* (bottom) trajectory, namely the continuation of the ground state. Recall that the one-particle spectrum in the harmonic potential consists of states with $E = 1, 2, 3, \ldots$ and $L = E-1, E-3, \ldots, -(E-3), -(E-1)$ (so the degeneracy equals the energy). Define complex coordinates by $z_j = (x_j + iy_j)/\sqrt{2}$. Since the center-of-mass motion is trivial, we will always concentrate on the relative motion only. For two anyons, the complete set of (not normalized) solutions may be written as

$$\psi_{ln}(z, \bar{z}) = \tilde{z}^{|2l+\nu|} \, {}_1F_1\left(-n, |2l+\nu|+1; z\bar{z}\right) \exp\left(-z\bar{z}/2\right) \tag{4}$$

with $z = z_1 - z_2$, l an integer, n a non-negative integer, and $\tilde{z}$ standing for z if $2l+\nu \geq 0$ and for $\bar{z}$ if $2l+\nu < 0$; the energy and the angular momentum are

$$E_{ln}(\nu) = |2l+\nu| + 2n + 1, \qquad L_{ln}(\nu) = L_l(\nu) = 2l + \nu, \tag{5}$$

respectively. Towers consist of states with the same l and different n; in accordance with the aforesaid, it is sufficient to consider the bottom states only, for which $n = 0$ and

$$E = |L| + 1 \tag{6}$$

irrespective of l. Indeed, all the states in question belong to one and the same trajectory, because the relation (3) does hold, in the form $E_0(\nu + 2l) = E_l(\nu)$ (and the same for the wave function). For $-1 < \nu < 1$ this is the ground state,

$$\psi_0(z, \bar{z}) = \tilde{z}^{|\nu|} \exp\left(-z\bar{z}/2\right), \tag{7}$$

$$E_0(\nu) = |\nu| + 1. \tag{8}$$

Thus, *all* bottom states are continuations of *one* ground state. Equation (6) exemplify our previous remark that a trajectory may become non-analytic when the relative angular momentum of one pair of particles (here L itself) becomes zero. By adding the pair potential $g^2/|z|^2$ to the Hamiltonian, (6) is changed to $E(L) = \sqrt{L^2 + g^2} + 1$, which demonstrates that singular interactions may lead to different behaviours.

We go now to the problem of three anyons, which is extremely interesting due to its nontriviality, on the one hand, and the possibility of a more or less exact analysis, on the other hand. Recall some results available. A state starting with (E, L) at $\nu = n$ (n integer) may reach one of the following points at $\nu = n+1$: $(E+3, L+3)$, $(E+1, L+3)$, $(E-1, L+3)$, $(E-3, L+3)$ [6, 7]. In other words, in any interval from a bosonic to a fermionic point or from a fermionic to a bosonic point, each state is characterized by an (average) slope

$$s = \frac{\Delta E}{\Delta \nu} = 3\frac{\Delta E}{\Delta L} = \pm 1 \quad \text{or} \quad \pm 3, \tag{9}$$

where ΔE is the change in energy, and $\Delta L = 3\Delta\nu = 3$ is the change in angular momentum. For the $s = \pm 3$ states the dependence $E(L)$ is linear, so that $s = dE/d\nu$ is the slope at any point in the interval, and their wave functions may be written down exactly [8, 9, 10]. For the $s = \pm 1$ states the dependence is nonlinear, so a slope of ± 1 is indeed only an average slope. Now, there are exact expressions for the multiplicities of states with all slopes [11]. If $\tilde{b}^n(E, L)$ denotes the number of bottom states in the relative motion spectrum that go from (E, L) to $(E + n, L + 3)$ as ν goes from $2m$ to $2m + 1$, i.e., from a bosonic to a fermionic point, then there are the asymptotic expressions for $E, L \gg 1$ (terms of order unity being omitted)

$$\tilde{b}^{+3}(E, L) = \frac{3L - E}{12} \qquad \left(\ \frac{E}{3} < L < E\right), \tag{10}$$

$$\tilde{b}^{+1}(E, L) = \frac{2E - |3L - E|}{12} \qquad \left(-\frac{E}{3} < L < E\right), \tag{11}$$

and the exact formulas $\tilde{b}^{-1}(E, L) = \tilde{b}^{+1}(E-1, -L-5)$, $\tilde{b}^{-3}(E, L) = \tilde{b}^{+3}(E-6, -L-6)$. In each of the formulas, L has to be within the respective interval specified and the equality $L \equiv E \pmod 2$ has to hold, otherwise $\tilde{b}^n(E, L) = 0$. Recall also that for all states at any statistics, there is the inequality $|L| < E$ [12].

