Nonnegative Feynman-Kac Kernels in Schrödinger's Interpolation Problem

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Abstract

The existing formulations of the Schrödinger interpolating dynamics, which is constrained by the prescribed input-output statistics data, utilize strictly positive Feynman-Kac kernels. This implies that the related Markov diffusion processes admit vanishing probability densities only at the boundaries of the spatial volume confining the process.

We extend the framework to encompass singular potentials and associated nonnegative Feynman-Kac-type kernels. It allows to deal with general nonnegative solutions of the Schrödinger boundary data problem. The resulting stochastic processes are capable of both developing and destroying nodes (zeros) of probability densities in the course of their evolution.

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1 From positive to nonnegative solutions of parabolic equations

We continue an investigation of the celebrated Schrödinger's boundary data problem, [1]-[11], of reconstructing the "most likely" evolution which interpolates between the prescribed input-output statistics data (usually analyzed in terms of nonvanishing probability densities) in a fixed finite-time interval, interpreted as a duration time of the process.

In the present paper we focus our attention again on stochastic Markov processes of diffusion-type (see Ref. [10] for a jump process alternative), which are associated with the temporally adjoint pair of parabolic partial differential equations:

$$\partial_t u(x,t) = \triangle u(x,t) - c(x,t)u(x,t) \tag{1}$$

$$\partial_t v(x,t) = -\triangle v(x,t) + c(x,t)v(x,t)$$

Here, $c(x,t)$ is a real function (left unspecified at the moment) and the solutions $u(x,t), v(x,t)$ are sought for in the time interval $[0,T]$ under the boundary conditions set at the time-interval borders:

$$\rho_0(x) = u(x,0)v(x,0) \tag{2}$$

$$\rho_T(x) = u(x,T)v(x,T)$$

$$\int_A \rho_0(x)dx = \rho_0(A)\ ,\ \int_B \rho_T(x)dx = \rho_T(B)$$

We assume that ρ is a probability measure with the density $\rho(x)$, and A, B stand for arbitrary Borel sets in the event space. In the above, suitable units were chosen to eliminate inessential in the present context (dimensional) parameters, and the process is supposed to live in/on R^1.

As emphasized in the previous publications, [8]-[11], the key ingredient of the formalism is to specify the function $c(x,t)$ such that $exp[-\int_0^t H(\tau)d\tau]$ can be viewed as a strongly continuous semigroup operator with the generator $H(t) = -\triangle + c(t)$, associated with the familiar [12] Feynman-Kac kernel:

$$(f, exp[-\int_0^t H(\tau)d\tau]g) = \int dy \int dx \overline{f}(y)k(y,0,x,t)g(x) = \tag{3}$$

$$\int \overline{f}(\omega(0))g(\omega(t))exp[-\int_0^t c(\omega(\tau),\tau)d\tau]d\mu_0(\omega)$$

Here f, g are complex functions, $\omega(t)$ denotes a sample path of the conventional Wiener process and $d\mu_0$ stands for the Wiener measure. Clearly, the kernel itself can be explicitly written in terms of the conditional Wiener measure $d\mu_{(x,t)}^{(y,s)}$ pinned at space-time points (y,s) and (x,t), $0 \leq s < t \leq T$:

$$k(y,s,x,t) = \int exp[-\int_s^t c(\omega(\tau),\tau)d\tau]\; d\mu_{(x,t)}^{(y,s)}(\omega) \tag{4}$$

As long as we do not impose any specific domain restrictions on the semigroup generator $H(\tau)$, the whole real line R^1 is accessible to the process. Various choices of the Dirichlet [12, 8] boundary conditions can be accounted for by the formula (3). If we replace R^1 by any open subset $\Omega \subset R^1$ with the boundary $\partial\Omega$, it amounts to confining Wiener sample paths of relevance to reside in (be interior to) Ω, which in turn needs an appropriate measure $d\mu_{(x,t)}^{(y,s)}(\omega \in \Omega)$ in (4). This is usually implemented by means of stopping times for the Wiener process, [7, 8, 13, 14, 15].

Let $f(x)$ and $g(x)$ be two real functions such that:

$$m_T(x,y) = f(x)k(x,0,y,T)g(y) \tag{5}$$

defines a bi-variate density of the probability measure:

$$m_T(A,B) = \int_A dx \int_B dy\; m_T(x,y) \tag{6}$$

i.e. a transition probability of the propagation from the Borel set A to the Borel set B to be accomplished in the time interval T. In particular, we need the marginal probability densities to be defined:

$$\rho_0(x) = m_T(x,\Omega)\;,\;\; \rho_T(y) = m_T(\Omega,y) \tag{7}$$

where $\Omega \subset R^1$ is a spatial area confining the process.

Formulas (5),(6), can be viewed as special cases of (3), so establishing an apparent link between the Schrödinger problem and the Feynman-Kac kernels, together with the related parabolic equations (1). Assuming that marginal probability measures (7) and their densities are given a priori, and a concrete Feynman-Kac kernel (4) (with or without Dirichlet domain restrictions) is specified, we are within the premises of the Schrödinger boundary data problem.

Let $\overline{\Omega} = \Omega \cup \partial\Omega$ be a closed subset of R^1, or R^1 itself. For all Borel sets (in the σ-field generated by all open subsets of $\overline{\Omega}$) we assume to have known $\rho_0(A)$ and $\rho_T(B)$, hence the respective densities as well. If the integral kernel $k(x,0,y,T)$ in the expression (5) is chosen to be *continuous* and *strictly positive* on $\overline{\Omega}$, then the integral equations (7) can be solved [5] with respect to the *unknown* functions $f(x)$ and $g(y)$. The solution comprises two nonzero, locally integrable functions of the same sign, which are unique up to a multiplicative constant.

If, in addition, the kernel $k(y,s,x,t)$, $0 \leq s < t \leq T$ is a *fundamental solution* [16] of the parabolic system (1) on R^1 (i.e. is a function which solves the forward equation in (x,t) variables, while the backward one in (y,s)), then we have defined a solution of the system (1) by:

$$u(x,t) \equiv f(x,t) = \int f(y)k(y,0,x,t)dy \tag{8}$$

$$v(x,t) \equiv g(x,t) = \int k(x,t,y,T)g(y)dy$$

Moreover, $\rho(x,t) = f(x,t)g(x,t)$ is propagated by the Markovian transition probability density:

$$p(y,s,x,t) = k(y,s,x,t)\frac{g(x,t)}{g(y,s)} \tag{9}$$

$$\rho(x,t) = \int \rho(y,s)p(y,s,x,t)dy$$

$$0 \leq s < t \leq T$$

$$\partial_t \rho = \triangle \rho - \nabla(b\rho)$$

$$b = b(x,t) = 2\frac{\nabla g(x,t)}{g(x,t)}$$

the result, which covers all traditional Smoluchowski diffusions [8, 17]. In that case, $c(x,t)$ is regarded as time-independent, and the corresponding stochastic process is homogeneous in time. The Dirichlet boundary data can be implemented as well, thus leading to the Smoluchowski diffusion processes with natural boundaries, [8]. Then, $k(y,s,x,t)$ stands for an appropriate Green function of the parabolic boundary-data problem, with the property to vanish at the boundaries $\partial\Omega$ of Ω.

Recently [11], an extension of the above formalism was elaborated to encompass situations when the involved Feynman-Kac kernels are strictly positive and continuous, but not necessarily fundamental solutions of (1) nor even differentiable. They still give rise to (8) and (9) under suitable regularity conditions for solutions of the parabolic system (1). Their existence is not in conflict with the fact that $k(y,s,x,t)$ itself needs not to be a solution of any differentiable equation.

