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Adelic Integrable Systems[1]

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Abstract

Incorporating the zonal spherical function (zsf) problems on real and p-adic hyperbolic planes into a Zakharov-Shabat integrable system setting, we find a wide class of integrable evolutions which respect the number-theoretic properties of the zsf problem. This means that at *all* times these real and p-adic systems can be unified into an adelic system with an S-matrix which involves (Dirichlet, Langlands, Shimura...) L-functions.

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1. Introduction

Scattering theory on real [1] and p-adic [2] symmetric spaces can be unified in an adelic context [2], [3]. This has the virtue of producing S-matrices involving the Riemann zeta function and of throwing new light on earlier work [4] concerning scattering on the noncompact finite-area fundamental domain of $SL(2,\mathbb{Z})$ on the real hyperbolic plane H_∞.

The real hyperbolic plane is a smooth manifold and as such quantum mechanics on H_∞ involves a second order Schrödinger *differential* equation. By contrast the p-adic hyperbolic planes H_p are discrete spaces (trees), and the corresponding Schrödinger equations are second order *difference* equations. The Jost functions, and therefore the S-matrices from all these *local* problems combine in adelic products, which then involve the Riemann zeta function [2].

At a given initial time consider all these ("S-wave") scattering problems and then let all of them undergo an *integrable* time evolution. In general such an evolution need not respect the number-theoretic endowment of the initial problem. In other words, even though at the initial time the real and p-adic scattering problems assembled into an interesting adelic scattering problem, at later times this need no longer be so. We want to explore here the conditions under which the integrable evolution respects adelizability and to see what kind of scattering problems can be obtained this way at later times. Specifically, we will incorporate the initial scattering problem into a Zakharov-Shabat (ZS) system and follow its integrable evolution. For the p-adic problems, time has to be discrete and for adelic purposes time then has to be discrete in the real problem as well. We will see that adelic products can be meaningful at later times in the evolution of such a system and that along with the Riemann zeta function involved in the adelic problem at the initial time, various (Dirichlet, Langlands, Shimura,...) L-functions [5] appear at later times.

2. Integrable Evolution of the p-adic Zonal Spherical Function Problem

As mentioned in the introduction, we consider "S-wave" scattering problems on local (real and p-adic) hyperbolic planes and embed them in integrable ZS systems. We start by setting up quantum mechanics on these hyperbolic planes and then finding S-wave solutions (i.e. solutions independent on the angular variable) thereof. Dealing with S-waves corresponds mathematically to the zonal spherical function (zsf) problems on these hyperbolic planes. We first consider the p-adic hyperbolic plane $H_p = SL(2,\mathbb{Q}_p)/SL(2,\mathbb{Z}_p)$. This H_p is a discrete space, a homogeneous Bruhat-Tits-Bethe tree of incidence number

$p+1$ (or equivalently, branching number p) and the radial coordinate is an integer, say n. The zsf w_n on H_p solves the familiar difference equation [2]

$$pw_{n+2} - \sqrt{p}\big(p^{\frac{ik}{2}} + p^{-\frac{ik}{2}}\big)w_{n+1} + w_n = 0. \tag{2.1}$$

It is this equation that we wish to embed in an integrable ZS system. We do this as follows. Consider the two-component ZS system governed by the equations

$$\begin{aligned} u_{n+1} &= zu_n + Q_n v_n \\ v_{n+1} &= P_n u_n + z^{-1} v_n \end{aligned} . \tag{2.2}$$

By decoupling this set of first order difference equations, one obtains separate second order difference equations for the two components of the ZS doublet. In particular v_n obeys

$$\frac{P_n}{P_{n+1}(1-P_nQ_n)}v_{n+2} - \frac{1}{1-P_nQ_n}\left[\frac{P_n}{P_{n+1}}z^{-1} + z\right]v_{n+1} + v_n = 0. \tag{2.3}$$

We achieve the embedding of the zsf equation (2.1) in the ZS system, by requiring that this v_n essentially reproduce the zsf w_n, specifically that for some real ν

$$w_n = p^{-\nu n} v_n. \tag{2.4}$$

For this to be the case, Q_n and P_n must obey the relations

$$\begin{aligned} &\frac{P_n}{P_{n+1}(1-P_nQ_n)}p^{2\nu} = p \\ &\frac{1}{1-P_nQ_n}\left[\frac{P_n}{P_{n+1}}z^{-1} + z\right]p^{\nu} = \sqrt{p}\big(p^{\frac{ik}{2}} + p^{-\frac{ik}{2}}\big). \end{aligned} \tag{2.5}$$

Here we must require Q_n and P_n to be z-independent and the relation between z and k to be $n-$independent. These requirements result in

$$z = p^{\frac{ik-\rho}{2}} \tag{2.6a}$$

where for convenience we introduced the new parameter

$$\rho = 2\,\nu - 1 \tag{2.6b}$$

with ν as in equation (2.4).

On account of the invariance of equation (2.1) under a sign change for k, in equation (2.6) k can just as well be replaced by $-k$. We opt for the relation (2.6a), as it stands. When used in eq.(2.5) it gives

$$\frac{P_{n+1}}{P_n} = p^{\rho}, \quad Q_n = 0 \tag{2.7}$$

so that

$$P_n = \sigma_p(0)\, p^{\rho n} \quad \text{and} \quad Q_n = 0, \tag{2.8}$$

where $\sigma_p(0)$ is an integration constant.

This determines, via (2.1) and (2.4)

$$\begin{aligned} u_n^0 &= p^{\frac{(ik-\rho)n}{2}}\, u_0^0 \\ v_n^0 &= \frac{p}{p+1}\left[p^{\frac{(ik+\rho)n}{2}}\, c^0(k;p) + p^{\frac{(-ik+\rho)n}{2}}\, c^0(-k;p)\right] \end{aligned} \tag{2.9}$$

where $c^0(k;p)$ is the Jost-Harish-Chandra function for the scattering problem on a tree [2],

$$c^0(k;p) = \frac{\zeta_p(ik)}{\zeta_p(ik+1)} = \frac{1-p^{-ik-1}}{1-p^{-ik}} \tag{2.10}$$

and

$$u_0^0 = \frac{p^{\frac{ik+\rho+2}{2}} - p^{\frac{-ik+\rho}{2}}}{\sigma_p(0)(p+1)} \tag{2.11}$$