As one goes along a trajectory, the slope will change at certain values of L, so any trajectory possesses a sequence of slopes and points of slope change. We make now the following statements.

(i) For each trajectory there exists an L_+ such that $s = +3$ if and only if $L > L_+$.

(ii) For each trajectory there exists an L_- such that $s = -3$ if and only if $L < L_-$.

(iii) The change of slope from/to ± 3 can occur at bosonic points only, while the change from -1 to $+1$ or vice versa can occur at both bosonic and fermionic points.

To prove statement (i), note first that the inequality $E > L$ and the fact that $dL/d\nu = 3$ force every trajectory to have $s = +3$ at least for some large positive values of L. Now, the explicit form of the $s = +3$ states [13, 8, 9] is such that if one of them exists for some $\tilde{\nu}$, it does exist with $s = +3$ for any $\nu > \tilde{\nu}$, that is for any $L > \tilde{L} = L(\tilde{\nu})$; in other words, if $s = +3$ at some point, then $s = +3$ everywhere to the right of that point. This concludes the proof. A "mirror reversed" reasoning proves statement (ii).

Statement (iii) is proved upon noticing that the change of slope from/to ± 3 always implies nonanalyticity in the function $E(L)$, because it is a change between a linear and a nonlinear dependence. Such nonanalyticity can only happen at bosonic points, as discussed in Sec. 1. This is also associated with the breakdown of regular perturbation theory[6] at such points [9, 14].

Note that since statements (i) and (ii) mean that each trajectory has $s = -3$ for some values of L and $s = +3$, for some others, it follows that each trajectory is indeed nonanalytic at least at one point. (This is of course equally true for the N-anyon problem, where the extreme slopes, with linear behavior, are $\pm N(N-1)/2$, and each trajectory possesses both of them just like in the case at hand; cf. [3].)

So far we may conclude that the behavior of each trajectory has the following features: At large negative L, slope -3, at some point (E_-, L_-) a change to ± 1, for $L_- < L < L_+$ some sequence of slopes ± 1 and finally a change to $+3$ at (E_+, L_+) and always $+3$ at $L > L_+$. Note that there is exactly one trajectory for which $L_- = L_+$, that is which has slopes $+3$ and -3 only: the one which contains the ground state near Bose statistics,

$$\psi_0 = \begin{cases} (\bar{z}_{12}\bar{z}_{23}\bar{z}_{31})^{-\nu} & \text{for } \nu < 0, \\ \\ (z_{12}z_{23}z_{31})^{\nu} & \text{for } \nu \geq 0, \end{cases} \tag{12}$$

where $z_{jk} = z_j - z_k$, and the overall Gaussian factor is understood.

We will now prove that

(iv) Each bottom trajectory may be unambiguously labeled by its (E_+, L_+) point.

Indeed, given an arbitrary bosonic point (E, L), it is easy to see that the number of bottom trajectories for which $(E_+, L_+) = (E, L)$, equals

$$\tilde{b}^{+3}(E, L) - \tilde{b}^{+3}(E-6, L-6) \leq 1,$$

where the inequality follows from eq. (10). The equality is reached, meaning that a trajectory with $(E_+, L_+) = (E, L)$ exists, for each (E, L) such that $\frac{E}{3} < L < E$ and $L \equiv E \pmod 2$. Like the equation itself, this statement is true up to terms of the order of unity. Fig. 1 shows the exact picture of the distribution of the (E_+, L_+) points on the plane (the bullets and triangles will be explained later).

3 The linear parts of the trajectories

We proceed to show how all the linear parts of the trajectories (and thus all the three-anyon linear states) are constructed as excitations of the "ground trajectory", the one which contains the ground state. By virtue of symmetry, it is enough to consider $s = +3$ states only. It is convenient to use the coordinates proposed in [16], which are

[6] The usual way to introduce perturbation theory is by replacing the anyonic boundary conditions by a ν dependent Aharonov-Bohm-type (perturbative) interaction.