Let us also mention that for time-independent potentials, $c(x,t) = c(x)$ for all $t \in [0,T]$, a number of generalizations is available [7, 8, 13, 14, 15, 18]-[21] to encompass the nodal sets of $\rho(x)$ and hence of the associated functions $f(x), g(x)$. The drift $b(x) = \nabla ln\rho(x) = \frac{\nabla\rho(x)}{\rho(x)}$ singularities do not prohibit the existence of a well defined Markov diffusion process (9), for which nodes are unattainable. In the considered framework they are allowed only at the boundaries of the connected spatial area Ω confinig the process.

The problem of relaxing the strict positivity (and/or continuity) demand for Feynman-Kac kernels is nontrivial [3, 4, 5] with respect to the eventual construction of the *unique* Markov process (9). To elucidate the nature of difficulties underlying this issue, let us consider quantally motivated examples of the parabolic dynamics (1).

2 Nonlinear parabolic dynamics with the fundamental solution

Let us choose the potential function $c(x,t)$ as follows:

$$c(x,t) = \frac{x^2}{2(1+t^2)^2} - \frac{1}{1+t^2} \tag{10}$$

for $x \in R^1, t \in [0,T]$. In view of its local Hölder continuity (cf. Ref. [11]) with exponent one, and its quadratic boundedness, the fundamental solution of the parabolic system is known to exist [16, 22, 23, 24]. It is constructed via the parametrix method [16]. Among an infinity of regular solutions of (1) with the potential (10), we can in particular identify [11] solutions of the Schrödinger boundary data problem for the familiar (quantal) evolution:

$$\rho_0(x) = (2\pi)^{-1/2} exp[-\frac{x^2}{2}] \longrightarrow \rho(x,t) = [2\pi(1+t^2)]^{-1/2} exp[-\frac{x^2}{2(1+t^2)}] \tag{11}$$

They read

$$u(x,t) \equiv f(x,t) = [2\pi(1+t^2)]^{-1/4} exp(-\frac{x^2}{4}\frac{1+t}{1+t^2} + \frac{1}{2} arctan\, t) \tag{12}$$

$$v(x,t) \equiv g(x,t) = [2\pi(1+t^2)]^{-1/4} exp(-\frac{x^2}{4}\frac{1-t}{1+t^2} - \frac{1}{2} arctan\, t)$$

and, while solving the nonlinear parabolic system (1) (with $c = \triangle\rho^{1/2}/\rho^{1/2}$), in addition they imply the validity of the Fokker-Planck equation:

$$\rho(x,t) = f(x,t)g(x,t) \rightarrow \partial_t \rho = \triangle\rho - \nabla(b\rho)$$

$$b(x,t) = 2\frac{\nabla g(x,t)}{g(x,t)} = -\frac{1-t}{1+t^2}\, x \tag{13}$$

Notice that $p(y,s,x,t) = k(y,s,x,t)\frac{g(x,t)}{g(y,s)}$ is a fundamental solution of the first and second Kolmogorov (e.g. Fokker-Planck) equations in the present case.

Let us recall that a concrete parabolic system corresponding to solutions (12) looks badly nonlinear. Our procedure, of first considering the linear system (but with the potential "belonging" to another, nonlinear one), and next identifying solutions of interest by means of the Schrödinger boundary data problem, allows to bypass this inherent difficulty. In connection with the previously mentioned quantal motivation of ours, let us define $g = exp(R+S), f = exp(R-S)$ where $R(x,t), S(x,t)$ are real functions. We immediately realize that (1), (10) provide for a parabolic alternative to the familiar Schrödinger equation and its temporal adjoint:

$$i\partial_t \psi = -\triangle \psi \tag{14}$$

$$i\partial_t \overline{\psi} = \triangle \overline{\psi}$$

with the Madelung factorization $\psi = exp(R+iS)$, $\overline{\psi} = exp(R-iS)$ involving the previously introduced real functions R and S.

3 Nonlinear parabolic dynamics with unattainable boundaries: the Green function

Things seem to be fairly transparent when the parabolic system (1) allows for fundamental solutions. However, even in this case complications arise if nodes of the probability density are admitted. The subsequent discussion has a quantal origin again, and comes from the free Schrödinger propagation (14) with the specific choice of the initial data:

$$\psi_0(x) = (2\pi)^{-1/4}\, x\, exp(-\frac{x^2}{4}) \longrightarrow \tag{15}$$

$$\psi(x,t) = (2\pi)^{-1/4} \frac{x}{(1+it)^{3/2}}\, exp[-\frac{x^2}{4(1+it)}]$$

such that our nonstationary dynamics example displays a stable node at $x=0$ for all times .

The parabolic system (1) in this case involves the potential function:

$$c(x,t) = \frac{\triangle \rho^{1/2}(x,t)}{\rho^{1/2}(x,t)} = \frac{x^2}{2(1+t^2)^2} - \frac{3}{1+t^2} \tag{16}$$

$$\rho(x,t) = (2\pi)^{-1/2}(1+t^2)^{-3/2}\, x^2\, exp[-\frac{x^2}{2(1+t^2)}]$$

The polar (Madelung) factorization of Schrödinger wave functions implies:

$$R(x,t) = ln\, \rho^{1/2}(x,t) \tag{17}$$

$$x>0 \rightarrow S(x,t) = S_+(x,t) = \frac{x^2}{4}\frac{t}{1+t^2} - \frac{3}{2} arctan\, t$$

$$x<0 \rightarrow S(x,t) = S_-(x,t) = \frac{x^2}{4}\frac{t}{1+t^2} - \frac{3}{2} arctan\, t\ +\ \pi$$

Although $S(x,t)$ is not defined at $x=0$, we can introduce continuous functions $f = exp(R-S)$ and $g = exp(R+S)$ by employing the step function $\epsilon(x) = 0$ if $x \geq 0$ and $\epsilon(x) = 1$ if $x < 0$. Then, the candidates for solutions of the parabolic system (1) with the potential (16) would read:

$$v(x,t) \equiv g(x,t) = (2\pi)^{-1/4}(1+t^2)^{-3/4}|x|\, exp(-\frac{x^2}{4}\frac{1-t}{1+t^2})\, exp[-\frac{3}{2}arctan\, t + \pi\epsilon(x)] \tag{18}$$

$$u(x,t) \equiv f(x,t) = (2\pi)^{-1/4}(1+t^2)^{-3/4}|x|\, exp(-\frac{x^2}{4}\frac{1+t}{1+t^2})\, exp[\frac{3}{2}arctan\, t - \pi\epsilon(x)]$$

For all $x \neq 0$ we can define the forward drift

$$b(x,t) = 2\frac{\nabla g(x,t)}{g(x,t)} = \frac{2}{x} - x\,\frac{1-t}{1+t^2} \tag{19}$$

which displays a singularity at $x=0$. Nonetheless, $(b\rho)(x,t)$ is a smooth function and the Fokker-Planck equation $\partial_t\rho = \triangle\rho - \nabla(b\rho)$ holds true on the whole real line R^1, for all $t \in [0,T]$. Notice that there is no current through $x=0$, since $v(x,t) = 2\nabla S(x,t) = \frac{xt}{1+t^2}$ vanishes at this point for all times.