We now wish to find an integrable time-evolution for this system. We will assume, throughout this paper, that time is discrete and its value will be indicated by a superscript, say m. Imposing the condition that at any later time the spatial dependence of the system should still be of the ZS type, leads to a compatibility relation between the time and space evolutions. Let us introduce matrix notation

$$\Psi_n^m = \begin{pmatrix} u_n^m \\ v_n^m \end{pmatrix} \quad R_n^m = \begin{pmatrix} z & Q_n^m \\ P_n^m & z^{-1} \end{pmatrix} \tag{2.12a}$$

where as explained the superscript m indicates time and the subscript n distance to the origin of the tree. The ZS equations at time m are then

$$\Psi_{n+1}^m = R_n^m \Psi_n^m \tag{2.12b}.$$

Introducing the time evolution matrix

$$M_n^m = \begin{pmatrix} A_n^m & B_n^m \\ C_n^m & D_n^m \end{pmatrix}, \tag{2.12c}$$

the time evolution of the system is governed by the equation

$$\Psi_n^{m+1} = M_n^m \Psi_n^m. \tag{2.12d}$$

The compatibility condition between the time evolution and the eigenvalue problem is

$$R_n^{m+1} M_n^m = M_{n+1}^m R_n^m. \tag{2.13}$$

In order to solve equations (2.11) we need to make some further assumptions, as otherwise we have six unknowns and four relations. We will expand the elements of the time-evolution matrix in powers of z, and retain only the first term. However, because of the way z and z^{-1} occur in R_n^m, these expansions will be chosen differently [6] for the four matrix elements of M_n^m:

$$\begin{array}{ll} A_n^m = a_n^m + z^{-2}\alpha_n^m & \qquad C_n^m = z c_n^m + z^{-1}\gamma_n^m \\ B_n^m = z b_n^m + z^{-1}\beta_n^m & \qquad D_n^m = d_n^m + z^2 \delta_n^m \end{array} . \tag{2.14}$$

As shown in detail in Appendix A, these assumptions lead to the following solution for the ZS system (2.12):

$$R_n^m = \begin{pmatrix} z & 0 \\ \sigma_p(m) p^{\rho(n-m)} & z^{-1} \end{pmatrix} \tag{2.15a}$$

and

$$M_n^m = \begin{pmatrix} a(m) + z^{-2}\,\alpha(m) & 0 \\ -p^{\rho(n-m-1)}\big[\sigma_p(m)\delta(m) z + \sigma_p(m+1)\alpha(m) z^{-1}\big] & d(m) + z^2 \delta(m) \end{pmatrix} \tag{2.15b}$$

where $\sigma_p(m)$ is an arbitrary function of the discrete time m and $a(m)$, $\alpha(m)$, $d(m)$ and $\delta(m)$ are equal respectively to a_0^m, α_0^m, d_0^m and δ_0^m of eq.(2.14) and must be related by:

$$\sigma_p(m+1)\big[\alpha(m)\, p^\rho + a(m)\big] = \sigma_p(m)\big[\delta(m) + d(m)\, p^\rho\big] \tag{2.15c}$$

Just like $\sigma_p(m)$, the quantities $a(m)$, $\alpha(m)$, $d(m)$ and $\delta(m)$ all depend, of course, on the Bethe lattice branching number p, but we choose not to explicitly indicate this dependence.

Corresponding to this solution, the ZS doublet, which at time $m = 0$ takes the form (2.9), evolves into:

$$
\begin{aligned}
u_n^m &= u_n^0 \prod_{j=0}^{m-1} [a(j) + p^{\rho - ik}\alpha(j)] \\
v_n^m &= \frac{p}{p+1} p^{\frac{\rho n}{2}} \Big[p^{\frac{ikn}{2}} c^0(k;p) \frac{\sigma_p(m)}{\sigma_p(0)} \prod_{j=0}^{m-1} [a(j)p^{-\rho} + \alpha(j)p^{-ik}] + \\
&\qquad + p^{-\frac{ikn}{2}} c^0(-k;p) \prod_{j=0}^{m-1} [d(j) + p^{ik-\rho}\delta(j)] \Big].
\end{aligned}
\tag{2.16}
$$

with u_n^0 given by eq. (2.9). For the S-matrix at $m \neq 0$ to be unitary, in the expression for v_n^m, $p^{\frac{ikn}{2}}$ and $p^{-\frac{ikn}{2}}$ must have complex conjugate coefficients, which requires that equation (2.15c) be replaced by the stronger pair of equations:

$$
\begin{aligned}
\sigma_p(m+1)\, \alpha(m)\, p^\rho &= \sigma_p(m)\, \delta(m) \\
\sigma_p(m)\, d(m)\, p^\rho &= \sigma_p(m+1)\, a(m),
\end{aligned}
\tag{2.15d}
$$

which, of course, imply eq.(2.15c).

With these relations one can eliminate $a(m)$ and $\alpha(m)$ and the formulae for u_n^m and v_n^m simplify considerably. To see this, it is convenient to renormalize u and v by removing from them an overall factor which depends *only* on the discrete time m. This way, we introduce the new functions $\tilde{u}_n^m$ and $\tilde{v}_n^m$ as follows:

$$
\begin{aligned}
\tilde{u}_n^m &= f(m)\, u_n^m \\
\tilde{v}_n^m &= g(m)\, v_n^m
\end{aligned}
\tag{2.17}
$$

with

$$
\begin{aligned}
f(m) &= \prod_{j=0}^{m-1} d^{-1}(j) \\
g(m) &= \frac{\sigma_p(m)}{\sigma_p(0)}\, f(m)
\end{aligned}
\tag{2.18}
$$

Then, the final simplified form of $\tilde{u}_n^m$ is

$$
\tilde{u}_n^m = \Lambda(m, \chi(p), k; p) u_n^0 \tag{2.19}
$$

and $\tilde{v}_n^m$ is given by the second equation (2.9) with the Jost-Harish-Chandra $c-$function $c^0(k;p)$ replaced by

$$
c^m(k;p) = c^0(k;p)\, \Lambda(m, \chi(p), k\,; p). \tag{2.20}
$$

In equations (2.19) and (2.20), the function Λ is given by

$$\Lambda(m, \chi(p), k\,; p) = \prod_{j=0}^{m-1} \left[1 - \chi^j(p)^{-ik}\right] \tag{2.21}$$

with

$$\chi^j(p) = -\frac{\delta(j)}{d(j)}\, p^{-\rho}. \tag{2.22}$$

In other words $\tilde{u}_n^m$ and the $c-$function both accrete the same factor Λ.

In the tendentious notation just used, it is clear that Λ is a candidate factor in an Euler product, provided only $\chi^j(p)$ has a "nice" $p-$dependence. By this we mean that in the simplest case $\chi^j(p)$ is a Dirichlet character, or something similar. We shall return to this point when we treat the adelic problem.