the discrete Fourier transform of the complex particle coordinates. The relative motion is described by

$$u = \frac{1}{\sqrt{3}}\left(z_1 + \eta z_2 + \eta^2 z_3\right), \qquad v = \frac{1}{\sqrt{3}}\left(z_1 + \eta^2 z_2 + \eta z_3\right), \tag{13}$$

where $\eta = \exp(2i\pi/3) = (-1 + i\sqrt{3})/2$. Introduce the creation and annihilation operators

$$a_u = \frac{1}{\sqrt{2}}(\bar{u} + \partial_u), \qquad a_u^\dagger = \frac{1}{\sqrt{2}}(u - \partial_{\bar{u}}), \tag{14}$$

$$b_u = \frac{1}{\sqrt{2}}(u + \partial_{\bar{u}}), \qquad b_u^\dagger = \frac{1}{\sqrt{2}}(\bar{u} - \partial_u), \tag{15}$$

and, with $u \to v$, the same relations for a_v, $a_v^\dagger$, b_v, $b_v^\dagger$. Then the relative Hamiltonian and angular momentum operator become

$$H = a_u^\dagger a_u + a_v^\dagger a_v + b_u^\dagger b_u + b_v^\dagger b_v + 2, \tag{16}$$

$$L = a_u^\dagger a_u + a_v^\dagger a_v - b_u^\dagger b_u - b_v^\dagger b_v. \tag{17}$$

The commutation relations are $[a_u, a_u^\dagger] = [b_u, b_u^\dagger] = [a_v, a_v^\dagger] = [b_v, b_v^\dagger] = 1$, and all other commutators vanish. Consequently, if ψ is a common eigenstate of H and L with quantum numbers (E, L), then $(a_u^\dagger)^k (a_v^\dagger)^l (b_u^\dagger)^m (b_v^\dagger)^n \psi$ is also a common eigenstate with $(E+k+l+m+n, L+k+l-m-n)$. However, to yield true anyonic eigenstates, a combination of the creation operators must be fully symmetric and produce wave functions that are not singular at the points where the positions of two particles coincide.

The pairs (u, v) and $(\bar{v}, \bar{u})$ define two equivalent irreducible representations of the permutation group S_3: for example, the pair (u, v) transforms, under the six possible permutations, into (u, v), $(\eta u, \eta^2 v)$, $(\eta^2 u, \eta v)$, (v, u), $(\eta v, \eta^2 u)$, $(\eta^2 v, \eta u)$, respectively. Therefore fully symmetric quantities are those of the form $u^k \bar{u}^l v^m \bar{v}^n + v^k \bar{v}^l u^m \bar{u}^n$ with $k - l - m + n \equiv 0 \pmod 3$. The pairs $(a_u^\dagger, a_v^\dagger)$, $(b_v^\dagger, b_u^\dagger)$, and $(\bar{v}, \bar{u})$ all transform like (u, v). Taking successive products of the above-mentioned representations of S_3 and decomposing them into irreducible representations, it is straightforward to prove that all symmetric polynomials in $a_u^\dagger, a_v^\dagger, b_u^\dagger, b_v^\dagger$ can be expressed as polynomials in the following basic symmetric polynomials:

$$\begin{array}{lrcl}
(k\ l\ m\ n) & & & \\
(0\,3\,0\,0): & c_{3,-3}^\dagger & = & (b_u^\dagger)^3 + (b_v^\dagger)^3, \\
(0\,1\,0\,1): & c_{2,-2}^\dagger & = & b_u^\dagger b_v^\dagger, \\
(0\,2\,1\,0): & c_{3,-1}^\dagger & = & (b_u^\dagger)^2 a_v^\dagger + (b_v^\dagger)^2 a_u^\dagger, \\
(1\,1\,0\,0): & c_{20}^\dagger & = & a_u^\dagger b_u^\dagger + a_v^\dagger b_v^\dagger, \\
(2\,0\,0\,1): & c_{31}^\dagger & = & (a_u^\dagger)^2 b_v^\dagger + (a_v^\dagger)^2 b_u^\dagger, \\
(1\,0\,1\,0): & c_{22}^\dagger & = & a_u^\dagger a_v^\dagger, \\
(3\,0\,0\,0): & c_{33}^\dagger & = & (a_u^\dagger)^3 + (a_v^\dagger)^3.
\end{array} \tag{18}$$

