

# Modular Covariance, PCT, Spin and Statistics

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**Abstract.** The notion of modular covariance is reviewed and the reconstruction of the Poincaré group extended to the low-dimensional case.

The relations with the PCT symmetry and the Spin and Statistics theorem are described.

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## Introduction

The first relation between some space-time transformations and the modular group of the von Neumann algebras associated with wedge regions was discovered by Bisognano and Wichmann in the particular case of Wightman fields [1,2]. They also proved that the modular conjugation implements both the space-time reflection w.r.t. the edge of the wedge and the charge conjugation. The analogous result for conformally covariant theories was then proven in [19].

Since the Bisognano-Wichmann relations are far more intrinsic in the algebraic setting than in the Wightman field approach, it is very natural to conjecture such relations to hold for the local algebras of a quantum field theory under general hypotheses.

Indeed the examples in [25] and in [27], where the modular operators are not associated with some covariant representation of the space-time symmetries of the theory, suggest that some kind of compactness condition (e.g. the split property) and Poincaré covariance could give such general hypotheses, but this conjecture is far from being proven.

In 1992, Borchers proposed a different approach to the problem. Instead of showing the Bisognano-Wichmann relations for a given covariant representation, his purpose was to reconstruct the space-time symmetries in terms of the modular operators. In particular he showed that for positive energy, translationally covariant, one- or two-dimensional theories, modular groups and translations give a covariant representation of the Poincaré group [3].

The theorem of Borchers had several consequences. In particular we quote the solution of the Bisognano-Wichmann conjecture for conformal theories [5,15], and the characterization of the conformal theories on  $S^1$  in terms of half-modular inclusions [26]. Converses of Borchers theorem are contained in [9].

Borchers' purpose may be pursued in terms of a different set of hypotheses, namely the geometrical meaning of some modular objects.

Buchholz and Summers [8] were able to reconstruct the translation group assuming that the modular conjugations of wedge regions implement space-time reflections [see also [4].

The reconstruction of the whole group of space-time symmetries for high-dimensional theories is studied in [6]. Neither translation covariance nor essential duality is assumed there, but the essence of the Bisognano-Wichmann prescription: the modular groups of the wedge regions should implement the correct space-time transformations. Such an assumption, which was called modular covariance, is sufficient to reconstruct a covariant representation of the Poincaré group and imply the second Bisognano-Wichmann property, namely the relation between modular conjugations and space-time reflections.

A related result in [22] shows that if the modular conjugation, resp. the modular group (dimension  $\geq 4$ ), implement a geometrical transformation whatsoever, then it implements the correct one. A classification of different, generalized forms of modular covariance is contained in [10].

Up to now conformal theories were the only place where both approaches completely solve the problem. Indeed Borchers theorem implies, as already mentioned, that conformal theories verify the Bisognano-Wichmann properties, while (conformal) modular covariance reconstructs the unique covariant, positive-energy representation of the conformal group ([6], see also Remark 1.6). On the other hand, when the reconstruction of the Poincaré group is concerned, modular covariance was confined to the high dimensional case and Borchers technique to the low dimensional one.

Here we show that the modular covariance assumption may reconstruct the Poincaré group in the one and two dimensional case either (cf. Section 1). Moreover, for any space-time dimension, such assumption may be weakened to a more intrinsic one if essential duality is assumed. More precisely, only the covariant action of the modular automorphism group associated with a wedge  $W$  on the algebras of subregions of  $W$  is requested.

In this way positive energy translational covariance is equivalent to (weak) modular covariance for low-dimensional theories satisfying essential duality.

As we mentioned before, the results of Bisognano and Wichmann are intimately tied with the PCT symmetry and therefore with the Spin and Statistics relation. Conversely, modular covariance properties give sufficient hypotheses for the PCT and Spin and Statistics theorems to hold, as it is shown in [16], [17], [21], [18], [7]. We present and discuss some of these results in Section 2.

## 1. Modular covariance on the Minkowski space

In this section we review some results about modular covariance studied in [6] and extend such results to low-dimensional Minkowski spaces, thus making them comparable with the theorem of Borchers about the Bisognano-Wichmann property and PCT symmetry in the two-dimensional space-time [3], see also [4].

Moreover we weaken the modular covariance assumption, requesting only that the intrinsic action of the modular automorphism group of a wedge on the algebras associated with some of its subregions has the prescribed geometrical meaning. As a counterpart, we assume essential duality, which follows when the stronger modular covariance is assumed [5].

We shall always consider local precosheaves on the wedges of the  $n$ -dimensional Minkowski space  $M$ ,  $n \geq 1$ , i.e. maps

$$W \rightarrow \mathcal{A}(W)$$

from the family  $\mathcal{W}$  of wedge regions in  $M$  (when  $n = 1$  wedge regions are, by definition, open half lines) to von Neumann algebras on a separable Hilbert space  $\mathcal{H}$  verifying the *isotony* property:

$$W_1 \subset W_2 \Rightarrow \mathcal{A}(W_1) \subset \mathcal{A}(W_2)$$

and the stronger form of locality called *essential duality*:

$$\mathcal{A}(W') = \mathcal{A}(W)' ,$$

where  $W'$  denotes the space-like complement of  $W$  (the interior of the complement when  $n = 1$ ).

We denote by  $\mathcal{P}_+$ , resp.  $\mathcal{P}_+^\uparrow$ , the proper, resp. proper orthochronous Poincaré group. If  $n = 1$ ,  $\mathcal{P}_+$ , resp.  $\mathcal{P}_+^\uparrow$  is the group of affine, resp. orientation preserving affine transformations.

