

Creation of spin 1/2 particles by an electric field in de Sitter space

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Abstract

In the present article we solve the Dirac equation in a de Sitter universe when a constant electric field is present. Using the Bogoliubov transformations, we compute the rate of spin 1/2 created particles by the electric field. We compare our results with the scalar case. We also analyze the behavior of the density of particles created in the limit $H=0$, when de Sitter background reduces to a flat space-time.

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During the last two decades, quantum field theory in curved spaces has been extensively discussed, mainly in the study of particle creation in vicinity of black holes as well as in expanding cosmological universes. Here we are talking about a regime where the spacetime curvature remains significantly less than the Planck scale, and therefore one may usefully employ the semiclassical approximation in which the gravitational field is treated classically and we analyze quantized elementary particles propagating on this classical background. This is perhaps the simplest theory that one can use for discussing quantum effects in the presence of gravitational fields in the early universe. In this framework we are interested in discussing spin 1/2 particle creation by a constant electric field in a (1+1) de Sitter space. This problem has been preceded by a series of articles by Lotze [1,2] where the author analyzes the influence of interacting fields on the rate of particles produced by the gravity. In particular, he considers the electromagnetic interaction $ie\bar{\Psi}\gamma^\mu A_\mu\Psi$ between the electron-positron field and the potential A_μ in expanding spatially flat Robertson-Walker Universes, and more recently, the problem of pair production by an electric field in (1+1)-dimensional de Sitter space has been discussed for the case of scalar particles [3]. Here, the idea was to establish a comparison with the results obtained after computing the rate of membranes created by an anti symmetric tensor field [4,5] In the present article, as a natural extension of the results reported in Ref. [3] we solve the Dirac equation in (1+1) de Sitter universe when a constant electric field is present. After identifying the the “in” and “out” vacuum states, we compute, via the Bogoliubov transformations, the rate of particles created by the electric and gravitational fields. We compare our results with those obtained in the scalar case.

The line element associated with a (1+1) de Sitter universe takes the simple form

$$ds^2 = -dt^2 + e^{2Ht} dx^2 \tag{1}$$

which, in terms of the conformal time $\eta = -\frac{1}{H}e^{-Ht}$ reduces to

$$\frac{1}{H^2\eta^2}(-d\eta^2 + dx^2) \tag{2}$$

Since we are interested in analyzing Dirac particle production by a constant electric field in a de Sitter background, in order to establish a comparison with the pure gravitational contribution, it is advisable to have as a previous result the rate of spin 1/2 pair production in de (1+1) Sitter universe. For this purpose we proceed to solve the Dirac equation in the (1+1) de Sitter space (2). The covariant generalization of the Dirac equation reads

$$(\gamma^\alpha(\partial_\alpha - \Gamma_\alpha) + m)\Psi = 0 \quad (3)$$

in the background field where γ^α are the curved Dirac matrices satisfying the commutation relation $\{\gamma^\alpha, \gamma^\beta\} = 2g^{\alpha\beta}$, Γ_α are the spinor connections, which for a diagonal metric as (1) reduces to

$$\Gamma_\lambda = -\frac{1}{4}g_{\mu\alpha}\Gamma_{\nu\lambda}^\alpha s^{\mu\nu}, \quad (4)$$

where

$$s^{\mu\nu} = \frac{1}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu) \quad (5)$$

Choosing to work in the diagonal tetrad gauge, we have that the gamma matrices take the form

$$\gamma^0 = -H\eta\bar{\gamma}^0, \quad \gamma^1 = -H\eta\bar{\gamma}^1 \quad (6)$$

where the flat gamma matrices $\bar{\gamma}^\alpha$ satisfy $\{\bar{\gamma}^\alpha, \bar{\gamma}^\beta\} = 2\eta^{\alpha\beta}$. Then, from (4), we obtain that the only non zero connection Γ_α reads

$$\Gamma_1 = -\frac{1}{2\eta}\bar{\gamma}^0\bar{\gamma}^1 \quad (7)$$

Since we are dealing with a (1+1) dimension space, we have that gamma matrices can be represented in terms of the Pauli matrices. In particular, we can make the following election:

$$\bar{\gamma}^0 = i\sigma^3, \quad \bar{\gamma}^1 = \sigma^1 \quad (8)$$

substituting the gamma matrices (6) as well as the spinor connection (7) into the Dirac equation (3), and introducing the spinor Φ related to Ψ as follows

$$\Phi = \eta^{-1/2}\Psi \quad (9)$$

we obtain that the problem of solving Eq. (3) reduces to find solutions of the coupled system of equations