The meaning of the subscripts is that

$$[H, c^\dagger_{pq}] = p\, c^\dagger_{pq}, \qquad [L, c^\dagger_{pq}] = q\, c^\dagger_{pq}, \tag{19}$$

so $c^\dagger_{pq}$ changes the energy by p and the angular momentum by q. It remains to check for possible singularities at $u = v$. All the $+3$ states have the form, up to the Gaussian factor $\exp(-\bar{u}u - \bar{v}v)$,

$$P(z_i, \bar{z}_i)\,[(z_1 - z_2)(z_2 - z_3)(z_3 - z_1)]^\nu = P(u, \bar{u}, v, \bar{v})(u^3 - v^3)^\nu$$

with P a polynomial and $\nu \geq 0$. Therefore the only source of singularity can be differentiation of the last factor. This immediately implies that $a^\dagger_u$, which involves $\partial_{\bar{u}}$ only, is always regular, and so is $a^\dagger_v$; hence, $c^\dagger_{22}$ and $c^\dagger_{33}$ are regular. Now,

$$b^\dagger_u (u^3 - v^3)^\nu = \frac{1}{\sqrt{2}}\bar{u}(u^3 - v^3)^\nu - \frac{3\nu}{\sqrt{2}} u^2 (u^3 - v^3)^{\nu-1}$$

shows that $b^\dagger_u$ by itself may be singular, due to the last term. Nevertheless,

$$\begin{aligned} c^\dagger_{20}(u^3 - v^3)^\nu &= \text{regular terms} - \frac{3\nu}{2} u^3 (u^3 - v^3)^{\nu-1} + \frac{3\nu}{2} v^3 (u^3 - v^3)^{\nu-1} \\ &= \text{regular terms} - \frac{3\nu}{2}(u^3 - v^3)^\nu \end{aligned}$$

is regular. Further,

$$c^\dagger_{31}(u^3 - v^3)^\nu = \text{regular terms} - \frac{3\nu}{2} u^2 v^2 (u^3 - v^3)^{\nu-1} + \frac{3\nu}{2} v^2 u^2 (u^3 - v^3)^{\nu-1},$$

but $c^\dagger_{3,-3}$, $c^\dagger_{2,-2}$ and $c^\dagger_{3,-1}$ are singular. A generic (not normalized) linear state of slope $s = +3$ may then be written as

$$\begin{aligned} \psi_{klmn}(u, v; \nu) &= (c^\dagger_{20})^k (c^\dagger_{31})^l (c^\dagger_{22})^m (c^\dagger_{33})^n \psi_0(u, v; \nu) \\ &= (c^\dagger_{20})^k (c^\dagger_{31})^l (c^\dagger_{22})^m (c^\dagger_{33})^n (u^3 - v^3)^\nu e^{-\bar{u}u - \bar{v}v}, \end{aligned} \tag{20}$$

with

$$E = 2k + 3l + 2m + 3n + 3\nu + 2, \tag{21}$$

$$L = l + 2m + 3n + 3\nu. \tag{22}$$

(For the -3 states, of course, $c^\dagger_{p,-q}$ would take the place of $c^\dagger_{pq}$.) This is a classification of *all* linear three-anyon states (cf. [5, 8, 17]). However, some of these states are tower excitations and some are different parts of the same trajectories. First, the operator $c^\dagger_{20}$ is nothing but the tower raising operator. It never produces singularities when acting on any state, linear or nonlinear [18], thus providing the tower structure of the whole spectrum. Hence put $k = 0$. Second, note that $\psi_{00mn}(u, v; \nu) = 2^{m+3n/2}(uv)^m(u^3 + v^3)^n(u^3 - v^3)^\nu e^{-\bar{u}u-\bar{v}v}$ and consequently

$$\begin{aligned}\psi_{0,0,m,n+2}(u, v; \nu) - 4\psi_{0,0,m+3,n}(u, v; \nu) &= 8[(u^3 + v^3)^2 - 4(uv)^3]\psi_{00mn}(u, v; \nu) \\ &= 8(u^3 - v^3)^2\psi_{00mn}(u, v; \nu) \\ &= 8\psi_{00mn}(u, v; \nu + 2);\end{aligned}$$

in other words, the wave functions (20), being linearly independent for $0 \leq \nu < 2$, are no longer so for $0 \leq \nu < \infty$, therefore some of them do not lead to new trajectories and should be excluded from the trajectory counting. As the last formula shows, it is sufficient to restrict to $n < 2$. Thus, a bottom trajectory may be labeled by three numbers $l, m = 0, 1, 2, \ldots$ and $n = 0, 1$. This is completely equivalent to the labeling by (E_+, L_+), because it follows from (21)–(22) that

$$E_+ = 3l + 2m + 3n + 2, \tag{23}$$

$$L_+ = l + 2m + 3n, \tag{24}$$

since the point of slope change to $+3$ here is $\nu = 0$, and it is straightforward to see that for any point (E_+, L_+) there is no more than one set $\{lmn\}$ such that these two equations are satisfied.