If  $W \in \mathcal{W}$ ,  $\Lambda_W$  denotes the one-parameter group of boosts preserving  $W$  (no rescaling is adopted here). When  $n = 1$  and  $W$  is a right half-line  $\Lambda_W$  is the one parameter group of dilations fixing the edge of  $W$ .  $\Lambda_W$  for left half lines is determined by the equation  $\Lambda_W(t) = \Lambda_{W'}(-t)$ . We also denote by  $r_W$  the element in  $\mathcal{P}_+$  corresponding to the reflection w.r.t. the edge of  $W$ .

The main result of this section is to show that under the weak modular covariance assumption (see below) we may construct a canonical representation of the proper Poincaré group acting on the local algebras consistently with the action of  $\mathcal{P}_+$  on  $M$ . In particular this gives a PCT operator, i.e. an antiunitary operator which corresponds to the PT transformation on  $M$  and implements the charge conjugation on superselection sectors (cf. [16,18]).

First we discuss the low-dimensional case ( $n = 1, 2$ ), which is interesting in itself and furnishes the basis for the weakening of the modular covariance assumption in the higher dimensional Minkowski space.

*The low-dimensional case,  $n \leq 2$ .*

**1.1 Theorem.** *Let  $\mathcal{A}$  be a precosheaf on the wedges of the  $n$ -dimensional Minkowski space,  $n \leq 2$ , satisfying essential duality. Assume also the existence of a vector  $\Omega$  (vacuum) cyclic for the algebras of all wedges, and weak modular covariance: if  $W_1 \supset W_2$  then*

$$\sigma_{W_1}^t(\mathcal{A}(W_2)) = \mathcal{A}(\Lambda_{W_1}(2\pi t)W_2) \tag{1.1}$$

where  $\sigma_W$  denotes the modular automorphism group of the algebra  $\mathcal{A}(W)$  associated with the state  $\omega := (\Omega, \cdot \Omega)$ .

Then there is a positive energy (anti)-unitary representation  $U$  of the Poincaré group  $\mathcal{P}_+$  determined by

$$\begin{aligned} U(\Lambda_W(t)) &= \Delta_W^{\frac{it}{2\pi}} \\ U(r_W) &= J_W \end{aligned}$$

The representation  $U$  implements precosheaf maps, i.e.

$$U(g)\mathcal{A}(W)U(g)^* = \mathcal{A}(gW) , \quad g \in \mathcal{P}_+ . \quad (1.2)$$

First we prove some lemmas concerning the the one-dimensional case, and adopt the subscript  $a$  when dealing with objects associated with the right half line (wedge)  $(a, +\infty)$ . First we observe that when  $a \leq b$ , weak modular covariance for the inclusion  $\mathcal{A}(a, +\infty) \supset \mathcal{A}(b, +\infty)$  implies  $\Delta_a^{it} \Delta_b^{is} \Delta_a^{-it} = \Delta_{\Lambda_a(2\pi t)b}^{is}$ . On the other hand, essential duality implies

$$\Delta_{(-\infty, a)}^{it} = \Delta_a^{-it} , \quad J_{(-\infty, a)} = J_a . \quad (1.3)$$

If  $a > b$ , we apply weak modular covariance for the inclusion  $\mathcal{A}(-\infty, a) \supset \mathcal{A}(-\infty, b)$ . Therefore we immediately get the following.

**1.2 Lemma.** *For all  $a, b, s, t \in \mathbf{R}$  we have*

$$\Delta_a^{it} \Delta_b^{is} \Delta_a^{-it} = \Delta_{\Lambda_a(2\pi t)b}^{is} .$$

Next step will be the construction of the translation group (cf. [4], Theorem 7.2 for an analogous construction). To this aim, let us consider the map from  $\mathbf{R}$  to the unitaries on  $\mathcal{H}$  given by

$$a \rightarrow T(a) := \Delta_0^{is_0} \Delta_a^{-is_0} ,$$

where  $s_0 := \frac{\log 2}{2\pi}$ .

**1.3 Proposition.** *The map  $a \rightarrow T(a)$  is a strongly continuous one-parameter group that implements translations, namely*

$$T(a)\mathcal{A}(W)T(a)^* = \mathcal{A}(W + a) . \quad (1.4)$$

**Proof.** A direct application of Lemma 1.2 and of the definition of  $T(a)$  gives

$$\begin{aligned} T(a)\Delta_b^{it}T(a)^* &= \Delta_{b+a}^{it} \\ T(a)^*\Delta_b^{it}T(a) &= \Delta_{b-a}^{it} . \end{aligned} \quad (1.5)$$

Now we show that

$$T(a)^n = T(na) , \quad n \in \mathbf{Z} \quad (1.6)$$

First we observe that, by (1.5), for any  $n \in \mathbf{N}$ ,

$$T(a)^n = T(a)^n \Delta_0^{is_0} \Delta_a^{-is_0} T(a)^{-n} = \Delta_{na}^{is_0} \Delta_{(n+1)a}^{-is_0} . \quad (1.7)$$

Then, assuming (1.6) for a given  $n$  and making use of (1.7) we have

$$T(a)^{n+1} = T(na)T(a) = \Delta_0^{is_0} \Delta_{na}^{-is_0} \Delta_{na}^{is_0} \Delta_{(n+1)a}^{-is_0} = T((n+1)a) ,$$

and (1.6) for  $n \in \mathbf{N}$  follows by induction. Since, by (1.5),

$$\begin{aligned} T(-a) &= T(-a)^* \Delta_0^{is_0} \Delta_{-a}^{-is_0} T(-a) \\ &= T(-a)^* \Delta_0^{is_0} T(-a) T(-a)^* \Delta_{-a}^{-is_0} T(-a) = T(a)^* , \end{aligned} \quad (1.8)$$

equation (1.6) holds for negative  $n$  too.