Finally, $\tilde{u}_n^m$ does not correspond to a scattering problem, since it does not contain an incoming wave. Yet, even $\tilde{u}_n^m$ has encoded in it an object $l^m(k,p)$, which in adelic considerations will play a role similar to that played by the Jost-Harish-Chandra function $c^m(k,p)$, which is encoded in the v's. This $l^m(k,p)$ is found by rewriting $\tilde{u}_n^m$ in the form

$$\tilde{u}_n^m = l^m(k,p) \frac{p^{\frac{ik(n+1)-\rho(n-1)+2}{2}}}{(p+1)\sigma_p(0)}. \tag{2.23}$$

Comparing with equations (2.19), (2.9)-(2.11), we then find

$$l^m(k,p) = \frac{\Lambda(m, \chi(p), k; p)}{1 - p^{-(ik+1)}}. \tag{2.24}$$

Next we have to deal with the place at infinity, which involves the real hyperbolic plane $H_\infty = SL(2, \mathbb{Q}_\infty)/SO(2, \mathbb{Q}_\infty)$ where $\mathbb{Q}_\infty \equiv \mathbb{R}$ is the field of real numbers.

3. Integrable evolution of the real Zonal Spherical Function Problem

The adelic partner of the zsf problems on the p-adic hyperbolic planes tackled in section 2, is the zsf problem on the real hyperbolic plane $H_\infty = SL(2,\mathbb{R})/SO(2,\mathbb{R})$. The corresponding eigenvalue equation for the radial Laplacian is:

$$w'' + 2\coth 2x\, w' + (k^2+1)\, w = 0 \tag{3.1}$$

This is no longer a difference equation, but rather a differential equation, ($w' \equiv \mathrm{d}w/\mathrm{d}x$), since, unlike its p-adic counterparts, the real hyperbolic plane is a continuous manifold and not a discrete Bruhat-Tits-Bethe tree. We therefore encounter a continuous radial coordinate x, while time m in the ZS system must stay discrete, for a proper match with the p-adic cases. The standard ZS problem in this case is [6]

$$\begin{aligned}(u^0)' &= i\zeta u^0 + q^0 v^0\\ (v^0)' &= P^0 u^0 - i\zeta v^0\end{aligned} \tag{3.2}$$

with ζ the counterpart of the spectral variable z. Specifically, ζ and z are related as

$$z = \mathrm{e}^{i\zeta}. \tag{3.3}$$

Were we to deal with this problem, we could easily find its solution and integrable time evolution. Yet expanding in ζ *not* being the same as expanding in z in section 2, the nice adelic match would be lost. Therefore, we shall consider a different, adelically better suited alternative, namely

$$\begin{aligned}(u^0)' &= z\, u^0 + Q^0(x)\, v^0\\ (v^0)' &= P^0(x)\, u^0 + z^{-1}\, v^0\end{aligned}\,. \tag{3.2$'$}$$

We will return to the original ZS problem (3.2) in Appendix C.

Just like in the discrete case, the second order equation obeyed by v^0 is

$$(v^0)'' - (v^0)'\Big[z + z^{-1} + \frac{(P^0)'}{P^0}\Big] + v^0\Big[1 - Q^0P^0 + z^{-1}\frac{(P^0)'}{P^0}\Big] = 0 \tag{3.4}$$

which, after a 'gauge' transformation similar to (2.3),

$$v^0(x) = w^0(x)\tau(x) \tag{3.5}$$

upon comparison with (3.1) yields the conditions

$$\begin{aligned}2\coth 2x &= 2\frac{\tau'}{\tau} - z - z^{-1} - \frac{(P^0)'}{P^0}\\ k^2 + 1 &= \frac{\tau''}{\tau} - \frac{\tau'}{\tau}\Big[z + z^{-1} + \frac{(P^0)'}{P^0}\Big] + 1 - P^0Q^0 + z^{-1}\frac{(P^0)'}{P^0}\end{aligned} \tag{3.6}$$

where P^0 and Q^0 are z-independent, and the relation between z and k has to be x-independent. Once these conditions are imposed, it follows that

$$\begin{aligned}P^0 &= \sigma(0)\mathrm{e}^{2\nu x}\\ Q^0 &= -\frac{1}{\sigma(0)\sinh^2 2x}\mathrm{e}^{-2\nu x}\end{aligned} \tag{3.7}$$

and

$$
\begin{aligned}
\tau(x) =& \tau_0(x)(\sinh 2x)^{1/2} \mathrm{e}^{(\nu + \frac{z+z^{-1}}{2})x} \\
z - z^{-1} =& - 2\nu \pm 2ik
\end{aligned}
\tag{3.8}
$$

with ν an arbitrary complex number. Again, everything is symmetric with respect to changing the sign of k, and we chose the $+$ sign in (3.8). The x-dependence in relations (3.7) shows that at large x only one of the two functions, P^0 and Q^0, survives, depending upon the real part of ν. In what follows, without any loss of generality, we assume $\mathrm{Re}\,\nu \geq 0$, so as to preserve the resemblance with the discrete case.

The $x \to \infty$ asymptotic form of the solution to (3.1) is

$$
\begin{aligned}
&u^0(x) \sim 2ik\, \frac{\tau(x)}{\sigma(0)} c^0\,(k;\infty) \mathrm{e}^{(ik-1-2\nu)x} \\
&v^0(x) \sim \tau(x) \big[c^0\,(k;\infty)\, \mathrm{e}^{(ik-1)x} + c^0\,(-k;\infty)\, \mathrm{e}^{-(ik+1)x} \big]
\end{aligned}
\tag{3.9}
$$

where

$$
c^0\,(k;\infty) = \pi^{-1/2} \frac{\Gamma\big[\frac{1}{2} ik\big]}{\Gamma\big[\frac{1}{2}(ik+1)\big]} = \frac{\zeta_\infty(ik)}{\zeta_\infty(ik+1)}
\tag{3.10}
$$

Next, we need to find an integrable time evolution for the system that is compatible with (3.1). The compatibility condition in this case is

$$
(M^m)' = R^{m+1} M^m - M^m R^m
\tag{3.11}
$$

and the superscript m again stands for discrete time. We shall solve equation (3.11) subject to the condition that at *all* times m, the asymptotic R matrix at large radial distances x have the form

$$
R^m \sim \begin{pmatrix} z & 0 \\ \sigma(m)\mathrm{e}^{2\nu x} & z^{-1} \end{pmatrix}.
\tag{3.12}
$$

This insures a proper adelic match with the p-adic cases treated in Section 2.