The set of bottom trajectories being *two*-parametric (plus a "double degeneracy" due to n) is due to the fact that a bottom state is identified by *three* quantum numbers (six degrees of freedom, minus two for the center of mass and one for the tower excitations). Two quantum numbers (say, E_+ and L_+) identify a trajectory, and the third one (L) chooses a point on that trajectory. To compare, in the two-anyon problem, only *one* quantum number, L, is enough to identify a bottom state, consequently there is only one trajectory. In general, for the N-anyon problem, the family of trajectories will be $(2N - 4)$-parametric. Let us return once more to Fig. 1; there are two copies of one and the same two-dimensional pattern, made of bullets ($n = 0$) and triangles ($n = 1$), and increasing m or l by 1 means moving within the pattern (3 units to the right and 3 units up, or 1 unit to the right and 3 units up, respectively).

Thus the counting of trajectories is complete, but to find the wave functions, certain modifications of eq. (20) are still necessary. First, we need only such functions for which $c_{20}\psi = 0$, where $c_{20} = a_u b_u + a_v b_v$ is the tower lowering operator, but since c_{20} does not commute with the $c^\dagger$'s, even for $k = 0$ some of the functions (20) will contain an admixture of non-bottom states with the same E and L. To correct for this, it turns out to be sufficient to replace $(c_{31}^\dagger)^l$ in (20) by another operator, writing

$$\begin{aligned}\psi_{lmn}(u, v; \nu) &= C_l^\dagger(c_{22}^\dagger)^m(c_{33}^\dagger)^n\psi_0(u, v; \nu) \\ &\propto C_l^\dagger(uv)^m(u^3 + v^3)^n(u^3 - v^3)^\nu e^{-\bar{u}u-\bar{v}v},\end{aligned} \tag{25}$$

where

$$C_l^\dagger = Q^l[(a_u^\dagger)^3 - (a_v^\dagger)^3]^l \tag{26}$$

and

$$Q = a_u b_v^\dagger - a_v b_u^\dagger \tag{27}$$

is Sen's supersymmetry operator [4, 18]. Clearly, $C_l^\dagger$ is symmetric, and it is straightforward to check that $c_{20}\psi_{lmn} = 0$ always [c_{20} commutes with Q and gives zero when acting on $f(u,v)e^{-\bar{u}u-\bar{v}v}$]. Also, C_l changes E and L by the same amount as $(c_{31}^\dagger)^l$, so that the formulas (23) and (24) remain valid.

Second, a wave function that is continuous at the point (E_+, L_+)—and thus corresponds to a trajectory—is, in general, not one of the form (25) but a linear combination of several functions of that form. Indeed, for any fixed l, m, n and $0 \le \nu < 2$ we have $(d+1)$ degenerate $s = +3$ states $\{\psi_{l,m-3j,n}(u,v;\nu+2j),\ j = 0,1,\ldots,d\}$, where $d = [m/3]$. These are linearly independent, but not orthogonal; and which of their linear combinations to choose as a basis in the relevant $(d+1)$-dimensional subspace, makes no difference anywhere except at the point $\nu = 0$. Here, the states with $j = 1,\ldots,d$ continue as linear to $\nu < 0$, and the basis in the relevant d-dimensional subspace may still be chosen at will, but one state "decouples", i.e., becomes nonlinear, and that is the one orthogonal to this subspace. In general, it will be not $\psi_{lmn}(u,v,0)$ but a sum $\sum_{j=0}^{d} a_j \psi_{l,m-3j,n}(u,v;\nu+2j)$, where the numbers a_j/a_0, $j = 1,\ldots,d$, are determined by the d orthogonality conditions. Consider the simplest example $l = 0$, $m = 3$, $n = 0$, which corresponds to the bosonic point $(8,6)$, see Fig. 2c. The two relevant states of the form (25) are $\psi_I \equiv \psi_{030}(u,v;\nu)$ and $\psi_{II} \equiv \psi_{000}(u,v;\nu+2)$. At $\nu > 0$, it makes no difference which two linear combinations of them to choose; but at $\nu = 0$, ψ_{II} continues to be linear, while the state which actually "decouples" and continues to $\nu < 0$ as nonlinear, is the one orthogonal to ψ_{II} at $\nu = 0$. An elementary calculation shows that it is $\psi_I + \frac{4}{11}\psi_{II}$. Such reasoning has to be repeated at each bosonic point with $m \ge 3$, and general expressions for the coefficients a_j apparently do not exist.