Equation (1.6) immediately imply that

$$\forall a, b \in \mathbf{Q}, \quad T(a)T(b) = T(a+b) . \quad (1.9)$$

Finally by Lemma 1.2 we have, if  $|a| < 1$ ,

$$\Delta_0^{is_0} \left( \Delta_0^{\frac{i}{2\pi} \log(a+1)} \Delta_1^{-\frac{i}{2\pi} \log(a+1)} \right) \Delta_0^{is_0} \left( \Delta_0^{\frac{i}{2\pi} \log(a+1)} \Delta_1^{-\frac{i}{2\pi} \log(a+1)} \right)^* = T(a)$$

which shows that  $T(a)$  is strongly continuous in a neighborhood of the origin therefore, by relation (1.9), it is a strongly continuous one-parameter group.

Now we prove (1.4). If  $b \geq 0$  we have

$$T(b)\mathcal{A}(a, +\infty)T(b)^* = \Delta_{a-b}^{is_0} \Delta_a^{-is_0} \mathcal{A}(a, +\infty) \Delta_a^{is_0} \Delta_{a-b}^{-is_0} = \mathcal{A}(a+b, +\infty)$$

where we have used  $T(b) = T(a-b)\Delta_0^{is_0}\Delta_b^{-is_0}T(a-b)^* = \Delta_{a-b}^{is_0}\Delta_a^{-is_0}$  and applied weak modular covariance for the inclusion  $(a-b, +\infty) \supseteq (a, +\infty)$ . If  $b < 0$  we may write  $T(b)$  as  $\Delta_{a+b}^{is_0}\Delta_{a+2b}^{-is_0}$  and then apply weak modular covariance for the inclusion  $(a+2b, +\infty) \supseteq (a, +\infty)$ . Essential duality implies (1.4) for left wedges too.  $\square$

**Proof of Theorem 1.1.** THE ONE-DIMENSIONAL CASE. First we show that the modular unitary groups associated with the wedge algebras generate a representation of  $\mathcal{P}_+^\uparrow$ . Since we already proved relation (1.5) and  $\mathcal{P}_+^\uparrow$  is the semidirect product of translations and dilations, it is sufficient to show that, for all  $a, b \in \mathbf{R}$  we have

$$\Delta_0^{it}T(a)\Delta_0^{-it} = T(e^{2\pi t}a) \quad (1.10)$$

Set  $c_t(a) = \Delta_0^{it}T(a)\Delta_0^{-it}T(e^{2\pi t}a)^*$ . By definition of  $T$  and by a repeated application of Lemma 1.2 we check that  $c_t(a)$  commutes with  $\Delta_b^{is}$ ,  $\forall a, b, s, t \in \mathbf{R}$ . As a consequence

$$\begin{aligned} c_t(a)c_t(b) &= \Delta_0^{it}T(a)\Delta_0^{-it}c_t(b)T(e^{2\pi t}a)^* \\ &= \Delta_0^{it}T(a+b)\Delta_0^{-it}T(e^{2\pi t}(a+b))^* = c_t(a+b) \end{aligned}$$

and, exploiting the dependence on  $t$ ,

$$\begin{aligned} c_{t+s}(a) &= \Delta_0^{it} (c_s(a)T(e^{2\pi s}a)) \Delta_0^{-it}T(e^{2\pi(t+s)}a)^* \\ &= c_s(a)c_t(e^{2\pi s}a) = c_t((e^{2\pi s} - 1)a)c_t(a)c_s(a) , \end{aligned}$$

where we used the centrality of  $c_t(a)$  in the group generated by the modular unitaries and the multiplicativity of  $c_t(\cdot)$ . Then, interchanging  $t$  with  $s$  in the previous equation, we get  $c_t((e^{2\pi s} - 1)a) = c_s((e^{2\pi t} - 1)a)$  hence, recalling that  $s_0 = \frac{\log 2}{2\pi}$ ,

$$c_t(a) = c_{s_0}((e^{2\pi t} - 1)a) \quad (1.11)$$

Now, making use of equations (1.5) and (1.8) we obtain

$$\begin{aligned} c_{s_0}(a) &= \Delta_0^{is_0} T(a) \Delta_0^{-is_0} \Delta_0^{is_0} \Delta_{-2a}^{-is_0} \\ &= \Delta_0^{is_0} \Delta_{-a}^{-is_0} T(a) = 1 \end{aligned} \quad \forall t \in \mathbf{R}$$

hence, by (1.11),  $c_t(\cdot) \equiv 1$ , namely (1.10) holds.

An analytic continuation argument (see e.g. [6], Proposition 2.7) shows that the generator of the translations is positive, therefore the theorem of Borchers ([3], see also [4]) applies and we get  $J_0 T(a) J_0 = T(-a)$ . This relation immediately imply that, setting  $U(r_0) = J_0$ , we get an (anti)-unitary representation of  $\mathcal{P}_+$ .