Our functions $f(x,t), g(x,t)$ are continuous on R^1, which however does not imply their differentiability. Indeed, they solve the parabolic system (1) with the potential (16) *not* on R^1 but on $(-\infty,0)\cup(0,+\infty)$. Hence, almost everywhere on R^1, with the exception of $x=0$.

An apparent obstacle arises because of this subtlety: these functions are *not* even weak solutions of (1), because of:

$$\int_{-\infty}^{+\infty} \partial_t f(x,t)\phi(x)dx + \int_{-\infty}^{+\infty} \nabla f(x,t)\nabla\phi(x)dx + \tag{20}$$

$$\frac{1}{2}\int_{-\infty}^{+\infty} c(x,t)f(x,t)\phi(x)dx\ \neq 0$$

for every test function ϕ such that $\phi(0) \neq 0$, continuous and with support on a chosen compact set (e.g. vanishing beyond this set).

One more obstacle arises, if we notice that $c(x,t)$, (16) permits the existence of the unique, bounded and strictly positive fundamental solution for the parabolic system (1). Then, while having singled out a fundamental solution and the boundary density data $\rho_0(x), \rho_T(x)$ consistent with (16), we can address the Schrödinger boundary data problem associated with (2),(3):

$$u(x,0)\int k(x,0,y,T)v(y,T)dy = \rho_0(x) \tag{21}$$

$$v(x,T)\int k(y,0,x,T)u(y,0)dy = \rho_T(x)$$

expecting that a unique solution $u(x,0), v(x,T)$ of this system of equations implies an identification $u(x,0) = f(x,0)$ and $v(x,T) = g(x,T)$.

However, it is not the case and our $f(x,t), g(x,t)$ do not come out as solutions of the Schrödinger problem, if considered on the whole real line R^1, on which the fundamental solution sets rules of the game.

Indeed, let us assume that (21) does hold true if we choose $u(x,0) = f(x,0), v(x,T) = g(x,T)$, with f and g defined by (18). Since, in particular we have

$$g(x,T)\int k(y,0,x,T)f(y,0)dy = g(x,T)f(x,T) \tag{22}$$

then for $x \neq 0$ there holds:

$$f(x,T) = \int k(y,0,x,T)f(y,0)dy \tag{23}$$

Both sides of the last identity represent continuous functions, hence the equality is valid point-wise (i.e. for every x). We know that $f(y,0)$ is continuous and bounded on R^1, and $k(y,0,x,T)$ is a fundamental solution of (1). Hence the right-hand-side of (23) represents a regular solution of the parabolic equation. Such solutions have continuous derivatives, while our left-hand-side function $f(x,T)$ certainly does not share this property. Consequently, our assumption leads to a contradiction and (23) is invalid in our case.

It means that the fundamental solution (e.g. the corresponding Feynman-Kac kernel) associated with (16) is inappropriate for the Schrödinger problem analysis, if the interpolating probability density is to have nodes (i.e. vanish at some points).

In our case, $x = 0$ is a stable node of $\rho(x,t)$, and is a time-independent repulsive obstacle for the stochastic process. An apparent way out of the situation comes by considering two non-communicating processes, which are separated by the unattainable barrier at $x = 0$, [15, 20, 8, 25].

The function

$$f_+(x,t) = (2\pi)^{-1/4}(1+t^2)^{-3/4}\, x\, exp(-\frac{x^2}{4}\frac{1+t}{1+t^2})\, exp(\frac{3}{2}arctan\, t) \tag{24}$$

$$x \in [0,\infty)\ ,\ t \in [0,T]$$

is a regular solution [16] of the first initial-boundary value problem for $\partial_t f = \triangle f - cf$ specified by:

$$f_+(x,0) = (2\pi)^{-1/4}\, x\, exp(-\frac{x^2}{4}) \tag{25}$$

$$f_+(0,t) = 0$$

Then, instead of the fundamental solution, we need to utilize the Green function of the problem. To distinguish it from the fundamental Feynman-Kac kernel k we shall denote this Green function k_+. Its existence is granted by the very existence of the fundamental solution for the considered potential (16), see Ref. [38].

The Green function $k_+(y,s,x,t)$ is a unique function such that for every ϕ continuous on $(0,\infty)$ and with a compact support, the function:

$$u(x,t) = \int_0^\infty k_+(y,s,x,t)\phi(y)dy \tag{26}$$

is a solution of $\partial_t u = \triangle u - c(x,t)u$ in $(0,\infty)\times(s,T)$, with the properties: $lim_{t\downarrow s} u(x,t) = \phi(x)$ for all $x \in [0,\infty)$, and $u(0,t) = 0$ for all $t \in (s,T]$. Moreover, for every $(y,s) \in (0,\infty)\times[0,T)$ the function k_+ is strictly positive in $(0,\infty)\times(s,T)$ and $k_+(y,s,0,t) = 0$ for all $t \in (s,T]$.

The uniqueness of solutions for the first initial-boundary value problem implies the validity of the semigroup composition rule for k_+ (Chapman-Kolmogorov identity, which anticipates the Markov property of the constructed stochastic process).

In this (uniqueness) connection a more detailed comment is necessary.

For all $s \in [0,T)$, the function $f_+(x,s)$ can be uniformly approximated by a sequence of continuous functions $\phi_n^s(x)$ such that, for each natural number n the support of ϕ_n^s is compact in $(0,\infty)$. There follows that the sequence of solutions of the first initial-boundary value problem $\partial_t u = \triangle u - cu$, given by:

$$u_n(x,t) = \int_0^\infty k_+(y,s,x,t)\phi_n^s(y)dy \tag{27}$$

is uniformly convergent to the solution $f_+(x,t)$. It implies that for any $s < t$ we have

$$f_+(x,t) = \int_0^\infty k_+(y,s,x,t)f_+(y,s)dy \tag{28}$$

Now, let us consider

$$g_+(x,t) = (2\pi)^{-1/4}(1+t^2)^{-3/4}\, x\, exp(-\frac{x^2}{4}\frac{1-t}{1+t^2}\; exp(-\frac{3}{2}arctan\, t) \tag{29}$$

which is the solution of the first initial-boundary value problem for the adjoint parabolic equation

$$\partial_t v = -\triangle v + cv \tag{30}$$

$$g_+(x,0) = (2\pi)^{-1/4}\, x\, exp(-\frac{x^2}{4})$$

$$g_+(0,t) = 0$$

for all $t \in [0,T]$. Let k_+^* denotes the Green function of this adjoint equation. For every continuous function ϕ with a compact support in $(0,\infty)$ the formula

$$v(y,s) = \int_0^\infty k_+^*(x,t,y,s)\phi(x)dx \tag{31}$$

with $s < t$, defines the solution of the first initial-boundary problem for the adjoint equation. The previous arguments (at least for $T < 1$, modulo appropriate rescalings) apply in this case as well. We conclude that there holds

$$g_+(y,s) = \int_0^\infty k_+^*(x,t,y,s)g_+(x,t)dx \tag{32}$$

But, we have [16]:

$$k_+^*(x,t,y,s) = k_+(y,s,x,t) \tag{33}$$

for all $x, y \in (0,\infty)$, and $k_+^*(x,t,0,s) = 0$. So, we can write:

$$g_+(y,s) = \int_0^\infty k_+(y,s,x,t)g_+(x,t)dx \tag{34}$$

for $y > 0$, while $g_+(y,s) = 0$ if $y = 0$.