With this choice, some straightforward calculations give the following form for the M-matrix

$$
M^m \sim
\begin{pmatrix}
a(m,k) - \frac{b(m,k)}{2ik}\sigma(m)\mathrm{e}^{2ikx} & b(m,k)\mathrm{e}^{2(ik-\nu)x} \\
\mathrm{e}^{2\nu x}\left[\frac{\sigma(m+1)a(m,k)-\sigma(m)d(m,k)}{2ik} + c(m,k)\mathrm{e}^{-2ikx} + \frac{b(m,k)}{4k^2}\mathrm{e}^{2ikx}\right] & d(m,k) + \frac{b(m,k)}{2ik}\sigma(m+1)\mathrm{e}^{2ikx}
\end{pmatrix}
\tag{3.13}
$$

where $a(m,k)$, $b(m,k)$, $c(m,k)$ and $d(m,k)$ all are x-independent. The complete derivation of these results is given in Appendix B. The ZS doublet at time m then has the asymptotic form

$$
\begin{aligned}
u^m(x) &\sim u^0(x)\prod_{j=0}^{m-1}\Big[a(j,k)+\frac{b(j,k)\sigma(j)}{2ik}\frac{c_-^j\,(k;\infty)}{c_+^j\,(k;\infty)}\Big]\\
v^m(x) &\sim \tau(x)\Big[\mathrm{e}^{(ik-1)x}\,c^0\,(k;\infty)\,\frac{\sigma(m)}{\sigma(0)}\prod_{j=0}^{m-1}\Big[a(j,k)+\frac{b(j,k)\sigma(j)}{2ik}\frac{c_-^j\,(k;\infty)}{c_+^j\,(k;\infty)}\Big]+\\
&\qquad + \mathrm{e}^{-(ik+1)x}\,c^0\,(-k;\infty)\prod_{j=0}^{m-1}\Big[d(j,k)+\frac{2ik\,c(j,k)}{\sigma(j)}\frac{c_+^j\,(k;\infty)}{c_-^j\,(k;\infty)}\Big]\Big]
\end{aligned}
\tag{3.14}
$$

where $c_+^j\,(k;\infty)$, $c_-^j\,(k;\infty)$ obey the recursion relations

$$
\begin{aligned}
\frac{c_+^{j+1}\,(k;\infty)}{\sigma(j+1)} &= a(j,k)\,\frac{c_+^j\,(k;\infty)}{\sigma(j)}+b(j,k)\,\frac{c_-^j\,(k;\infty)}{2ik}\\
c_-^{j+1}\,(k;\infty) &= d(j,k)\,c_-^j\,(k;\infty)+2ik\,c(j,k)\,\frac{c_+^j\,(k;\infty)}{\sigma(j)}
\end{aligned}
\tag{3.15}
$$

This is precisely the real analog of the p-adic equation (2.16). Again we enforce S-matrix unitarity, by imposing

$$
\frac{b(j,k)\sigma(j+1)}{2ik} = -\frac{2ik\,c(j,k)}{\sigma(j)} \tag{3.16a}
$$

and

$$
\sigma(j)\,d(j,k) = \sigma(j+1)\,a(j,k) \tag{3.16b}
$$

and, therefore, $[c_-^j\,(k;\infty)]^* = c_+^j\,(k;\infty)$. Here again, $\sigma(m)$, $a(m,k)$, $b(m,k)$, $c(m,k)$ and $d(m,k)$ are different from those in section 2, as now they correspond to $p=\infty$. Then

$$
v^m(x) \sim \tau(x)\Big[\mathrm{e}^{(ik-1)x}\,c^0(k;\infty)\prod_{j=0}^{m-1}\Delta(j,k)+\mathrm{e}^{-(ik+1)x}\,c^0(-k;\infty)\prod_{j=0}^{m-1}\Delta^*(j,k)\Big] \tag{3.14a}
$$

where

$$
\Delta(j,k) = d(j,k)\Big[1+2ik\,\frac{c(j,k)}{\sigma(j)d(j,k)}\,\frac{c_+^j\,(k;\infty)}{c_-^j\,(k;\infty)}\Big] \tag{3.17}
$$

and we see that

$$c^m(k;\infty) = c^0(k;\infty)\,\Lambda(m,k\,;\infty) \tag{3.18}$$

where

$$\Lambda(m,k\,;\infty) = \prod_{j=0}^{m-1} \Delta(j,k) \tag{3.19}$$

As we shall see in the next section this has precisely the right form for an adelic product formula.

The asymptotics of $u^m(x)$ yields the $p=\infty$ counterpart of the l^m of section 2. The same $\Lambda(m,k;\infty)$ function (3.19) is involved. Everything parallels the p-adic case, as expected.

For completeness, in Appendix C we discuss the integrable evolution of the real zsf problem with ZS system of type (3.2).

4. Integrable Evolution of the Adelic Zonal Spherical Function Problem

In sections 2 and 3 we treated an infinite set of ZS problems, one for each Bruhat-Tits tree with finite prime branching number p and one for the ZS problem on the real hyperbolic plane, the case $p=\infty$. We found similar evolutions in discrete time m for, the Jost-Harish-Chandra c-functions in all these cases. Specifically

$$c^m(k;v) = c^0(k;v)\,\Lambda(m,\chi^m(v),k;v) \tag{4.1}$$

with $c^0(k;v)$ the ordinary Jost-Harish-Chandra c-function for the local hyperbolic planes (following arithmetic usage, v labels the places of the field $\mathbb{Q}$ of rational numbers and runs over all finite primes and the infinite prime $v=\infty$, which denotes the place at which completion of the rationals yields the ordinary real numbers) and

$$c^0(k;v) = \frac{\zeta_v(ik)}{\zeta_v(ik+1)}, \tag{4.2a}$$

$$\zeta_p(ik) = \frac{1}{1-p^{-ik}}, \tag{4.2b}$$

$$\zeta_\infty(ik) = \pi^{-\frac{ik}{2}}\Gamma(\frac{ik}{2}), \tag{4.2c}$$

$$\Lambda(m,\chi^m,k;p)=\prod_{j=0}^{m-1}[1-\chi^m(p)p^{-ik}], \tag{4.2d}$$

$$\Lambda(m,\chi^m,k;\infty)=\prod_{j=0}^{m-1}\Delta(j,k). \tag{4.2e}$$

One passes from these local evolutions to the adelic evolution, by performing the adelic product over all the finite and infinite places v. At time $m=0$ this yields the familiar result

$$c^0_{\mathbb{A}}(k)=\frac{\zeta_{\mathbb{A}}(ik)}{\zeta_{\mathbb{A}}(ik+1)}, \tag{4.3a}$$

where

$$\zeta_{\mathbb{A}}(s)=\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s), \tag{4.3b}$$