We conclude this proof showing that  $U(g)$  implements a precosheaf map for any  $g \in \mathcal{P}_+$ . Since we already proved (1.4), it is sufficient to show (1.2) only for  $g = \Lambda_0^{it}$ ,  $t \in \mathbf{R}$  and for  $g = r_0$ . We prove the relation when  $g = \Lambda_0(t)$  and  $W = (a, +\infty)$  is a right wedge. The proofs for the other cases are analogous.

$$\begin{aligned} U(\Lambda_0(t)) \mathcal{A}(a, +\infty) U(\Lambda_0(t))^* &= \Delta_0^{\frac{it}{2\pi}} T(a) \mathcal{A}(0, +\infty) T(a)^* \Delta_0^{-\frac{it}{2\pi}} \\ &= T(\Lambda_0(t)a) \mathcal{A}(0, +\infty) T(\Lambda_0(t)a)^* = \mathcal{A}(\Lambda_0(t)a, +\infty) \end{aligned}$$

**THE TWO-DIMENSIONAL CASE.** First we construct a positive energy, covariant representation of the group of translations. Set  $\tau^\pm = \tau(a, \pm a)$  where  $\tau(v)$ ,  $v \in \mathbf{R}^2$ , denotes the translation by  $v$  on  $\mathbf{R}^2$  and the first coordinate is the time coordinate, and set  $W_a^+ := \tau^+(a)W_0$ ,  $W_a^- := \tau^-(a)W_0'$ , where  $W_0$  is the right wedge whose edge is the origin. Then we may consider the one-dimensional sub-precosheaves given by  $(\mathcal{A}(W_a^+), \mathcal{A}(W_a^+)')$  and by  $(\mathcal{A}(W_a^-), \mathcal{A}(W_a^-)')$ . For such one-dimensional precosheaves weak modular covariance holds, hence, by Proposition 1.3, we get two one-parameter translation groups  $T^\pm(a)$  such that  $T^\pm(a) \mathcal{A}(W_b^\pm) T^\pm(a)^* = \mathcal{A}(W_{a+b}^\pm)$ . Recalling equations (1.3) and (1.10), we get

$$\Delta_{W_0}^{it} T^\pm(a) \Delta_{W_0}^{-it} = T^\pm(e^{\pm 2\pi t} a) . \quad (1.12)$$

First we show that these two light-like translations implement precosheaf maps, i.e.

$$T^+(a) \mathcal{A}(W) T^+(a)^* = \mathcal{A}(\tau^+(a)W) \quad (1.13)$$

$$T^-(a) \mathcal{A}(W) T^-(a)^* = \mathcal{A}(\tau^-(a)W) \quad (1.14)$$

Observe that if  $W_1 \supseteq W_2 \supseteq W_3$  then, for each  $t \in \mathbf{R}$ ,

$$\Delta_1^{it} \Delta_2^{-it} \mathcal{A}(W_3) \Delta_2^{it} \Delta_1^{-it} = \mathcal{A}(\tau((\Lambda_1(2\pi t) - 1)v) W_3) \quad (1.15)$$

where  $\Delta_i^{is}$ , resp.  $\Lambda_i(s)$  denotes the modular unitary group, resp. the one parameter group of boosts associated with the wedge  $W_i$ ,  $i = 1, 2$  and  $v \in \mathbf{R}^2$  is defined by  $W_2 = W_1 + v$ . Let  $(t, x) \in \mathbf{R}^2$  be the edge of  $W$ . If  $W$  is a right wedge and  $x \geq t$ , we may apply equation (1.15) to the inclusion  $W_{t-2|a|}^+ \supseteq W_t^+ \supseteq W$ ,  $a \in \mathbf{R}$ . Since

$$T^+(a) = \Delta_{W_{t-2|a|}^+}^{is(a)} \Delta_{W_t^+}^{-is(a)}$$

where  $s(a) = (2\pi)^{-1} \log 3/2$  if  $a \geq 0$  and  $s(a) = (2\pi)^{-1} \log 1/2$  if  $a < 0$ , we get (1.13). If  $W$  is a left wedge and  $x \geq t$ , then  $W_{t-2|a|}^+ \supseteq W_t^+ \supseteq W'$ , therefore (1.13) for  $W$  follows dualizing the analogous relation for  $W'$ . Finally if  $x < t$  we may consider the inclusion  $(W_{t+2|a|}^+)' \supseteq (W_t^+)' \supseteq W$  when  $W$  is a left wedge or the inclusion  $(W_{t+2|a|}^+)' \supseteq (W_t^+)' \supseteq W'$  when  $W$  is a right wedge, and (1.13) again follows. Equation (1.14) is proven in the same way.

Relations (1.13) and (1.14) implies that the multiplicative commutator

$$c(s, t) := T^+(s)T^-(t)T^+(-s)T^-(-t) \quad (1.16)$$

commutes with  $\Delta_W^{it}$  for any  $W \in \mathcal{W}$ ,  $t \in \mathbf{R}$ . Now we show that  $c(t, s) \equiv 1$ . On the one hand, recalling equation (1.12), we get

$$c(s, t) = \Delta_{W_0}^{ir} c(s, t) \Delta_{W_0}^{ir} = c(e^{2\pi r} s, e^{-2\pi r} t) . \quad (1.17)$$

On the other hand, multiplying equation (1.16) by  $(T^+(s)T^-(t))^*$  on the left and by  $T^+(s)T^-(t)$  on the right we get  $c(s, t) = c(-s, -t)$ . This, together with (1.17), implies that  $c(s, t)$  depends on only one variable:

$$c(s, t) = c(1, st) =: c(st) . \quad (1.18)$$

Now, a direct computation show that  $c(t)$  is a one parameter group. Then equation (1.16) reads

$$T^+(s)T^-(t) = c(st)T^-(t)T^+(s) ,$$

namely we get a representation of the Heisenberg group. Since the generators of  $T^+(\cdot)$  and  $T^-(\cdot)$  are positive,  $c(\cdot) \equiv 1$ . Indeed decomposing the representation of the Heisenberg group along its center, any irreducible direct summand for which  $c(t) \neq 1$  would give a representation of the canonical commutation relations with positive generators, which is impossible.