All that finally allows us to introduce the transition probability density of the Markov process respecting the stable repulsive boundary at $x = 0$ as follows:

$$p_+(y,s,x,t) = k_+(y,s,x,t)\frac{g_+(x,t)}{g_+(y,s)} \tag{34}$$

$$y \in (0,\infty)\ ,\ x \in [0,\infty)\ ,\ 0 \le s < t \le T$$

The behaviour of p_+ at $y = 0$ is to some extent irrelevant, and may involve a discontinuity. But, an innocent modification on the set of measure zero is allowed, and we choose $p(0,s,x,t) = \delta(x)$.

It completes the definition of the transition probability density of the Markov process, which is consistent with the dynamics of $\rho(x,t)$. For all $x \in [0,\infty)$, we have $\rho(x,t) = \int_0^\infty p_+(y,s,x,t)\rho(y,s)dy$, $0 \le s < t \le T$ and also $\int_0^\infty p_+(y,s,x,t)dx = 1$ for all $y \in [0,\infty)$.

However, in view of $b_+(x,t) = 2\frac{\nabla g_+(x,t)}{g_+(x,t} = \frac{1}{x} - x\frac{1-t}{1+t^2}$, which is singular at $x = 0$, the density $p_+(y,s,x,t)$ *cannot* by itself be a Green function for the associated Fokker-Planck equation $\partial_t \rho = \triangle \rho - \nabla(b\rho)$, if considered on the whole of R^1. The equation $\partial_t p_+ = \triangle p_+ - \nabla(bp_+)$ holds true in the open set $(0,\infty) \times (0,T)$.

By combining the known results [13, 15, 18]-[21] about the unattainability of nodes by the diffusion process on R_+ (respectively, on R_-), we conclude that $p_+(y,s,x,t)$ (p_-, respectively) is a transition probability density of the diffusion with the density $\rho(x,t)$, (16),forward drift $b(x,t)$, (19) for which $x = 0$ is an inaccessible repelling barrier. It remains in conformity with situations met in the conservative $c(x,t) = c(x)$ cases, when an (ergodic [25]) decomposition into the non-communicating due to nodes processes, is generic.

On both semi-axes, the respective strictly positive, continuous (domain-restricted Feynman-Kac) kernels are defined almost everywhere, except for the barrier location $x = 0$, where they vanish. Thus, if considered on $(-\infty, 0]$ or $[0, +\infty)$, the respective integral kernels are *non-negative* , and no longer strictly positive. Moreover, they seem to need to be considered separately on R_+ and R_-.

It is instructive to add that the existence of a node at time $t = 0$ does not automatically imply its survival for times $t > 0$, and in reverse (while in the present context of non-negative kernels).

4 The "Wiener exclusion"

The conventional definition of the Feynman-Kac kernel (in the conservative case)

$$exp[-t(-\triangle + c)](y,x) = \int exp[-\int_0^t c(\omega(\tau))d\tau]d\mu_{(x,t)}^{(y,0)}(\omega) \tag{35}$$

comprises all sample paths of the Wiener process on R^1, providing merely for their nontrivial redistribution by means of the Feynman-Kac weight $exp[-\int_0^t c(\omega(\tau))d\tau]$ assigned to each sample path $\omega(s) : \omega(0) = y, \omega(t) = x$.

Assume that $c(x)$ is bounded from below and locally (i.e. on compact sets) bounded from above. Then, the kernel is strictly positive and continuous [12].

For $c = 0$ we deal with the conditional Wiener measure

$$exp(t\triangle)(y,x) = \mu_{(x,t)}^{(y,0)}[\omega(s) \in R^1; 0 \leq s \leq t] = \tag{36}$$

$$\mu[\omega(s) \in R^1; \omega(0) = y, \omega(t) = x; 0 \leq s \leq t]$$

pinned at space-time points $(y,0)$ and (x,t).

The previous discussion indicates that R_- is inaccessible for all sample paths originating from R_+. In reverse, R_+ is inaccessible for those from R_-. As well, we may confine the process to an arbitrary closed subset $\Omega \subset R^1$, or enforce it to avoid ("Wiener exclusion" of Ref. [26]) certain areas in R^1.

In this context, it is instructive to know that [12] for an arbitrary open set Ω, there holds:

$$exp(t\triangle_\Omega)(y,x) = \mu_{(x,t)}^{(y,0)}[\omega(s) \in \Omega, 0 \leq s \leq t] \tag{37}$$

which is at the same time a definition of the operator $-\triangle_\Omega$, i.e. the Laplacian with Dirichlet boundary conditions, and that of the associated semigroup kernel. This formula provides us with the conditional Wiener measure which is *confined* to the interior of a given open set,[27, 28, 8].

We can introduce an analogous measure, which is confined to the *exterior* of a given closed subset $S \subset R^1$. In case of not too bad sets (like an exterior of an interval in R^1 or a ball in R^n, the corresponding integral kernel in known [12] to be positive and continuous. Technically, if S is a (regular) closed set such that the Lebesgue measure of ∂S is zero, then:

$$exp(t\triangle_{R\backslash S})(y,x) = \mu_{(x,t)}^{(y,0)}[\omega(s) \notin S; 0 \leq s \leq t] \tag{38}$$

The Feynman-Kac spatial redistribution of Brownian paths can be extended to cases (37), (38) through the general formula valid for any $f, g \in L^2(\Omega)$, where Ω is any open set of interest (hence $R\backslash S$, in particular):

$$(f, exp(-tH_\Omega)g) = \int_\Omega \overline{f}(\omega(0))g(\omega(t))exp[-\int_0^t c(\omega(\tau))d\tau]d\mu_0(\omega) \tag{39}$$

It gives rise to the integral kernel comprising the restricted Wiener path integration, which is defined at least almost everywhere in x, y. Then, its continuity is not automatically granted. We can also utilize a concept of the first exit time T_Ω for the sample path started inside Ω (or outside S)

$$T_\Omega(\omega) = inf[t > 0, X_t(\omega) \notin \Omega] \tag{40}$$

where X_t is the random variable of the process. Then, we can write, [13, 14, 8]

$$exp(-tH_\Omega)(y,x) = \int exp[-\int_0^t c(\omega(\tau))d\tau]d\mu_{(x,t)}^{(y,0)}[\omega; t < T_\Omega] = \tag{41}$$

$$\int_\Omega exp[-\int_0^t c(\omega(\tau))d\tau]d\mu_{(x,t)}^{(y,0)}(\omega)$$

It is an integration restricted to these Brownian paths, which while originating from $y \in \Omega$ at time $t = 0$ are conditioned to reach $x \in \Omega$ at time $t > 0$ without crossing (but possibly touching) the boundary ∂S of S. The contribution from paths which would touch the boundary without crossing, for at least one instant $s \in [0,t]$ is of Wiener measure zero, [27].

In case of processes with unattainable boundaries, with probability 1, there is no sample path which could possibly reach the barrier at any instant $s < \infty$.

The above discussion made an implicit use of the integrability property

$$\int_0^t c(\omega(s))ds < \infty \tag{42}$$

for $\omega \in R^1, 0 \leq s \leq t.$, in which case the corresponding integral kernel (for bounded from below potentials) is strictly positive. Then, if certain areas are inaccessible to the process, it occurs excusively [18]-[21] due to the drift singularities, which are capable of "pushing" the sample paths away from the barriers.

The previous procedure can be extended to the singular [29]-[37] potentials, which are allowed to diverge. Their study was in part motivated by the so called Klauder's phenomenon (and the related issue of the ground state degeneracy of quantal Hamiltonians), and had received a considerable attention in the literature.