$\zeta(s)$ being the ordinary Riemann zeta function and $\zeta_{\mathbb{A}}(s)$ the adelic zeta function which obeys the simple functional equation

$$\zeta_{\mathbb{A}}(1-s)=\zeta_{\mathbb{A}}(s). \tag{4.4}$$

At time $m=0$, an interesting *adelic* problem (see [2]) thus unifies all the *local* problems considered in sections 2 and 3. Does this adelic unification persist in the course of the time evolution? In general it does not, but one can constrain the p-dependence of the integration constants which appear in equation (4.2), in such a manner that the adelic unification remain meaningful at *all times* m. By inspecting equation (4.2), it becomes evident that the p-dependence of the $\chi^m(p)$ must be such as to allow an Euler product to be formed at each time m. The simplest way to insure this is, to fix, at each time m, the p-dependence of $\chi^m(p)$ to be that of a Dirichlet character modulo some integer r_m, which can depend on the discrete time m. Then the adelic Λ-function,

$$\Lambda_{\mathbb{A}}(m,\chi,k)=\prod_v\Lambda(m,\chi^m(v),k;v), \tag{4.5}$$

the product of all local Λ-functions of equation (4.2), evolves in a simple fashion:

$$\Lambda_{\mathbb{A}}(m+1,\chi,k)=L_{\mathbb{A}}(\chi^m,ik)\,\Lambda_{\mathbb{A}}(m,\chi,k) \tag{4.6}$$

with $L_{\mathbb{A}}$ the adelic Dirichlet L-function corresponding to the Dirichlet character χ^m:

$$L_{\mathbb{A}}(\chi^m,ik)=L_\infty(\chi^m,ik)\,L(\chi^m,ik). \tag{4.7}$$

Here

$$L(\chi^m, ik) = \prod_p \frac{1}{1-\chi^m(p)} \tag{4.8}$$

is the Dirichlet L-function corresponding to the character χ^m and $L_\infty(\chi, ik)$ its gamma factor. Specifically [7], if the exponent ϵ of the character χ^m is defined by

$$(-1)^{\epsilon_m} = \frac{\chi^m(-1)}{\chi^m(1)}, \tag{4.9}$$

then

$$L_\infty(\chi^m, ik) = (\frac{r_m}{\pi})^{\frac{ik}{2}}\,\Gamma(\frac{ik+\epsilon_m}{2}) \tag{4.10}$$

It is now evident that the time evolution in the adelic case amounts to the accretion at time m of a factor $L_{\mathbb{A}}(\chi^m, ik)$ by the adelic Jost-Harish-Chandra c-function.

A similar adelic treatment can also be given to the other component, the u-component of the ZS doublet, more precisely to the l^m function encoded in it. From the local l^m functions (2.24) we can construct an adelic l^m function, by forming the Euler product and including the place at infinity. This adelic l-function then keeps accreting the same $L_{\mathbb{A}}(\chi^m, ik)$ factors as the c^m-function. The evolution of the adelic ZS system is thus fully (i.e. for both components of the ZS doublet) determined by a single adelic $\Lambda_{\mathbb{A}}$ function.

All this can be considerably generalized, by weakening the condition that an adelic amplitude be obtained at *all times m*. If instead, we only require an adelic amplitude at all *even* values of the discrete time m, as if though time steps of the adelic system were twice longer than those of the local systems, then Langlands-type L-functions for $GL(2)$ can be accreted. To see how this comes about, let us consider a cusp form f of weight k, which is an eigenfunction of all Hecke operators $T(p)$. Let a_n be the Fourier coefficients of f and let $a_1 = 1$. Then the local Langlands L-functions corresponding to f are

$$L_p(s, f) = \frac{1}{1 - a_p\, p^{-s} + p^{k-1-2s}} = \frac{1}{(1-\mu_p\, p^{-s+\frac{k-1}{2}})(1-\nu_p\, p^{-s+\frac{k-1}{2}})} \tag{4.11a}$$

with

$$\mu_p\, \nu_p = 1 \qquad \mu_p + \nu_p = a_p\, p^{\frac{1-k}{2}}, \tag{4.11b}$$

so that setting

$$\chi^{2j+1}(p) = \mu_p\, p^{\frac{k-1}{2}} \tag{4.12a}$$

and

$$\chi^{2j+2}(p) = \nu_p\, p^{\frac{k-1}{2}}, \tag{4.12b}$$

we pick up a $GL(2)$ Langlands L-factor at each positive *even* value of the discrete time m. Similarly requiring the adelic time to correspond to every third, fourth or higher step of the ordinary "local" time, we can accrete higher Langlands L-functions ($GL(3)$,...), Shimura symmetric squares of L-functions, etc...

It can also happen that the L-function accretion is irregular, say at time $m = 1$ we accrete a Dirichlet L-function, then at time $m = 3$ a $GL(2)$ Langlands L-function, then at times $m = 4, 5, 6$ again Dirichlet L-functions, and so on without any visible Dirichlet-Langlands-Shimura-... pattern. All the local $\chi^m(p)$ at each time m are given in terms of the *arbitrary* local parameters of the time evolution ($d^m(p)$ and $\delta^m(p)$ in equation (2.21.b)). So, there is no mechanism directing the arithmetic evolution of the system. In particular the adelizability at all times was obtained by restricting the integration constants and not by explicit dynamical constraints. What we have found, is an infinite family of integrable systems, which together give rise to adelic integrable systems with the just explained degree of arbitrariness.

5. Conclusions

We have studied integrable ZS systems which for one of the components of the ZS doublet, reduce at the initial time to the zsf problem on a (local) real or p-adic hyperbolic plane. We have found that it is possible to so coordinate the integrable evolutions of these systems, that at all later times as well, meaningful adelic Jost functions are obtained. These adelic Jost functions involved L-functions of various kinds, Dirichlet, Langlands, Shimura... Though the appearance of these number-theoretic functions is interesting in its own right, it is far from fully understood. First of all, it is not clear what replaces at later times the adelic symmetric space on which the adelic scattering problem at the initial time is defined. Moreover, as we saw, there is a lot of freedom in the order in which the various types of L-functions get accreted at later times. It would be interesting if a dynamical principle could be found to determine the "arithmetic evolution" of the system. This principle would ultimately have to account for the arithmetically meaningful p-dependence of the $\chi^m(p)$ assumed in section 4 to make an adelic evolution possible.