The rest of the proof is completely analogous to the one-dimensional case.  $\square$

**1.4 Remark.** First we compare our result with that of Borchers [3] and observe that, for a precosheaf on the wedges of the 1 or 2 dimensional space-time satisfying essential duality, weak modular covariance is equivalent to the existence of positive energy translations, and both assumptions imply the thesis of Theorem 1.1.

Indeed we used Borchers theorem in the proof of Theorem 1.1 to show that the relation  $J_0 T(a) J_0 = T(-a)$  holds in the one-dimensional case. We can give an alternate proof of this relation. Analogously to [17, Proposition 2.6], the operator  $\Delta_0^{1/2} T(a) \Delta_0^{-1/2}$  is densely defined and extended by  $J_0 T(a) J_0$ , therefore it is sufficient to check the relation

$$\Delta_0^{1/2} T(a) \Delta_0^{-1/2} \subset T(-a) \quad (1.19)$$

in any positive energy representation of  $\mathcal{P}_+^\uparrow$ . On the other hand, if  $H$  is the generator of  $T(a)$ , the one-parameter groups  $t \rightarrow \Delta_0^{\frac{it}{2\pi}}$  and  $s \rightarrow e^{is \log H}$  give a representation of the Weyl commutation relations. Therefore since all representations of these commutation relations are multiples of the Schrödinger representation, it is sufficient to check relation (1.19) in one representation, e.g. the free field of mass  $m$ , where it is known to hold. We now define the algebras associated with double cones (open intervals in the one-dimensional case):

$$\mathcal{A}(\mathcal{O}) := \bigcap_{W \supset \mathcal{O}} \mathcal{A}(W) .$$

The kernel of the representation  $U$  is either a group of light-like translations (in a two-dimensional theory) or the whole group of translations. In the first case we get a one-dimensional theory as a degenerate case of a two-dimensional one, in the latter the precosheaf consists of only two algebras,  $\mathcal{A}(W_R)$  and  $\mathcal{A}(W'_R) = \mathcal{A}(W_R)'$ , where  $W_R$  is (any) right wedge, and the algebras of all double cones coincide with the center of  $\mathcal{A}(W_R)$ . In particular local algebras may be trivial. Conversely, if the algebra  $\bigvee_t T^\pm(t) \mathcal{A}(\mathcal{O}) T^\pm(t)^*$  is irreducible, then  $\Omega$  is cyclic for  $\mathcal{A}(\mathcal{O})$  (Reeh-Schlieder theorem) and this implies that the algebras of wedge regions are generated by the algebras of double cones. In this case, either  $U$  is injective, or  $\mathcal{P}_+^\uparrow$  is in the kernel of  $U$  and all the algebras  $\mathcal{A}(W)$  coincide with a Maximal Abelian Sub-Algebra of  $\mathcal{B}(\mathcal{H})$  (cf. [6]).

*The  $n$ -dimensional case,  $n \geq 3$ .* Here we consider a precosheaf on wedges of von Neumann algebras in  $\mathcal{B}(\mathcal{H})$ ,  $\mathcal{H}$  a separable Hilbert space, and assume essential duality. Then set

$$\mathcal{A}(\mathcal{O}) := \bigcap_{W \supset \mathcal{O}} \mathcal{A}(W) \quad \mathcal{O} \in \mathcal{K}$$

where  $\mathcal{K}$  denotes the family of double cones in  $M$ , and require that there exists a common cyclic vector  $\Omega$  for the algebras associated with double cones. The weak

modular covariance assumption now takes this form:

$$\sigma_W^t(\mathcal{A}(\mathcal{O})) = \mathcal{A}(\Lambda_W(2\pi t)\mathcal{O}) \quad (1.20)$$

where  $\sigma_W$  denotes the modular automorphism group of the algebra  $\mathcal{A}(W)$  associated with the state  $\omega := (\Omega, \cdot \Omega)$ .

**1.5 Corollary.** *Let  $\mathcal{A}$  be a precosheaf on the wedges of the  $n$ -dimensional Minkowski space satisfying the mentioned properties.*

*Then there is a positive energy (anti)-unitary representation  $U$  of the Poincaré group  $\mathcal{P}_+$  determined by*

$$\begin{aligned} U(\Lambda_W(t)) &= \Delta_W^{\frac{it}{2\pi}} \\ U(r_W) &= J_W \end{aligned}$$

*The representation  $U$  implements precosheaf maps, i.e.*

$$U(g)\mathcal{A}(W)U(g)^* = \mathcal{A}(gW) , \quad g \in \mathcal{P}_+ \quad (1.21)$$

*and, as a consequence,*

$$U(g)\mathcal{A}(\mathcal{O})U(g)^* = \mathcal{A}(g\mathcal{O}) , \quad g \in \mathcal{P}_+ .$$

**Proof.** First we notice that, for each  $\mathcal{O} \in \mathcal{K}$ ,  $\mathcal{O} \subset W$ ,

$$\bigvee_{t \in \mathbf{R}} \mathcal{A}(\Lambda_W(t)\mathcal{O}) = \mathcal{A}(W)$$

because the first von Neumann algebra is a globally invariant subalgebra of the latter w.r.t. the modular group, and admits  $\Omega$  as a cyclic vector. Therefore, whenever  $W_1 \supset W_2$ , weak modular covariance implies

$$\sigma_{W_1}^t(\mathcal{A}(W_2)) = \mathcal{A}(\Lambda_{W_1}(2\pi t)W_2) .$$

Then the one and two dimensional cases follow by Theorem 1.1. In the following we discuss the higher dimensional case.