In principle, if S is a closed set in R^1 like before, and $c(x) < \infty$ for all $x \in \Omega = R\backslash S$, while $c(x) = \infty$ for $x \in S$, then depending on how severe the singularity is, we can formulate a criterion to grant the exclusion of certain sample paths of the process and hence to limit an availability of certain spatial areas to the random motion. Namely, in case of (42) nothing specific happens, but if we have

$$\int_0^t c(\omega(\tau))d\tau = \infty \tag{43}$$

for $\omega(\tau) \in S$ for some $\tau \in [0,t]$, then the "Wiener exclusion" certainly appears: we are left with contributions from these sample paths only for which (43) does not occur. Unless the respective set is of Wiener measure zero.

The area Ω comprising the relevant sample paths is then selected as follows:

$$\Omega = [\omega; \int_0^t c(\omega(\tau))d\tau < \infty] \tag{44}$$

In particular, the criterion (44) excludes from considerations sample paths which cross S and so would establish a communication between the distinct connected components of Ω.

The singular set S can be chosen to be of Lebesgue measure zero and contain a finite set of points dividing R into a finite number of open connected components. With each open and connected subset $\Omega \subset R^1$ we can [29] associate a strictly positive Feynman-Kac kernel, which can be expected to display continuity.

Since the respective potentials diverge on S, their behaviour in a close neighbourhood of nodes is quite indicative. For, if ω_S is a Wiener process sample path which is bound to cross a node at $0 \leq s \leq t$, then the corresponding contribution to the path integral vanishes. Such paths are thus excluded from consideration. If their subset is sizable (of nonzero Wiener measure), then the eliminated contribution

$$\int_{\omega_S} exp[-\int_0^t c(\omega_S(\tau))d\tau]d\mu_{(x,t)}^{(y,0)}(\omega) = 0 \tag{45}$$

is substantial in the general formula (35).

At the same time, we get involved a nontrivial domain property of the semigroup generator $H = -\triangle + c$ resulting in the so called ground state degeneracy [29, 30, 31]. Let us recall (Theorem 25.15 in Ref. [12]) that if c is bounded from below and locally bounded from above, then the ground state function of $H = -\triangle + c$ is everywhere strictly positive and thus bounded away from zero on every compact set.

5 Singular potentials and the ground state degeneracy

Our further discussion will concentrate mainly on singular perturbations of the harmonic potential. Therefore, some basic features of the respective parabolic problem are worth invoking. The eigenvalue problem (the temporally adjoint parabolic system now trivializes):

$$-\triangle g + (x^2 - E)g = 0 = \triangle f - (x^2 - E)f \tag{46}$$

has well known solutions labeled by $E_n = 2n + 1$ with $n = 0, 1, 2, ...$ In particular, $g_0(x) = f_0(x) = \frac{1}{\pi^{1/4}} exp(-\frac{x^2}{2})$ is the unique nondegenerate ground state solution. The corresponding Feynman-Kac kernel reads

$$exp(-tH)(y,x) = k(y,0,x,t) = k_t(y,x) =$$

$$(\pi)^{-1/2}(1 - exp(-t))^{-1/2} exp[-\frac{x^2 - y^2}{2} - \frac{(y exp(-t) - x)^2}{2}] \tag{47}$$

$$\partial_t k = -\triangle_x k + (x^2 - 1)k$$

and the invariant probability density $\rho(x) = f(x)g(x) = (\pi)^{-1/2} exp(-x^2)$ is preserved in the course of the time-homogeneous diffusion process with the transition probability density

$$p(y,s,x,t) = k_{t-s}(y,x)\frac{g(x)}{g(y)} \tag{48}$$

We have $p(y,s,x,t) = p(y,0,x,t-s)$.

Notice the necessity of the eigenvalue correction of the potential both in (35) and (36), which is indispensable to reconcile the functional form of the forward drift $b(x) = 2\nabla lng(x) = -2x$ with the general expression for the corresponding (to the diffusion process) parabolic system potential

$$c = c(x,t) = \partial_t ln\, g + \frac{1}{2}(\frac{b^2}{2} + \nabla b) \tag{49}$$

which equals $c(x) = x^2 - 1$ in our case.

Let us pass to the singular (degenerate) problems.

Example 1: The potential [7]:

$$c(x) = x^2 + \frac{\gamma^2}{x^{2(\gamma+1)}} - \frac{\gamma(\gamma+1)}{|x|^{2+\gamma}} - \frac{2\gamma}{|x|^\gamma} \tag{50}$$

with $\gamma > 0, x \in R^1$, is singular at $x = 0$ and is a well defined even function if otherwise. We can give [7] a solution to the stationary parabolic system

$$-\triangle g + (c-1)g = 0 = \triangle f - (c-1)f \tag{51}$$

in terms of

$$g(x) = f(x) = exp[-(\frac{1}{|x|^\gamma} + \frac{x^2}{2})] \tag{52}$$

In this case, an invariant density up to normalization reads $\rho(x) = (fg)(x) = g^2(x)$ and is integrable on R^1. It vanishes at $x = 0$ and at both spatial infinities.

By independent arguments [15]-[21] we know that a Markov diffusion process preserving $\rho(x)$ can be consistently defined. The node $x = 0$ is unattainable in view of the appropriate singularity of the forward drift:

$$b(x) = 2\nabla ln\, g(x) = sgn\, x\, \frac{2\gamma}{|x|^{1+\gamma}} - 2x \tag{53}$$

which pushes sample paths away from the node. Hence, there is no communication (realised by sample paths of the process) between R_+ and R_-. Like in case of $n > 1$ eigenfunctions of the harmonic oscillator, we deal with the totally disjoint (ergodic, [25]) components of the would-be global [20] diffusion.

This feature is nicely manifested in the apparent domain degeneracy of the associated semigroup generator $H = -\triangle + (c-1)$. Namely, $Hg = 0$ is simply an eigenvalue problem. Let us define $g_+(x,t) = g(x,t)$ for $x > 0$ and $g_+(x,t) = 0$ for

$x \leq 0$, and $g_-(x,t) = 0$ for $x \geq 0$ while $g_-(x,t) = g(x,t)$ for $x < 0$. The same procedure can be repeated for $f \to f_\pm$.

The functions g_+ and g_- (f_+ and f_- respectively) belong to $L^2(R^1)$, and are orthogonal on R^1, while corresponding to the same eigenvalue. It is an obvious spectral degeneracy of the respective generator H. As mentioned before, semigroups $exp(-tH)$ with strictly positive kernels do not have [29] generators with the ground state degeneracy. On the other hand, if S is the (singular) set of Lebesgue measure zero and $R\backslash S$ has m or more connected components, then there always exists [29] a positive $c(x)$ in $L^1_{loc}(R\backslash S)$ such that the ground state of $H = -\triangle + c$ is m-fold degenerate.

This phenomenon we encounter in connection with (50). Recall that L^1_{loc} comprises equivalence classes of functions which are integrable on compact sets (e.g. locally).