4 The nonlinear parts of the trajectories

We have shown that each trajectory is uniquely identified by its (E_+, L_+) point; the point being given, one should in principle be able to reconstruct the behavior of the whole trajectory, and in particular to find all the bosonic and fermionic points it passes through. In fact, it would be sufficient to find (E_-, L_-) and the behavior between that point and (E_+, L_+) [where $E(L)$ is nonlinear]. To see whether this can be achieved, we plot the low-lying trajectories, using the exact multiplicities given in [11], on Fig. 2a–f. (There are six different values of $L(0)$ and consequently six plots, one for each class of trajectories.) The nonlinear pieces are shown schematically only. The continuity of the trajectories is confirmed by the fact that at bosonic and fermionic points, where they cross, the total number of trajectories coming from the left and from the right is always the same. The main question is how to identify which piece on the left is a continuation of which piece on the right. If at a certain crossing point perturbation theory works and the first-order corrections to energy are different for all the states involved, then the rule is that the order of trajectories, by increasing energy, on the right is opposite to that on the left (because the first-order corrections have opposite signs). This argument does not work (a) if the first-order corrections to some of the states are equal and (b) when perturbation theory breaks down, at certain bosonic points, as mentioned above. A way out then is to find the wave functions numerically, using the method of [16] and to identify them by comparing their limits as the crossing point is approached. As it appears, the above rule—the trajectory which is the n-th from above on the left is the n-th from below on the right—holds for all crossings analyzed. In other words, all crossings appear to be *true* crossings. When this rule is used, it becomes possible to identify the trajectories completely without actually finding the wave functions. It turns out that for all the trajectories there is at most one point of slope change, apart from (E_+, L_+) and (E_-, L_-), which we denote by (E_0, L_0), the change being from -1 to $+1$ as L increases through L_0; clearly, $L_- \leq L_0 \leq L_+$. A generic behavior of a trajectory is, therefore,

$$\stackrel{-3}{\to} (E_-, L_-) \stackrel{-1}{\to} (E_0, L_0) \stackrel{+1}{\to} (E_+, L_+) \stackrel{+3}{\to}$$

(numbers on top of the arrows meaning slopes), but in fact it may be $L_0 = L_-$ or $L_0 = L_+$ (for the ground state, $L_- = L_0 = L_+ = 0$).

Table 1 shows the points of slope change for all the trajectories with $E_+ \leq 12$.

E_-	L_-	E_0	L_0	E_+	L_+
2	0	2	0	2	0
6	−4	4	2	4	2
7	−5	5	1	5	1
5	−3	4	0	5	3
4	−2	4	−2	6	4
11	−9	7	3	7	3
5	−1	5	−1	7	5
8	−4	7	−1	8	2
8	−2	7	1	8	4
8	−6	6	0	8	6
9	−7	7	−1	9	5

E_-	L_-	E_0	L_0	E_+	L_+
9	−5	7	1	9	7
10	−8	8	−2	10	4
10	−6	8	0	10	6
10	−4	8	2	10	8
11	−3	10	0	11	3
13	−7	10	2	11	5
13	−11	9	1	11	7
7	−3	7	−3	11	9
14	−12	10	0	12	6
14	−10	10	2	12	8
14	−8	10	4	12	10

Table 1. The behavior of the low-lying trajectories.

The regularity is not obvious here, but it becomes visible when one looks at sufficiently high-lying trajectories. Introduce the quantities

$$n_+ = \frac{L_+ - L_0}{6}, \qquad n_- = \frac{L_0 - L_-}{6}, \tag{28}$$

that is, the "numbers of revolutions", or the numbers of bosonic points that a trajectory passes through having slopes $+1$ and -1, respectively; these may be integer or half-integer, as L_0 may correspond to either a bosonic or a fermionic point. The following obvious relations hold

$$E_0 = E_+ - 2n_+, \qquad L_0 = L_+ - 6n_-, \tag{29}$$
$$E_- = E_+ - 2(n_+ - n_-), \quad L_- = L_+ - 6(n_+ + n_-). \tag{30}$$