Let us fix a wedge  $W_0$  and consider the two-dimensional precosheaf on the translated of  $W_0$ . Applying Theorem 1.1 we get a two-parameter group of translations  $T(x)$ ,  $x \in \mathbf{R}^2$ , such that  $T(x)\mathcal{A}(W_0)T(x)^* = \mathcal{A}(W_0 + x)$  and  $\Delta_{W_0}^{it}T(x)\Delta_{W_0}^{-it} = T(\Theta(2\pi t)x)$ ,

where  $\Theta(t)$  is the matrix  $\begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$ . Then, since the algebra of any double cone

may be translated into  $\mathcal{A}(W_0)$  by a translation  $T(x)$  for some  $x$  in  $\mathbf{R}^2$ , weak modular covariance implies strong modular covariance namely, for all  $\mathcal{O} \in \mathcal{K}$ ,

$$\Delta_{W_0}^{it}\mathcal{A}(\mathcal{O})\Delta_{W_0}^{-it} = \mathcal{A}(\Lambda_{W_0}(2\pi t)\mathcal{O}) .$$

Now the results in [6] gives a representation  $U$  of the universal covering of  $\mathcal{P}_+^\uparrow$  such that any  $U(g)$  implements a precosheaf map. Then, according to [17],  $J_W$  has the correct commutation relations with the modular unitaries and this implies that  $U$  is indeed a representation of  $\mathcal{P}_+^\uparrow$  and extends to an (anti)-unitary representation of  $\mathcal{P}_+$ .  $\square$

**1.6 Remark.** The same argument used in Lemma 2.5 in [6] shows that either  $U$  is injective or its kernel is  $\mathcal{P}_+^\uparrow$ . Indeed, in order to obtain Corollary 1.5 from weak modular covariance we had to assume the vacuum vector to be cyclic for the algebras associated with double cones, therefore the kernel of  $U$  cannot consist of the translations only. This case would be allowed if we assumed strong modular covariance and cyclicity of  $\Omega$  for the wedge algebras only, and can be interpreted as a theory on a different space-time [7].

Now we compare Corollary 1.5 with Theorem 3.2 in [17] and notice that, for a local precosheaf on wedges, strong modular covariance is equivalent to weak modular covariance plus essential duality. Another equivalent formulation is the following:

$$\begin{aligned}\sigma_{\mathcal{A}(W)}^t(\mathcal{A}(\mathcal{O})) &= \mathcal{A}(\Lambda_W(2\pi t)\mathcal{O}) , & \mathcal{O} \subset W \\ \sigma_{\mathcal{A}(W)'}^t(\mathcal{A}(\mathcal{O})) &= \mathcal{A}(\Lambda_W(-2\pi t)\mathcal{O}) , & \mathcal{O} \subset W' .\end{aligned}$$

Indeed this formulation still involve only the action of the modular automorphisms on local subalgebras, and implies essential duality as in [5]. Moreover, it can easily be adapted to conformal theories. Given a local net on the double cones of the  $n$ -dimensional Minkowski space, we require that

$$\begin{aligned}\sigma_{\mathcal{A}(\mathcal{O}_1)}^t(\mathcal{A}(\mathcal{O}_2)) &= \mathcal{A}(\Lambda_{\mathcal{O}_1}(2\pi t)\mathcal{O}_2) , & \mathcal{O}_2 \subset \mathcal{O}_1 \\ \sigma_{\mathcal{A}(\mathcal{O}_1)'}^t(\mathcal{A}(\mathcal{O}_2)) &= \mathcal{A}(\Lambda_{\mathcal{O}_1}(-2\pi t)\mathcal{O}_2) , & \mathcal{O}_2 \subset \mathcal{O}_1' .\end{aligned}$$

where the last equation holds whenever  $t$  is in a suitable neighborhood of the origin, namely  $\Lambda_{\mathcal{O}_1}(-2\pi s)\mathcal{O}_2$  is well defined for any  $s$  between 0 and  $t$ . In this case, the same arguments used in this section together with the analysis in Section 1 of [5] imply that the net extends to a conformally covariant precosheaf on the double cones of the universal covering of the Dirac-Weyl compactification of the Minkowski space.

## 2. PCT invariance and Spin-Statistics relation

Modular covariance properties play an important role in the proofs of PCT theorem and Spin and Statistics relation which are given in [16,17,21,18,7]. In particular the geometrical meaning of the modular conjugation associated with wedge algebras is an essential tool. As proved in Section 1, such a property follows from the weak modular covariance. In conformal theories such properties need not to be assumed, since they follow from conformal covariance [5].

In this section we review the main steps from modular covariance to PCT and Spin-Statistics results.