Example 2: The canonical (in the context of Refs. [29]-[34]) choice of the centrifugal potential:

$$c_E(x) = x^2 + \frac{2\gamma}{x^2} - E \tag{54}$$

generates a well known spectral solution [33, 34] for $Hg = [-\triangle + c_E(x)]g$. The eigenvalues:

$$E_n = 4n + 2 + (1 + 8\gamma)^{1/2} \tag{55}$$

with $n = 0, 1, 2, ...$ and $\gamma > -\frac{1}{8} \Rightarrow (1 + 8\gamma)^{1/2} > \sqrt{2}$, are associated with the eigenfunctions of the form:

$$g_n(x) = x^{(2\gamma+1)/2}\, exp(-\frac{x^2}{2})\, L_n^\alpha(x^2) \tag{56}$$

$$\alpha = (1 + 8\gamma)^{1/2}$$

$$L_n^\alpha(x^2) = \sum_{\nu=0}^{n} \frac{(n+\alpha)!}{(n-\nu)!(\alpha+\nu)!} \frac{(-x^2)^\nu}{\nu!} \longrightarrow$$

$$L_0^\alpha(x^2) = 1 \ , \ L_1^\alpha(x^2) = -x^2 + \alpha + 1$$

It demonstrates an apparent double degeneracy of both the ground state and of the whole eigenspace of the generator H. The singularity at $x = 0$ does not prevent the definition of $H = -\triangle + x^2 + \frac{2\gamma}{x^2}$ since this operator is densely defined on an appropriate subspace of $L^2(R^1)$. This singularity is sufficiently severe to decouple $(-\infty, 0)$ from $(0, \infty)$ so that $L^2(-\infty, 0)$ and $L^2(0, \infty)$ are the invariant subspaces of H with the resulting overall double degeneracy.

Potentials of the form, [12, 26, 29]:

$$c(x) = x^2 + [dist(x, \partial\Omega)]^{-3} \tag{57}$$

where $\partial\Omega$ can be identified with ∂S, and S is a closed subset in R^1 of any (zero or nonzero) Lebesgue measure, have properties generic to the Klauder's phenomenon. Because the Wiener paths are known to be Hölder continuous of any order $\frac{1}{2}-\epsilon, \epsilon > 0$ and of order $\frac{1}{3}$ in particular, there holds $\int_0^t c(\omega(\tau))d\tau = \infty$ if $\omega(\tau) \in S$ for some τ. Conversely, $\int_0^c(\omega(\tau))d\tau < \infty$ if ω never hits S. This implies that the relevant contributions to:

$$(f, exp[-t(-\triangle + c)]g) = \int \overline{f}(\omega(0))g(\omega(t))exp[-\int_0^t c(\omega(\tau))d\tau]d\mu_0(\omega) \tag{58}$$

come only from the subset of paths defined by

$$Q_t = [\omega;\ \int_0^t c(\omega(\tau))d\tau < \infty] \tag{59}$$

The above argument might seem inapplicable to the centrifugal problem. However it is not so. In the discussion of the divergence of certain integrals of the Wiener process, in the context of Klauder's phenomenon, it has been proven [36] that for almost every path from $x = 1$ to $x = -1$ (crossing the singularity point $x = 0$) there holds $\int_{t-\delta}^{t+\delta} |\omega(\tau)|^{-1}d\tau = \infty$ for any $\delta > 0$.

To be more explicit: if $\tau_1 = \tau_1(\omega)$ is the first time such that the Wiener process $W(t) = W(t, \omega)$ attains the level (location on R^1) $W(\tau_1) = 1$, then the integral over any right-hand-side neighbourhood $(\tau_1, \tau_1 + \delta)$ of τ_1 diverges:

$$\int_{\tau_1}^{\tau_1+\delta} c(\omega(t) - 1)dt = \infty \tag{60}$$

if $\int_{-1}^{+1} c(x)dx = \infty$.
In case of the left-hand-neighbourhood of τ_1, we have

$$\int_{\tau_1-\delta}^{\tau_1} c(\omega(t) - 1)dt = \infty \tag{61}$$

if $\int_{-1}^{0} xc(x)dx = \infty$.

All that holds true in case of the centrifugal potential, thus proving that the only subset of sample paths, which matters in (58) is (59). Obviously, Q_t does not include neither paths crossing $x = 0$ nor those which might hit (touch) $x = 0$ at any instant. The singularity is sufficiently severe to create an unattainable repulsive boundary for all possible processes, which we can associate with the spectral solution (55),(56).

6 The singular potential in action

After the previous analysis one might be left with an impression that the appearence of the stable barrier at $x = 0$ persisting for all $t \in [0, T]$, is a consequence of the initial data choice $\psi_0(0) = 0$ for the involved quantum Schrödinger picture dynamics. In general it is not so. For example, $\psi_0(x) = x^2 exp(-x^2/4)$ which vanishes at $x = 0$, does not vanish anymore for times $t > 0$ of the free evolution.

On the other hand, somewhat surprisingly from the parabolic (intuition) viewpoint, the node can be dynamically developed from the nonvanishing initial data and lead to the nonvanishing terminal data.

Let us consider a complex function:

$$\psi(x,t) = (1+it)^{-1/2} exp[-\frac{x^2}{4(1+it)}]\,[\frac{x^2}{2(1+it)^2} + \frac{it}{1+it}] \tag{62}$$

which solves the free Schrödinger equation with the initial data $\psi(x,0) = \frac{x^2}{2} exp(-\frac{x^2}{4})$. It vanishes at $x = 0$ exclusively at the initial instant $t = 0$ of the evolution.

Obviously, there is nothing to prevent us from considering

$$\Psi(x,t) = \psi(x, t-\alpha) \tag{63}$$

for $\alpha > 0$. It solves the same free equation, but with nonvanishing initial data. However, the node is developed in the course of this evolution at time $t = \alpha$ and instantaneously desintegrated for times $t > \alpha$. Here, the Schrödinger boundary data problem would obviously involve two strictly positive probability densities $\rho_0(x) = |\Psi(x,0)|^2$ and $\rho_T(x) = |\Psi(x,T)|^2$, $T > \alpha$. It would suggest to utilize the theory [11], based on strictly positive Feynman-Kac kernels, to analyze the corresponding interpolating process. However, this tool is certainly inappropriate and cannot reproduce the a priori known dynamics, with the node arising at the intermediate time instant.

To handle the issue by means of a parabolic system, which we can always associate with a quantum Schrödinger picture dynamics, let us evaluate the potential $c(x,t)$ appropriate for (1).

In view of

$$\rho(x,t) = const\ (1+t^2)^{-5/2} exp[-\frac{x^2}{2(1+t^2)}]\,[\frac{x^4}{4} - x^2t^2 + t^2(1+t^2)] \tag{64}$$

we have (while setting $w^{1/2}(x,t) = [\frac{x^2}{4} - x^2t^2 + t^2(1+t^2)]$):

$$c(x,t) = \frac{\triangle \rho^{1/2}(x,t)}{\rho^{1/2}(x,t)} = \frac{1}{4}(-\frac{x}{1+t^2} + \nabla w)^2 + \frac{1}{2}(-\frac{1}{1+t^2} + \triangle ln\ w) = \tag{65}$$

$$\frac{1}{4}\frac{x^2}{(1+t^2)^2}-\frac{1}{2}\frac{3x^2-2t^2x}{\frac{x^4}{4}-t^2x^2+t^2(1+t^2)}-\frac{1}{2(1+t^2)}+$$
$$\frac{1}{2}\frac{3x^2-2t^2}{\frac{x^4}{4}-t^2x^2+t^2(1+t^2)}-\frac{1}{4}(\frac{x^3-2t^2x}{\frac{x^4}{4}-t^2x^2+t^2(1+t^2)})^2$$

The expression looks desparately discouraging, but its $t \downarrow 0$ (i.e. the initial data) limit is quite familiar and displays a centrifugal singularity at $x = 0$:

$$c(x,t) = \frac{\triangle\rho^{1/2}(x,0)}{\rho^{1/2}(x,0)} = \frac{x^2}{4} + \frac{2}{x^2} - \frac{5}{2} \tag{66}$$

Since the original, dimensional expression for the centrifugal eigenvalue problem is [33]:

$$(-\frac{1}{2}\triangle + \frac{m^2}{2}x^2 + \frac{\gamma}{x^2})g = Eg \tag{67}$$
$$E_n = m[2n+1+\frac{1}{2}(1+8\gamma)^{1/2}]$$

with $n = 0, 1, ...$, an obvious adjustment of constants $m = 1/2, \gamma = 1$ allows to identify $E = 5/2$ as the $n = 0$ eigenvalue of the centrifugal Hamiltonian $H = -\triangle + \frac{x^2}{4} + \frac{2}{x^2}$.