Appendix A. Derivation of the solution (2.15)

In this appendix we find the solution for the time evolution (2.13) of the ZS system (2.12), when the z-dependence of the M-matrix elements is as given in eq.(2.11). Inserting (2.14) into (2.13) we obtain

$$
\begin{aligned}
a_n^m + Q_n^{m+1} c_n^m &= a_{n+1}^m + P_n^m b_{n+1}^m \\
\alpha_n^m + Q_n^{m+1} \gamma_n^m &= \alpha_{n+1}^m + P_n^m \beta_{n+1}^m & \qquad b_n^m + Q_n^{m+1}\delta_n^m &= 0 \\
\delta_n^m + P_n^{m+1} b_n^m &= \delta_{n+1}^m + Q_n^m c_{n+1}^m & \gamma_n^m + P_n^{m+1}\alpha_n^m &= 0 \\
d_n^m + P_n^{m+1} \beta_n^m &= d_{n+1}^m + Q_n^m \gamma_{n+1}^m & \beta_n^m + Q_{n-1}^m \alpha_n^m &= 0 \\
\beta_n^m + Q_n^{m+1} d_n^m &= b_{n+1}^m + Q_n^m a_{n+1}^m & c_n^m + P_{n-1}^m \delta_n^m &= 0 \\
c_n^m + P_n^{m+1} a_n^m &= \gamma_{n+1}^m + P_n^m d_{n+1}^m
\end{aligned}
\tag{A.1}
$$

From these equations, one readily finds

$$
\begin{aligned}
a_n^m &= Q_n^{m+1} P_{n-1}^m \delta_n^m + a(m) \\
b_n^m &= -Q_n^{m+1} \delta_n^m & \qquad \beta_n^m &= -Q_{n-1}^m \alpha_n^m \\
c_n^m &= -P_{n-1}^m \delta_n^m & \gamma_n^m &= -P_n^{m+1} \alpha_n^m \\
d_n^m &= Q_{n-1}^m P_n^{m+1} \alpha_n^m + d(m)
\end{aligned}
\tag{A.2a}
$$

$$
\alpha_n^m = \left[\prod_{i=0}^{n-1} \frac{\left(1 - Q_i^{m+1} P_i^{m+1}\right)}{\left(1 - Q_n^m P_n^m\right)} \right] \alpha(m) \quad \delta_n^m = \left[\prod_{i=0}^{n-1} \frac{\left(1 - Q_i^{m+1} P_i^{m+1}\right)}{\left(1 - Q_n^m P_n^m\right)} \right] \delta(m)
\tag{A.2b}
$$

and two coupled equations for Q_n^m and P_n^m:

$$
\begin{aligned}
&Q_n^m a(m) - Q_n^{m+1} d(m) = \\
&\qquad \left[\prod_{i=0}^{n-1} \frac{\left(1 - Q_i^{m+1} P_i^{m+1}\right)}{\left(1 - Q_i^m P_i^m\right)} \right] \left(1 - Q_n^{m+1} P_n^{m+1}\right) \left(Q_{n+1}^{m+1} \delta(m) - Q_{n-1}^m \alpha(m)\right)
\end{aligned}
\tag{A.3a}
$$

$$
\begin{aligned}
&P_n^m d(m) - P_n^{m+1} a(m) = \\
&\qquad \left[\prod_{i=0}^{n-1} \frac{\left(1 - Q_i^{m+1} P_i^{m+1}\right)}{\left(1 - Q_i^m P_i^m\right)} \right] \left(1 - Q_n^{m+1} P_n^{m+1}\right) \left(P_{n+1}^{m+1} \alpha(m) - P_{n-1}^m \delta(m)\right)
\end{aligned}
\tag{A.3b}
$$

where $a(m) = a_0^m$, $\alpha(m) = \alpha_0^m$, $d(m) = d_0^m$ and $\delta(m) = \delta_0^m$ are arbitrary functions depending on the discrete time m and on the lattice branching number p, but we choose not to show the latter dependence explicitly.

Dividing the two equations (A.3), we have

$$
\begin{aligned}
&\alpha(m)a(m)\big(Q^m_{n-1}P^{m+1}_n - Q^m_n P^{m+1}_{n+1}\big) + \delta(m)d(m)\big(Q^{m+1}_{n+1}P^m_n - Q^{m+1}_n P^m_{n-1}\big)+\\
&+\alpha(m)d(m)\big(Q^{m+1}_n P^{m+1}_{n+1} - Q^m_{n-1}P^m_n\big) + \delta(m)a(m)\big(Q^m_n P^m_{n-1} - Q^{m+1}_{n+1}P^{m+1}_n\big) = 0
\end{aligned}
\tag{A.4}
$$

From eq.(A.4) we recognize the particular solution:

$$
Q^{m+1}_{n+1} = \frac{\sigma_p(m)}{\sigma_p(m+1)}Q^m_n \qquad P^{m+1}_{n+1} = P^m_n\frac{\sigma_p(m+1)}{\sigma_p(m)} \tag{A.5a}
$$

with

$$
\big[\delta(m)d(m)(\sigma_p(m))^2 - \alpha(m)a(m)(\sigma_p(m+1))^2\big]\big(Q^m_{n-1}\,P^m_{n-1} - Q^m_n\,P^m_n\big) = 0 \tag{A.5b}
$$

There might be some further solutions of this type, but those would require that Q^m_n and P^m_n also satisfy some supplementary conditions (such as $P^m_{n-1} = P^m_{n+1}$, for example), which have to be reflected at time $m = 0$, and which are not satisfied in our case, so we will not concern ourselves with them.

Since $n \geq 0$ and Q^0_n vanishes in our case, (see (2.8)), this solution will produce a Q^m_n which vanishes for $n > m$ at any finite time m. In other words $Q^m_n = 0$ asymptotically at all times. As we are interested in the Jost functions, i.e. the asymptotic scattering regime, this means that for our purposes Q^m_n can be set to zero at all times m and all radial distances n, $Q^m_n = 0$. A nice feature of such a choice of solution is that it preserves exactly, not only asymptotically, the p-adic zsf-problem structure for the equation describing v^m_n at all times.

Imposing then

$$
Q^m_n = 0 \tag{A.6a}
$$

eqs. (A.5) are solved by

$$
P^m_n = \sigma_p(m)p^{\rho(n-m)} \tag{A.6b}
$$

This yields precisely the R-matrix of eq. (2.15a). Inserting eqs.(A.6) into eqs.(A.2) yields the M-matrix (2.15b). Once we impose (A.6a), eq.(A.5b) becomes an identity. So far we have only used eq.(A.4) which is the ratio of the two eqs.(A.3). Eq.(A.3a) is now an identity, too, and we thus require that eq.(A.3b) be obeyed. Inserting eqs.(A.6) into (A.3b) one immediately finds eq.(2.15c). We thus showed that the equations (2.15) do indeed solve the consistency equation (2.13).