*Poincaré covariant theories* Let  $\mathcal{A}$  be a precosheaf on the wedges of the  $n$ -dimensional Minkowski space  $M$  of von Neumann algebras in  $\mathcal{B}(\mathcal{H})$ ,  $\mathcal{H}$  a separable Hilbert space. We assume the same properties as in Section 1, namely essential duality, the existence of a common cyclic vector  $\Omega$  for the algebras associated with double cones and weak modular covariance as stated in equation (1.20).

Then we consider the quasi-local C\*-algebra  $\mathcal{A}_0$  given by the inductive C\*-limit of the algebras associated with double cones, and localized representations of  $\mathcal{A}_0$ , namely representations which are unitarily equivalent to the identity representation if restricted to the causal complement of any space-like cone. Therefore, given any space-like cone  $\mathcal{S}$ , we may identify the Hilbert space of  $\pi$  with  $\mathcal{H}$  in such a way that  $\pi|_{\mathcal{A}(\mathcal{S}')} = id|_{\mathcal{A}(\mathcal{S}')}$ . The resulting representation is called localized morphism. Localized morphisms are transportable, namely may be localized in any space-like cone by a unitary conjugation.

If  $\rho$  is localized in  $\mathcal{S} \subset W$ , then  $\rho$  restricted to  $\mathcal{A}_0 \cap \mathcal{A}(W)$  extends to an endomorphism  $\rho_W$  of  $\mathcal{A}(W)$ . Two localized morphisms  $\rho$  and  $\sigma$  are said *locally conjugate* if, once localized in the same space-like cone  $\mathcal{S}$ , for any  $W \supset \mathcal{S}$   $\rho_W$  and  $\sigma_W$  are conjugate as endomorphisms of the von Neumann algebra  $\mathcal{A}(W)$  [24]. If  $\rho$  has finite statistics, we shall call *global conjugate* the conjugate in the sense of Doplicher, Haag and Roberts [11,12].

A localized morphism  $\rho$  is (Poincaré) covariant if there exists a positive energy representation  $U_\rho$  of the universal covering of the Poincaré group such that  $\rho \cdot U(g) = adU_\rho(g) \cdot \rho$ . We recall that if regularity holds for  $\mathcal{A}$  (cf. [16], Section 5), any localized morphism  $\rho$  with finite statistics is Poincaré covariant.

We have already shown in Section 1 that the modular conjugation  $J_W$  associated with any wedge  $W$  gives a (partial) PT operator, namely implements the space-time reflection w.r.t. the edge of  $W$ . The rest of the PCT theorem is contained in the following:

**2.1 Theorem.** *Let  $\mathcal{A}$  be a precosheaf on the wedges of the  $n$ -dimensional Minkowski space verifying essential duality and weak modular covariance, and let  $\rho$  be a localized*

covariant morphism. Then, if  $j$  denotes the modular anti-automorphism associated with any given wedge,  $\text{adj} \cdot \rho \cdot j$  is a local conjugate of  $\rho$  and, if  $\rho$  has finite statistics, it is a global conjugate of  $\rho$ .

**Proof.** The local conjugate property is proven in [16] and does not depend on the covariance of  $\rho$ , but only on the geometrical meaning of the modular conjugation. The proof for the global conjugate property is contained in [7], and is a corollary of the analogous statement in [18] where positive energy and covariance play a major role.  $\square$

Now we restrict to the high dimensional case ( $n \geq 3$ ). If  $\rho$  is an irreducible covariant morphism with finite statistics, the statistics parameter  $\lambda_\rho$  [11] and the spin  $s_\rho := U_\rho(2\pi)$  are two scalar quantities. The index statistics theorem [23] states that  $|\lambda_\rho| = \text{Ind}(\rho)^{-1/2}$ , where  $\text{Ind}(\rho)$  is the Jones index of the inclusion  $\rho_W(\mathcal{A}(W)) \subset \mathcal{A}(W)$ ,  $W$  containing the localization region of  $\rho$ . Setting  $\lambda_\rho = |\lambda_\rho|\kappa_\rho$ , the following theorem gives the spin-statistics relation.

**2.2 Theorem.** *Let  $\mathcal{A}$  be a precosheaf on the wedges of the  $n$ -dimensional Minkowski space,  $n \geq 3$ , verifying essential duality and weak modular covariance, and let  $\rho$  be an irreducible covariant morphism with finite statistics. Then  $\kappa_\rho = s_\rho$ .*

**Proof.** If  $n \geq 4$ , the proof is contained in [17] (cf. also [21]). Indeed in this case (or when  $\rho$  is localized in a double cone) one may construct the Doplicher Roberts field algebra [13], and the theorem follows by the equality between the statistics operator and  $U(2\pi)$  on such algebra. When  $n = 3$  and  $\rho$  is localized in a space-like cone, the previous technique does not apply and we refer to the proof in [7], which is a natural extension of the arguments in [18].  $\square$

*Conformal theories on  $S^1$*  Let  $\mathcal{A}$  be a local precosheaf on the intervals of  $S^1$  of von Neumann algebras on a separable Hilbert space  $\mathcal{H}$ , where by interval we mean an open non empty connected subset of  $S^1$  such that the interior  $I'$  of its complement is non empty too. Following [18] we assume conformal covariance and the existence of a unique conformally invariant vector  $\Omega$  cyclic for the algebra generated by the  $\mathcal{A}(I)$ 's. These properties imply that both the modular groups and the modular conjugations have a geometrical meaning.