$$\left(\frac{d}{d\eta} + \frac{im}{H\eta}\right)\Phi_1 + k_x\Phi_2 = 0 \quad (10)$$

$$\left(-\frac{d}{d\eta} + \frac{im}{H\eta}\right)\Phi_2 + k_x\Phi_1 = 0 \quad (11)$$

where k_x is the eigenvalue of the linear momentum operator $-i\partial_x$ consequently, we can write down the spinor Φ as $\Phi = \Phi_0(\eta)e^{ik_x x}$ Substituting (11) into (10) and vice-versa, we arrive at

$$\left(\frac{d^2}{d\eta^2} + \left(\frac{m^2}{H^2} \mp \frac{im}{H}\right)\frac{1}{\eta^2} + k_x^2\right) \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = 0 \quad (12)$$

The solutions of the system of equations (12) can be expressed in terms of cylinder functions $Z_\nu(z)$ as follows

$$\Phi_1 = \eta^{1/2}Z_\nu(k_x\eta), \quad \Phi_2 = -\eta^{1/2}Z_{\nu-1}(k_x\eta) \quad (13)$$

where the parameter ν is given by the expression

$$\nu = \frac{1}{2} + \frac{im}{H} \quad (14)$$

In order to analyze quantum effects in the background field (2) we have to define vacuum states in the asymptotic regions. The notion of positive and negative frequency states is associated with the existence of a timelike Killing vector. In expanding cosmological Universes that vector does not exist, and we have to appeal to another methods for obtaining a reasonable definition of the vacuum state. Among the different approaches we have that the so-called adiabatic method is perhaps the most popular. [6,7] In our particular case, since the asymptotic behavior for scalar fields in the background field (2) was discussed in Ref. [3], we already know the asymptotic form that our in and out vacua should take.

Nevertheless, a natural criterion for choosing the initial and final states is to look at the behavior of the quasiclassical modes described by the Hamilton-Jacobi equation

$$g^{\alpha\beta} S_{,\alpha} S_{,\beta} + m^2 = 0 \quad (15)$$

which in the metric (2) can be written as a sum

$$S = T(\eta) + F(x) \quad (16)$$

and, after introducing the constant of separation k associated with the space variable x , we arrive at the first order differential equation:

$$k^2 + \frac{m^2}{H^2 \eta^2} = (S_{,\eta})^2 \quad (17)$$

Then the quasiclassical behavior of the solution of the Dirac equation (3) in the metric (2) is

$$\Psi \rightarrow \Psi_0(x) \exp(\pm i \int \sqrt{k^2 + \frac{m^2}{H^2 \eta^2}} d\eta) = \Psi_0(x) \frac{\exp(\pm i k \sqrt{\eta^2 + \frac{m^2}{k^2 H^2}} \eta^{\pm im/H})}{\left(\frac{m}{kH} + \sqrt{\eta^2 + \frac{m^2}{k^2 H^2}}\right)^{im/H}} \quad (18)$$

then, as $\eta \rightarrow -\infty$ ($t \rightarrow \infty$) we find

$$\Psi \rightarrow \Psi_0(x) \exp(\mp i k \eta) \quad (19)$$

and as $\eta \rightarrow 0$ ($t \rightarrow \infty$)

$$\Psi \rightarrow \Psi_0(x) \left(\frac{2m}{Hk}\right)^{-im/H} \eta^{\pm im/H} = \Psi_0(x) \left(\frac{2m}{Hk}\right)^{-im/H} e^{\mp imt/2} \quad (20)$$

where the upper and lower signs in (19) and (20) are associated with positive and negative frequencies respectively. Then, looking at the asymptotic behavior of the Hankel functions,

$$H_\nu^{(2)}(x) \approx \sqrt{\frac{2}{\pi x}} e^{-i(x - \nu\pi/2 - \pi/4)} \quad x \rightarrow \infty \quad (21)$$

we have that the “in” state as $\eta \rightarrow -\infty$ can be expressed in terms of $H_\nu^{(2)}(x)$ as

$$\Phi_1 = \eta^{1/2} H_\nu^{(2)}(k_x \eta), \quad (22)$$

substituting (22) into (32) we readily find that $\Phi_2 = -\eta^{1/2}H_{\nu-1}^{(2)}(k_x\eta)$. The expression (22) corresponds to the truly de Sitter invariant vacuum as it was pointed out by Bunch and Davies [9]. From the definition of $H_\nu^{(2)}(z)$ in terms of the Bessel function $J_\nu(z)$

$$H_\nu^{(2)}(z) = \frac{e^{i\pi\nu}J_\nu(z) - J_{-\nu}(z)}{i \sin \pi\nu} \quad (23)$$