Appendix B. Derivation of equation (3.13)

We will find here an exact solution for the time-evolution equation (3.11) of the real ZS system (3.2'). The form of the $M-$matrix is the same as in the previous paragraph

$$M^m(x) = \begin{pmatrix} A^m(x,k) & B^m(x,k) \\ C^m(x,k) & D^m(x,k) \end{pmatrix}. \tag{B.1}$$

With the choice (3.12) for the $R-$matrix, equation (3.11) produces the following equations

$$\begin{aligned} (A^m)' &= -B^m \sigma(m)\, \mathrm{e}^{2\nu x} \\ (B^m)' &= B^m\, (z - z^{-1}) \\ (C^m)' &= -C^m\, (z - z^{-1}) + \mathrm{e}^{2\nu x} \big[A^m \sigma(m+1) - D^m \sigma(m) \big] \\ (D^m)' &= B^m \sigma(m+1)\, \mathrm{e}^{2\nu x} \end{aligned} \tag{B.2}$$

It is readily seen that the second equation of this system yields

$$B^m(x,k) = b(m,k)\, \mathrm{e}^{2(ik-\nu)x} \tag{B.3}$$

which, when used in the first and fourth equation gives in turn

$$A^m(x,k) = a(m,k) - \frac{b(m,k)}{2ik} \sigma(m) \mathrm{e}^{2ikx} \tag{B.4}$$

and

$$D^m(x,k) = d(m,k) + \frac{b(m,k)}{2ik} \sigma(m+1) \mathrm{e}^{2ikx}. \tag{B.5}$$

Finally a straightforward integration of the third equation in the system (B.2) results in

$$\begin{aligned} C^m(x,k) = & \frac{b(m,k)}{4k^2} \sigma(m)\sigma(m+1)\, \mathrm{e}^{2(ik+\nu)x} + c(m,k)\, \mathrm{e}^{2(\nu-ik)x} + \\ & + \frac{\sigma(m+1)a(m,k) - \sigma(m)d(m,k)}{2ik}\, \mathrm{e}^{2\nu x} \end{aligned} \tag{B.6}$$

where all the 'constants of integration' $a(m,k), b(m,k), c(m,k)$, and $d(m,k)$ are arbitrary functions of z, and implicitly of k,(see equation (3.8)).

Appendix C. The ZS system (3.2)

Let us now return to the Z.S. system (3.2) mentioned at the beginning of section 3.

Just like in the 'exact' case treated in section 3, we get a second order differential equation for v^0

$$(v^0)'' - \frac{(P^0)'}{P^0}(v^0)' + \left[\zeta^2 - i\zeta\frac{(P^0)'}{P^0} - Q^0P^0\right]v^0 = 0 \tag{C.1}$$

Again ζ is related to the spectral parameter k of equation (3.1). The gauge transformation

$$v^0(x) = \rho(x)t^0(x). \tag{C.2}$$

and the same logic that led to equations (3.6), now yields

$$Q^0(x) = \frac{\sigma(0)}{\sinh^2 2x}\,\mathrm{e}^{-2\nu x} \quad \text{and} \quad P^0(x) = -\frac{1}{\sigma(0)}\,\mathrm{e}^{2\nu x} \tag{C.3}$$

with

$$k = \pm\big(\zeta - i\nu\big) \quad \text{and} \quad \rho(x) = \rho_0\mathrm{e}^{\nu x}\sqrt{\sinh 2x} \tag{C.4}$$

Using the known solution (3.9), (3.10) to (3.1), we now look for an integrable time evolution of the system which at all times is compatible with the zsf equation (3.1). That is, we wish to find a solution to equation (3.11), but, this time, by expanding in ζ. Again, because we want to preserve as much as possible the form of the R^m-matrix, we will make the following assumptions:

$$Q^m(x) = \frac{\sigma(m)}{\sinh^2 2x}\,\mathrm{e}^{-2\nu x} \quad \text{and} \quad P^m(x) = \tau(m)\,\mathrm{e}^{2\nu x} \tag{C.5}$$

These assumptions together with the expansion

$$\begin{aligned} A^m(x,\zeta) &= a(m,x)\zeta + \alpha(m,x) \qquad & B^m(x,\zeta) &= b(m,x)\zeta + \beta(m,x) \\ C^m(x,\zeta) &= c(m,x)\zeta + \gamma(m,x) \qquad & D^m(x,\zeta) &= d(m,x)\zeta + \delta(m,x) \end{aligned} \tag{C.6}$$

lead to the system:

$$
\begin{aligned}
a^m(x)' &= \sigma(m+1)c^m(x)\,\frac{\mathrm{e}^{-2\nu x}}{\sinh^2 2x} - \tau(m)b^m(x)\,\mathrm{e}^{2\nu x} \\
d^m(x)' &= \tau(m+1)b^m(x)\,\mathrm{e}^{2\nu x} - \sigma(m)c^m(x)\,\frac{\mathrm{e}^{-2\nu x}}{\sinh^2 2x} \\
\alpha^m(x)' &= \sigma(m+1)\gamma^m(x)\,\frac{\mathrm{e}^{-2\nu x}}{\sinh^2 2x} - \tau(m)\beta^m(x)\,\mathrm{e}^{2\nu x} \\
\delta^m(x)' &= \tau(m+1)\beta^m(x)\,\mathrm{e}^{2\nu x} - \sigma(m)\gamma^m(x)\,\frac{\mathrm{e}^{-2\nu x}}{\sinh^2 2x} \\
2i\beta^m(x) &= \big[a^m(x)\sigma(m) - d^m(x)\sigma(m+1)\big]\,\frac{\mathrm{e}^{-2\nu x}}{\sinh^2 2x} \\
2i\gamma^m(x) &= \big[a^m(x)\tau(m+1) - d^m(x)\tau(m)\big]\,\mathrm{e}^{2\nu x} \\
\beta^m(x)' &= \big[\delta^m(x)\sigma(m+1) - \alpha^m(x)\sigma(m)\big]\,\frac{\mathrm{e}^{-2\nu x}}{\sinh^2 2x} \\
\gamma^m(x)' &= \big[\alpha^m(x)\tau(m+1) - \delta^m(x)\tau(m)\big]\,\mathrm{e}^{2\nu x} \\
b^m(x) &= 0 \quad c^m(x) = 0
\end{aligned}
\tag{C.7}
$$