Then we consider the universal  $C^*$ -algebra  $C^*(\mathcal{A})$  associated with the precosheaf  $\mathcal{A}$  (cf. [14]), identifying the local algebras with the corresponding subalgebras of  $C^*(\mathcal{A})$ . It turns out that the representations of the precosheaf  $\mathcal{A}$  are in one-to-one correspondence with the localized endomorphisms of  $C^*(\mathcal{A})$ , namely the endomorphisms  $\rho$  of  $\mathcal{A}$  such that, for some interval  $I$ ,  $\rho|_{\mathcal{A}(I')} = \text{id}|_{\mathcal{A}(I')}$ .

If  $\rho$  is a localized covariant endomorphism then finite statistics is equivalent to finite index. When  $\rho$  is irreducible, the index-statistics correspondence holds and the univalence  $s_\rho := U_\rho(2\pi)$  is a well defined complex number of modulus one.

**2.3 Theorem.** *Let  $\mathcal{A}$  be a conformal precosheaf on  $S^1$ , and let  $\rho$  be a covariant, finite-statistics, irreducible endomorphism of the universal algebra  $C^*(\mathcal{A})$ . Then  $j \cdot \rho \cdot j$  gives a conjugate endomorphism for  $\rho$ , where  $j$  is the modular anti-automorphism associated with an interval  $I$ , and the spin-statistics relation holds, namely  $\kappa_\rho = s_\rho$ .*

We refer to [18] for the proof of this statement (an independent proof of this theorem based on different ideas, namely the reconstruction of local fields, has been given recently by Jörss [20]).

Further generalizations of the techniques developed in [18] are contained in [7], where spin-statistics relations for conformal theories on higher-dimensional space-times or for theories on a different space-time are proven.

## References

- [1] Bisognano J., Wichmann E., “*On the duality condition for a Hermitian scalar field*”, J. Math. Phys. **16** (1975), 985-1007.
- [2] Bisognano J., Wichmann E., “*On the duality condition for quantum fields*”, J. Math. Phys. **17** (1976), 303-321.
- [3] Borchers H.J., “*The CPT theorem in two-dimensional theories of local observables*”, Commun. Math. Phys. **143** (1992), 315.
- [4] Borchers H.J., “*On the use of modular groups in quantum field theory*” Ann. Inst. H. Poincaré, this Issue.
- [5] Brunetti R., Guido D., Longo R., “*Modular structure and duality in conformal quantum field theory*”, Commun. Math. Phys., **156** (1993), 201-219.
- [6] Brunetti R., Guido D., Longo R., “*Group cohomology, modular theory and space-time symmetries*”, Rev. Math. Phys., **7** (1994), 57-71.
- [7] Brunetti R., Guido D., Longo R., in preparation.
- [8] Buchholz D., Summers S.J., “*An algebraic characterization of vacuum states in Minkowski space*”, Commun. Math. Phys. **155** (1993), 442-458.
- [9] Davidson D., “*Endomorphism Semigroups and Lightlike Translations*”, preprint (hep-th/9412094).
- [10] Davidson D., in preparation
- [11] Doplicher S., Haag R., Roberts J.E., “*Local observables and particle statistics I*”, Commun. Math. Phys. **23** (1971), 199-230.
- [12] Doplicher S., Haag R., Roberts J.E., “*Local observables and particle statistics II*”, Commun. Math. Phys. **35** (1974), 49-85.
- [13] Doplicher S., Roberts J.E., “*Why there is a field algebra with a compact gauge group describing the superselection structure in particle physics*”, Commun. Math. Phys. **131** (1990), 51-107.

- [14] Fredenhagen K., “*Generalization of the theory of superselection sectors*”, in *The algebraic Theory of Superselection Sectors*, D. Kastler ed., World Scientific, Singapore 1990.
- [15] Gabbiani F., Fröhlich J., “*Operator algebras and Conformal Field Theory*”, *Commun. Math. Phys.* **155** (1993), 569-640.
- [16] Guido D., Longo R., “*Relativistic invariance and charge conjugation in quantum field theory*”, *Commun. Math. Phys.* **148** (1992), 521-551.
- [17] Guido D., Longo R., “*An algebraic Spin and Statistics theorem. I*”, to appear in *Commun. Math. Phys.*
- [18] Guido D., Longo R., “*The Conformal Spin and Statistics theorem*”, preprint (hep-th/9505059).
- [19] Hislop P., Longo R., “*Modular structure of the local algebras associated with the free massless scalar field theory*”, *Commun. Math. Phys.* **84** (1982), 84.
- [20] Jörss M., “*The construction of pointlike fields from conformal Haag-Kastler nets*”, preprint (hep-th/9506016)
- [21] Kuckert B., “*A new approach to Spin & Statistics*”, preprint (hep-th/9412130).
- [22] Kuckert B., “*Borchers’ commutation relations and modular symmetries*”, preprint.
- [23] Longo R., “*Index of subfactors and statistics of quantum fields. I*”, *Commun. Math. Phys.* **126** (1989), 217-247.
- [24] Longo R., “*Index of subfactors and statistics of quantum fields. II*”, *Commun. Math. Phys.* **130** (1990), 285-309.
- [25] Oksak A.I., Todorov I.T., “*Invalidity of TCP-theorem for infinite-component fields*”, *Commun. Math. Phys.* **11** (1968), 125-130.
- [26] Wiesbrock H.V., “*Conformal quantum field theory and half-sided modular inclusions of von Neumann algebras*”, *Commun. Math. Phys.* **158**, (1993) 537-544.
- [27] Yngvason J., “*A note on essential duality*”, *Lett. Math. Phys.* **31** (1994) 127-141.