The results, deduced from the numerical analysis, are the following:

$$\frac{E_+}{3} < L_+ < \frac{E_+}{2}: \qquad n_+ = \frac{3L_+ - E_+}{4}, \quad n_- = \frac{E_+ - 2L_+}{2}, \quad L_0 \text{ is fermionic}, \tag{31}$$
$$\frac{E_+}{2} < L_+ < E_+: \qquad n_+ = \frac{E_+}{8}, \quad n_- = \frac{E_+}{8}, \quad L_0 \text{ is bosonic}. \tag{32}$$

The formulas, as well as the inequality signs, are valid asymptotically for $E \gg 1$, in the same sense as (10)–(11). One can verify the compatibility of (31)–(32) with (10)–(11); also, a simple consistency check comes from the symmetry requirement—a trajectory $(E_-, L_-) \to (E_0, L_0) \to (E_+, L_+)$ must have its partner $(E_+, -L_+) \to (E_0, -L_0) \to (E_-, -L_-)$: this is satisfied as well. Note that n_+ depends continuously on L_+, but there is a discontinuity in n_- at the point $L_+ = E_+/2$. Also, the exact results exhibit certain periodic structure with period 8 by E_+ and 4 by L_+. Explaining these features remains an open item.

A semiclassical interpretation—rather rough, although—of some features of the trajectories may be adduced. When two particles are close together and the third one is far from them, that is, say, $|\rho_{12}| \ll |\rho_3|$ $[\rho_{12} = z_1 - z_2,\ \rho_3 = (z_1 + z_2)/2 - z_3]$, one gets two independent anyonic oscillators, one (ρ_{12}) with the statistics parameter ν, the other (ρ_3) with the parameter 2ν [5, 15]. The energy in this approximation is $|2l_{12} + \nu| + 2n_{12} + 1 + |2l_3 + \nu| + 2n_3 + 1$, where in general $l_3 \gg l_{12}$; as ν increases through $-2l_{12}$, the slope changes from -3 to -1 (if $l_3 < 0$) or from $+1$ to $+3$ (if $l_3 > 0$); the other two points of change, however, can not be described correctly, nor can the formulas (31)–(32) be explained. Perhaps a more accurate semiclassical approximation would be able to explain them.

5 Conclusion

We have developed the concept of trajectories for anyons, noting that many different many-anyon states are in fact continuations of each other. We discuss in particular the harmonic oscillator external potential. Apart from the center-of-mass and tower excitations, in the two-anyon problem there is only one trajectory. For the three-anyon problem, we have worked out the classification and the main features of the trajectories: The slope of a trajectory always changes as $-3 \to -1 \to +1 \to +3$ (where the $+1$ and/or -1 pieces may be missing), and the point (E_+, L_+) of the last change may be used to label the trajectory. The wave function corresponding to the linear part of the trajectory can be written down exactly by applying excitation operators to the ground trajectory. Concerning the nonlinear part, we conjecture the formulas, based on a numerical analysis, expressing the lengths of intervals of slope -1 and $+1$ in terms of (E_+, L_+).

Qualitatively, the picture will be the same for the N-anyon problem: Again, each trajectory will have the points (E_+, L_+) and (E_-, L_-) such that its $E(L)$ dependence is linear with slopes $\pm N(N-1)/2$ for $L > L_+$ ($L < L_-$) and nonlinear in between. Here the family of trajectories will be $(2N-4)$-parametric. It is quite plausible that in this case as well, the sequence of slopes will be regular, from $-N(N-1)/2$ up to $+N(N-1)/2$ in steps of two. However, to gain a more precise understanding of the behavior of the trajectories remains an open problem.

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Figure captions

Fig. 1. The distribution in the (E, L) plane of the points (E_+, L_+) at which the trajectories turn linear with slope $s = +3$. The bullets define the set $E_+ = 3\ell + 2m + 2$, $L_+ = \ell + 2m$, and the triangles define the set $E_+ = 3\ell + 2m + 5$, $L_+ = \ell + 2m + 3$, with $\ell = 0, 1, \ldots$ and $m = 0, 1, \ldots$ in both cases.

Fig. 2, a–f. The six classes of trajectories in the (E, L) plane, with $L(0) = -2, -1, 0, 1, 2, 3$; solid and dashed vertical lines mean Bose and Fermi statistics, respectively, dashes show the multiplicities of the linear states.