In a straightforward manner, the solution for this system is found to be

$$
\begin{aligned}
a^m(x) &= 2ia(m) \quad \alpha^m(x) = -\frac{a(m)}{2}\big[\tau(m+1)\sigma(m+1) - \tau(m)\sigma(m)\big]\,\coth 2x + \alpha(m) \\
d^m(x) &= 2id(m) \quad \delta^m(x) = \frac{d(m)}{2}\big[\tau(m+1)\sigma(m+1) - \tau(m)\sigma(m)\big]\,\coth 2x + \delta(m) \\
b^m(x) &= 0 \quad \beta^m(x) = \big[a(m)\sigma(m) - d(m)\sigma(m+1)\big]\,\frac{\mathrm{e}^{-2\nu x}}{\sinh^2 2x} \\
c^m(x) &= 0 \quad \gamma^m(x) = \big[a(m)\tau(m+1) - d(m)\tau(m)\big]\,\mathrm{e}^{2\nu x}
\end{aligned}
\tag{C.8}
$$

where the 'constants of integration' $a(m), d(m), \alpha(m)$ and $\delta(m)$ are subject to either the compatibility conditions

$$
\begin{aligned}
\alpha(m)\tau(m+1) - \delta(m)\tau(m) &= 2\nu\big[a(m)\tau(m+1) - d(m)\tau(m)\big] \\
a(m)\tau(m+1) + d(m)\tau(m) &= 0 \\
\sigma(m+1)\delta(m) - \sigma(m)\alpha(m) &= -2\nu\big[a(m)\sigma(m) - d(m)\sigma(m+1)\big] \\
\big[\tau(m+1)\sigma(m+1) - \tau(m)\sigma(m)\big]^2 &= 8\big[\tau(m+1)\sigma(m+1) + \tau(m)\sigma(m)\big]
\end{aligned}
\tag{C.9}
$$

or to the conditions

$$
\begin{aligned}
\tau(m+1)\sigma(m+1) &= \tau(m)\sigma(m) \\
\frac{a(m)}{d(m)} = \frac{\alpha(m)}{\delta(m)} &= \frac{\sigma(m+1)}{\sigma(m)}
\end{aligned}
\tag{C.10}
$$

.

From the first three equations (C.9) we have

$$\begin{aligned} d(m) &= -a(m)\frac{\tau(m+1)}{\tau(m)} \\ \alpha(m) &= 2\nu a(m) \\ \delta(m) &= -2\nu a(m)\frac{\tau(m+1)}{\tau(m)} \end{aligned} \tag{C.11}$$

whereas the last one determines the time-evolution of the R-matrix:

$$\Big[\pi(m+1) - \pi(m)\Big]^2 = 8\Big[\pi(m) + \pi(m+1)\Big] \tag{C.12}$$

where we used the notation $\pi(m) = \sigma(m)\,\tau(m)$. Notice that $d(m)$, $\alpha(m)$, and $\delta(m)$ are all proportional to $a(m)$. Because of equation (C.8) so are then $\beta(m)$ and $\gamma(m)$. Thus $a(m)$ is an overall time-dependent normalization of the time evolution matrix M^m. Without any loss of generality we henceforth set $a(m)$ constant, say $a(m) = 1/2$.

We therefore start with equation (C.12) and introduce the function $l(m)$ by

$$\pi(m) = l^2(m) - 1 \tag{C.13}$$

Then equation (C.12) becomes a quadratic equation for $l(m+1)^2$ for a given $l(m)$. Its two solutions are

$$l^2(m+1) = \big[l(m) \pm 2\big]^2 \tag{C.14}$$

The R-matrix at time $m = 0$ determines $\pi(0) = -1$ so that

$$l(0) = 0. \tag{C.15}$$

The corresponding solutions of (C.9), written in matrix form are:

$$R_{\pm}^{m+1} = \begin{pmatrix} ik - \nu & \frac{\sigma(m+1)}{\sinh^2 2x}\, e^{-2\nu x} \\ \frac{[l(m)\pm 1][l(m)\pm 3]}{\sigma(m+1)}\, e^{2\nu x} & -ik + \nu \end{pmatrix} \tag{C.16}$$

and

$$M_{\pm}^{m} = 2\,a(m) \begin{pmatrix} ik - \Big[1 \pm l(m)\Big] \coth 2x & \frac{\sigma(m)}{\sinh^2 r2x}\, e^{-2\nu x}\, \frac{l(m)\pm 1}{l(m)\mp 1} \\ \frac{[l(m)\pm 1][l(m)\pm 3]}{\sigma(m+1)}\, e^{2\nu x} & -\Big[ik + \Big[1 \pm l(m)\Big]\coth 2x\Big]\, \frac{\sigma(m)}{\sigma(m+1)}\, \frac{l(m)\pm 3}{l(m)\mp 1} \end{pmatrix}. \tag{C.17}$$

For the alternate set (C.10) of compatibility conditions one finds

$$\pi(m+1) = \pi(m) \tag{C.18}$$

and the corresponding solutions

$$R^{m+1} = R^m \quad \text{and} \quad M^m = \begin{pmatrix} a(m)\big(ik-\nu\big) & 0 \\ 0 & d(m)\big(ik-\nu\big) \end{pmatrix}. \tag{C.19}$$

Thus, at any time there are five possible evolutions, the four (due to the *two* sign ambiguities in the quadratic equation (C.14),)

$$l(m) \rightarrow \; l(m+1)_{\pm} \; = \pm l(m) \pm 2 \tag{C.20}$$

and the solution (C.18), (C.19). For simplicity we choose the time evolution corresponding to $l(m) \rightarrow l(m)+2$ so as to avoid 'returns', 'reflections', or 'stagnations'. Then $l(m) = 2m$ and the asymptotic solution of the Z-S system is

$$\begin{aligned} u^m(x) &= -\, 2ik\sigma(0)\, e^{-\nu x}\sqrt{\sinh 2x}\, c^m(k)\, e^{(ik-1)x} \qquad \text{(C.21)} \\ v^m(x) &= \big(2m-1\big)\big(2m+1\big)\frac{\sigma(0)}{\sigma(m)} e^{\nu x}\sqrt{\sinh 2x}\left[c^m(k)\, e^{(ik-1)x} + c^m(-k)\, e^{(-ik-1)x}\right] \end{aligned}$$

The Jost functions at time m are given by expressions that are similar to those we obtained for the p-adic case:

$$c^m(k) = c^0(k;\infty)\prod_{l=1}^{m}\left[ik - \big(2l-1\big)\right] \tag{C.22}$$

with $c^0(k;\infty)$ given by equation (3.10), and the conjugate relation for $c^m(-k)$. Although the form of (C.22) is similar to the one in (2.22), it is not suited for adelization. This is evident from the appearance of the factors linear in K as opposed to the expected gamma factors. Presumably this is due to the expansion in ζ, undertaken here, which is different from the z-expansion in section 3. It is amusing to note that the linear factors in (C.22) are themselves "gamma-like" functions of one order lower in the Barnes hierarchy [8].

We note that the second order equation obeyed by $w^m(x)$ can be brought to the form

$$(w^m)'' + 2\coth 2x\,(w^m)' + \left(k^2 + 1 - \frac{4m^2}{\sinh^2 2x}\right) w^m = 0 \tag{C.23}$$

or, equivalently, with $w^m(x) = \frac{z(x)}{\sqrt{\sinh 2x}}$

$$z'' + \left(k^2 - \frac{4m^2 - 1}{\sinh^2 2x}\right) z = 0 \tag{C.24}$$

which is the eigenvalue problem for the $P_k^m(x)$, i.e. the associated Legendre functions [1].

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