A peculiarity of the considered example is that it enables us to achieve an explicit insight into an emergence of the centrifugal singularity and its subsequent destruction (decay) for times $t > \alpha$, due to the free quantum evolution.

In view of the degeneracy of the ground-state eigenfunction $\frac{x^2}{2}exp(-\frac{x^2}{4})$ of the centrifugal Hamiltonian, we deal here with the gradually decreasing communication between R_+ and R_-, which results in the emergence of the completely separated (disjoint) sets $(-\infty, 0)$ and $(0, +\infty)$ at $t = \alpha$, followed by the gradual increase of the communiaction for times $t > \alpha$. By "communication" we understand that the set of sample paths crossing $x = 0$ forms a subset of nonzero Wiener measure.

It also involves a generalisation (cf. also Refs. [11, 37, 38]) to time-dependent Feynman-Kac kernels:

$$(f, exp[-\int_s^t H(\tau)d\tau]g) = \int \overline{f}(\omega(s))g(\omega(t))exp[-\int_s^t c(\omega(\tau),\tau)d\tau]\; d\mu_0(\omega) \tag{68}$$

$$Q_{s,t} = [\omega;\int_s^t c(\omega(\tau),\tau)d\tau < \infty]$$

The finiteness condition $\int_s^t c(\omega(\tau),\tau)d\tau < \infty$, surely does not hold true, [36], if $\delta > 0$ is sufficiently small, cf. (60),(61).

Let us mention that a few interesting mathematical questions are left aside in the present paper. They deserve a separate, thorough study on their own. For example, even in case of conventional Feynman-Kac kernels, the weakest possible criterions allowing for their continuity in spatial variables are not yet established. An issue of the continuity of the kernel in case of general singular potentials, needs an investigation as well.

7 Nonnegative solutions of parabolic equations and the Schrödinger boundary data problem according to R. Fortet

As emphasized before, one of motivations for our analysis was the quantal observation (exploiting the Born statistical interpretation postulate as the principal building block of the theory) that the temporally adjoint pair of Schrödinger equations:

$$i\partial_t\psi(x,t) = -D\triangle\psi(x,t) + \frac{1}{2mD}V(x)\psi(x,t) \tag{69}$$

$$i\partial_t\overline{\psi}(x,t) = D\triangle\overline{\psi}(x,t) - \frac{1}{2mD}V(x)\overline{\psi}(x,t)$$

by means of the polar decomposition $\psi = exp(R+iS), \overline{\psi} = exp(R-iS)$ can be transformed into the (hopelessly looking at the first glance) nonlinearly coupled parabolic system of the form (1):

$$\partial_t\theta_* = D\triangle\theta_* - \frac{1}{2mD}(2Q-V)\theta_* \tag{70}$$

$$\partial_t\theta = -D\triangle\theta + \frac{1}{2mD}(2Q-V)\theta$$

$$Q = 2mD^2\frac{\triangle\rho^{1/2}}{\rho^{1/2}}$$

$$\rho(x,t) = \theta_*(x,t)\theta(x,t) = \overline{\psi}(x,t)\psi(x,t)$$

for real functions $\theta = exp(R+S), \theta_* = exp(R-S)$. In the above $\hbar/2m = D$ can be set to restore the traditional notation.

While searching for a probabilistic meaning of the system (70), we had in fact assumed (see also Refs. [9, 11]) to have in hands a solution of (69), so that $\rho(x,t)$ was known. Our next step amounts to replacing the nonlinear parabolic system (70) by a linear one (1), (with $D = m = 1$) where for each given functional choice of

$c(x,t) = \frac{1}{2mD}(2Q - V)$, *all* allowed solutions $u(x,t), v(x,t)$ were sought for, including the fundamental one. At this point the crucial role of the respective Feynman-Kac kernel was disclosed. Effectively, we have invoked the Schrödinger boundary data problem to pick up the unique solution from among an infinity of others, such that the arising probability measure dynamics (if any) is consistent with the a priori known probability density boundary data.

In this indirect way, a definite justification for the probabilistic significance of the nonlinear parabolic system has been achieved. The procedure proves to be consistent, thus allowing to investigate fundamental solutions, Green functions and other solutions of (70). Even, if viewed as the nonlinearly coupled parabolic system per se. See e.g. also Refs. [7, 39] for a discussion of the "propagation of chaos" in a system of interacting (coupled) diffusion processes

If the Feynman-Kac kernels are strictly positive, the respective solutions are strictly positive as well, except for the boundaries $\partial\Omega$ of the spatial area confining the process. Once we admit the *nonnegative* Feynman-Kac kernels, we fall into another theoretical framework, this of nonnegative solutions of linear (and nonlinear) parabolic equations, [40]. Since we address the general time-independent potentials, and as we have seen the time-dependent domain properties of semigroup generators are involved, we touch upon an almost undeveloped theory of construction of Markov processes associated with time-dependent Dirichlet forms and spaces, [41, 42]. The standard theory of forms [21] does not work in case of time-inhomogeneous evolutions.

On the other hand, the Schrödinger problem itself needs an extension to nonnegative Feynman-Kac kernels. The strongest (uniqueness of solution) result was established [5] for strictly positive kernel functions, a demand which needs to be relaxed for our purposes.

Hence, our major assumption must be that the kernel is a nonnegative and continuous function. One can try to relax the continuity condition as well. Moreover, it appears that the kernel may not be a function, and nonetheless one can expect that the main features of our analysis would persist.

Beurling [4] has attempted to relax the strict positivity condition in more than one spatial dimension, with a partial success only. An earlier analysis due to Fortet [3] is of particular importance in our one-dimensional context. He addressed an issue of the existence and uniqueness of nonnegative solutions of the Schrödinger boundary data problem, under an assumption that the kernel is continuous and nonnegative in one spatial dimension.

Fortet's integral equations:

$$\rho_1(x) = f(x) \int_{\Omega_2} k(x,y) g(y) dy \tag{71}$$

$$\rho_2(x) = g(y) \int_{\Omega_1} f(x)k(x,y)dx$$

where Ω_1, Ω_2 are finite (or not) intervals in R^1, and $\int_{\Omega_1} \rho_1(x)dx = \int_{\Omega_2} \rho_2(y)dy > 0$, while $k(x,y) \geq 0$, $\rho_1(x) \geq 0, \rho_2(y) \geq 0$, were to be solved with respect to the unknown functions $f(x), g(y)$, defined respectively on Ω_1, Ω_2.