and the behavior for small values of z

$$J_\nu(z) \rightarrow \frac{z^\nu}{2^\nu\Gamma(\nu+1)}, \quad (24)$$

and as $\eta \rightarrow 0$ we have that in the vicinity of $\eta = 0$ the spinor component Φ_1 takes the form

$$\Phi_1 \rightarrow \frac{\eta^{1/2}}{i\pi\nu} \left(\frac{(k_x\eta)^\nu}{2^\nu} e^{i\pi\nu}\Gamma(1-\nu) - 2^\nu\Gamma(\nu+1)(k_x\eta)^{-\nu} \right) \quad (25)$$

then, looking at the quasiclassical asymptotic behavior of the “out” states (20) and the linear decomposition between positive and negative frequency modes

$$\Phi_{in,k}^+ = \alpha_k \Phi_{out,k}^+ + \beta_k \Phi_{out,k}^- \quad (26)$$

we arrive at

$$|\alpha_k|^2 = e^{2\pi m/H} |\beta_k|^2 \quad (27)$$

following Mishima and Nakayama [10], we can already establish that the spectrum of particles created is a thermal one with temperature $T = H/2\pi$. From $|\alpha_k|^2 + |\beta_k|^2 = 1$, one readily gets

$$|\beta_k|^2 = \frac{1}{e^{2\pi m/H} + 1} \quad (28)$$

then, we obtain a thermal Fermi-Dirac distribution for $|\beta_k|^2$, which for $m \gg H$ gives as result that the number of particles per unit coordinate volume is $\frac{dN}{dx} \approx e^{-2\pi m/H} \frac{dk}{2\pi}$. Notice that for the Dirac case, we do not need to impose any restriction on the relation between m and H in order to obtain well defined “out” states.

Let us extend our problem to the case when a constant electric field minimally coupled to the spinor field is present. Then the Dirac equation can be written as

$$(\gamma^\alpha(\partial_\alpha - \Gamma_\alpha) + m - ie\gamma^\alpha A_\alpha)\Psi = 0 \quad (29)$$

where the vector potential A_μ associated with a constant electric field E_0 can be written in the comoving time η as follows

$$A_0 = 0, \quad A_1 = -\frac{E_0}{H}e^{Ht} = \frac{E_0}{H^2\eta} \quad (30)$$

substituting (30) and Γ_α given by (7) into (29) we find

$$\left(\bar{\gamma}^0\partial_\eta + \bar{\gamma}^1\partial_x - \frac{m}{H\eta} - \frac{ieE_0}{H^2\eta}\bar{\gamma}^1\right)\Phi = 0 \quad (31)$$

using the gamma's representation (8) we find that eq. (31) yields

$$\left(\frac{d}{d\eta} + \frac{im}{H\eta}\right)\Phi_1 + \left(k_x - \frac{eE_0}{H^2\eta}\right)\Phi_2 = 0 \quad (32)$$

$$\left(-\frac{d}{d\eta} + \frac{im}{H\eta}\right)\Phi_2 + \left(k_x - \frac{eE_0}{H^2\eta}\right)\Phi_1 = 0 \quad (33)$$

the system of equations (32)-(33) can be written in the matrix form

$$\left[\frac{d}{d\eta} + \frac{im}{H\eta}\sigma_3 + i\sigma_2\left(k_x - \frac{eE_0}{H^2\eta}\right)\right]\begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = 0 \quad (34)$$

and with the help of the similarity transformation T , acting on the Pauli matrices σ_i as $T\sigma_iT^{-1} = \sigma'_i$

$$T = \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} = (1 - i\sigma_1) \quad (35)$$

and the auxiliary spinor $\Theta = \eta^{1/2}\Phi$, we reduce the coupled system of equations (32)-(33) to the Whittaker equations

$$\left(\frac{d^2}{d\rho^2} - \frac{1}{4} + \left(\frac{ieE_0}{H^2} \pm \frac{1}{2}\right)\frac{1}{\rho} + \frac{+e^2E_0^2/H^4 + m^2/H^2 + 1/4}{\rho^2}\right)\begin{pmatrix} \Theta_1 \\ \Theta_2 \end{pmatrix} = 0 \quad (36)$$

having as a solution

$$M_{\lambda,\mu}(\rho) = \Theta_{1,2} = \eta^{1/2} \Phi_{1,2} = c_{1,2} \rho^{\mu+1/2} e^{-\rho/2} M(\mu - \lambda + \frac{1}{2}, 2\mu + 1, \rho) \quad (37)$$

where we have introduced the auxiliary variable $\rho = 2ik_x\eta$, and the parameters μ and λ read

$$\mu = \frac{1}{H} \sqrt{-e^2 E_0^2 / H^2 - m^2} = i |\mu|, \quad \lambda = \pm \frac{1}{2} + \frac{ieE_0}{H^2} \quad (38)$$

where the upper and lower signs in (38) and (37) correspond to Φ_1 and Φ_2 respectively. A second set of solutions of the Eq. (36) can be expressed in terms of the Whittaker functions $W_{\lambda,\mu}(z)$ [8]

$$W_{\lambda,\mu}(z) = z^{\mu+1/2} e^{-z/2} U(\frac{1}{2} - \lambda + \mu, 2\mu + 1, z) \quad (39)$$