All functions $k(x,y), \rho_1(x), \rho_2(y)$ are by assumption real and measurable (and integrable) for $x \in \Omega_1, y \in \Omega_2$. There are however the additional assumptions which must be respected:
(i) $k(x,y)$ is continuous, bounded from the above and nonnegative almost everywhere in $\Omega_1 \times \Omega_2$, i.e. except for a set S of measure zero comprising both $x's$ and $y's$ in R^1,
(ii) $\rho_1(x)$ and $\rho_2(y)$ are continuous,
(iii) let $\overline{A}$ be a closed interval in Ω_1. For a nonnegative continuous in Ω_2 functions $g(y)$ we demand that *if* the integral $G(x) = \int_{\Omega_2} k(x,y)g(y)dy$ is finite almost everywhere on an *open* subset of Ω_1 containing $\overline{A}$, then this integral is uniformly convergent on $\overline{A}$. Analogously with respect to $y \in \overline{B} \subset \Omega$, with $f(x) \rightarrow F(y)$ on Ω_2.

Under these hypotheses, the integral equations (71) admit a unique solution given in terms of two functions:
(1) $f(x)$ which is strictly positive and continuous almost everywhere, except for the set of zeros of $\rho_1(x)$,
(2) $g(y)$ which has the *same* zeros as $f(x)$, is strictly positive almost everywhere and measurable.

Hence, $g(y)$ is not necessarily continuous, and one should realise that we must have granted the existence of $\nabla ln\ g(x,t)$ as the drift field. If the function $g(x)$ is continuous, all that fits to our previous discussion. However, even in this case some additional restrictions on $k(x,y)$ are necessary to guarrantee a differentiability of $f(x,t) = \int k(y,0,x,t)f(y)dy$ and $g(x,t) = \int k(x,t,y,T)g(y)dy$, and make them solutions of the time-adjoint parabolic system once we set $k(x,y) = k(y,s,x,t), 0 \leq s < t \leq T$ and select the appropriate Feynman-Kac kernel.

In connection with the previous centrifugal example, the above conditions appear to be too restrictive. Let us therefore invoke another result due to Fortet. Namely, if the above condition (iii) is replaced by the demand:

$$\int_{\Omega_2} \frac{\rho_2(y)}{[\int_{\Omega_1} k(z,y)\rho_1(z)dz]}\, dy \ < \ \infty \tag{72}$$

then a unique nonnegative solution of the integral equations (71) comprises a continuous function $f(x)$ whose zeros coincide with those of $\rho_1(x)$. The function $g(y)$ is measurable and has zeros of $\rho_2(x)$, which are not necessarily in common with those of ρ_1.

This result opens a number of interesting propagation scenarios, and deserves a careful analysis (with the generalization prospects) in higher dimensions, cf. also for a related discussion in Refs. [43, 44]. Notice, that our centrifugal example exhibits in its simplest version an intriguing feature of Fortet's analysis: we are capable of producing probability densities $\rho_1(x)$ (initial) and $\rho_2(y)$ (terminal), which have *non-coinciding* sets of zeros. The inquiry into the corresponding Schrödinger's interpolating dynamics is quite an appealing problem.

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References

[1] E. Schrödinger, Ann. Inst. Henri Poincare, 2, (1932), 269

[2] S. Bernstein, Verh. Intern. Math. Kongr. (Zürich), (1932),288

[3] R. Fortet, J. Math. Pures Appl., 9, (1940), 83

[4] A. Beurling, Ann. Math. 72, (1960), 189

[5] B. Jamison, Z. Wahrsch. verw. Geb. 30, (1974), 65

[6] J. C. Zambrini, J. Math. Phys. 27, (1986), 3207

[7] M. Nagasawa, "Schrödinger Equations and Diffusion Theory", Birkhäuser, Basel, 1993

[8] Ph. Blanchard, P.Garbaczewski, Phys. Rev. E 49, (1994), 3815

[9] P.Garbaczewski, R. Olkiewicz, Phys. Rev. A 51, (1995), 3445

[10] P.Garbaczewski, J. R. Klauder, R. Olkiewicz, Phys. Rev. E 51, (1995), 4114

[11] P. Garbaczewski, R. Olkiewicz, "Feynman-Kac Kernels in Markovian Representations of the Schrödinger Interpolating Dynamics", subm. for publ., quant-ph/9505012 preprint

[12] B. Simon, "Functional Integration and Quantum Physics", Academic Press, New York, 1979

[13] R. Carmona, in Taniguchi Symposium , PMMP, Katata 1985, ed. K. Itô, N. Ikeda, Academic, Boston, 1987

[14] M. Nagasawa, Prob. Theory Relat. Fields, 82, (1989), 109

[15] Ph. Blanchard, S. Golin, Commun. Math. Phys. 109, (1987), 421

[16] A. Friedman, "Partial Differential Equations of Parabolic type", Prentice-Hall, Englewood, NJ, 1964

[17] P.Garbaczewski, Phys. Lett. A 178, (1993), 7

[18] Ph. Blanchard, Ph. Combe, W. Zheng, "Mathematical and Physical Aspects of Stochastic Mechanics", Lecture Notes in Physics vol. 281, Springer-Verlag, Berlin, 1987

[19] E. Nelson, "Quantum Fluctuations", Princeton University Press, Princeton, 1985

[20] E. Carlen, Commun. Math. Phys. 94, (1984), 293

[21] M. Fukushima, in: "Mathematics + Physics", ed. L.Streit, World Scientific, Singapore, 1985, vol. 1

[22] A. M. Ilin, A. S. Kalashnikov, O. A. Oleinik, Usp. Mat. Nauk (in Russian), 27, (1962), 65

[23] M. Krzyżański, A. Szybiak, Lincei-Rend. Sc. fis. mat. e nat. 28, (1959), 26

[24] P. Besala, Ann. Polon. Math. 29, (1975), 403

[25] S. Albeverio, R. Høegh-Krohn, Z. Warsch. verw. Geb. 40, (1977), 1

[26] B. Simon, Adv. Math. 30, (1978), 268

[27] J. Ginibre, in: "Statistical Mechanics and Field Theory", ed. C. DeWitt, R. Stora, Gordon and Breach, New York, 1971

[28] O. Bratteli, D. W. Robinson, "Operator Algebras and Quantum Statistical Mechanics", Springer-Verlag, Berlin, 1981, vol. II

[29] W. Faris, B. Simon, Duke Math. Journ. 42, (1975), 559

[30] J. R. Klauder, Acta Phys. Austr. Suppl. 11, (1973), 341

[31] H. Ezawa, J. R. Klauder, L.A.Shepp, J. Math. Phys. 16, (1975), 783

[32] E. M. Harrell, Ann. Phys. (NY), 105, (1977), 379

[33] G. Parisi, F. Zirilli, J. Math. Phys. 14, (1973), 243

[34] F. Calogero, J. Math. Phys. 10, (1969), 2191

[35] B. Simon, J. Funct. Anal. 14, (1973), 293

[36] L. A. Shepp, J. R. Klauder, H. Ezawa, Ann. Inst. Fourier-Grenoble, 24, (1974), 189

[37] H. Ezawa, J. R. Klauder, L. A. Shepp, Ann. Phys. (NY), 18, (1974), 65

[38] M. Freidlin, "Functional Integration and Partial Differential Equations", Princeton University Press, Princeton, 1985

[39] H. P. McKean, Proc. Nat. Acad. Sci. 56, (1966), 1907

[40] D. G. Aronson, Ann. Scuola Norm. Sup.-Pisa, 6, (1968), 607

[41] Y. Oshima, Osaka J. Math., 29, (1992), 103

[42] Y. Oshima, Forum Math. 4, (1992), 395

[43] T. Wallstrom, Phys. Rev. A 49, (1994), 1613

[44] E. A. Carlen, M. I. Loffredo, Phys. Lett. A 141, (1989), 9