Using the quasiclassical asymptotae, we can choose positive and negative frequency modes.

In fact, looking at the asymptotic behavior for large values of $|z|$

$$W_{\lambda,\mu}(z) \rightarrow e^{-z/2} z^\lambda \quad (40)$$

and for small z

$$M_{\lambda,\mu}(z) \rightarrow z^{\mu+\frac{1}{2}} \quad (41)$$

we obtain that $\Phi_{in,k}^+ = cW_{\lambda,\mu}(z)$, where c is a constant of normalization, and $\Phi_{out,k}^+ \sim M_{\lambda,\mu}(z)$. Then, using the relation [8]

$$M_{\lambda,\mu}(z) = \Gamma(2\mu + 1) e^{i\pi\lambda} \left[\frac{W_{-\lambda,\mu}(e^{i\pi}z)}{\Gamma(\mu - \lambda + \frac{1}{2})} + \frac{W_{\lambda,\mu}(z)}{\Gamma(\mu + \lambda + \frac{1}{2})} \exp \left[i\pi(\lambda - \mu - \frac{1}{2}) \right] \right] \quad (42)$$

$$-\frac{3\pi}{2} < \arg z < \frac{\pi}{2}; \quad 2\mu \neq -1, -2, \dots \quad (43)$$

from Eq. (42), and the relation [8]

$$|\Gamma(iy)|^2 = \frac{\pi}{y \sinh \pi y} \quad (44)$$

we arrive at

$$\frac{|\alpha_k|^2}{|\beta_k|^2} = \frac{|\sinh \pi(|\mu| - eE_0/H^2)| e^{2\pi(|\mu| - eE_0/H^2)}|}{|\sinh \pi(|\mu| + eE_0/H^2)|} \quad (45)$$

the expression (45) takes simpler form when $|\mu \pm eE_0/H^2| \gg 1$ Taking into account the relation $|\alpha_k|^2 + |\beta_k|^2 = 1$ we finally obtain the thermal distribution

$$|\beta_k|^2 = \frac{1}{1 + e^{2\pi|\mu| - 2\pi eE_0/H^2}} \approx e^{(-\frac{2\pi}{H^2}[(e^2 E_0^2 + m^2 H^2)^{1/2} - eE_0])} \quad (46)$$

which shows that the temperature depends on the intensity of the electric field E_0 . Also we have, as for the scalar case, that $|\beta_k|^2$ changes according the sign of k_x . In fact, from (32)-(33), it becomes clear that changing the sign of k_x is equivalent to replace e by $-e$ in (46).

It is interesting to analyze the form expression (46) takes in the limit $H \rightarrow 0$. In this case we have that the line element (1) reduces to a 2-d Minkowski (flat) metric.

Since the exponent of $|\beta_k|^2$ in (46) can be rewritten as:

$$\frac{-2\pi}{H^2} [(e^2 E_0^2 + m^2 H^2)^{1/2} - eE_0] = \frac{-2\pi eE_0}{H^2} \left[\left(1 + \frac{m^2 H^2}{e^2 E_0^2}\right)^{1/2} - 1 \right] \quad (47)$$

and in the limit when H goes to zero, we have

$$\lim_{H \rightarrow 0} \left[\left(1 + \frac{m^2 H^2}{e^2 E_0^2}\right)^{1/2} - 1 \right] \rightarrow \frac{m^2 H^2}{2e^2 E_0^2} \quad (48)$$

we readily obtain that in the flat limit the density of particles created reduces to

$$|\beta_k|_{\text{flat}}^2 = \lim_{H \rightarrow 0} |\beta_k|^2 = e^{-\pi m^2/(eE_0)} \quad (49)$$

which is the result obtained by Narozhny and Nikishov [11,12] in analyzing the phenomenon of pair creation by a strong constant electric field when the transverse linear momentum p_\perp is equated to zero. It is worth mentioning that expression (49) is proportional to the first term of the series obtained by Schwinger [13], giving the probability, per unit time, and per unit volume, that a pair is created by the constant electric field.

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