# Gauge Invariant Density and Temperature Perturbations in the Quasi–Newtonian Formulation

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## Abstract

We here give an improved formalism for calculating the evolution of density fluctuations and temperature perturbations in flat universes. Our equations are general enough to treat the perturbations in collisionless relics like massive neutrinos. We find this formulation to be simpler to use than gauge dependent and other gauge–invariant formalisms. We show how to calculate temperature fluctuations (including multipole moments) and transfer functions, including the case of collisionless relics like massive neutrinos. We call this formalism "quasi-Newtonian" because the equations for the potential and cold matter fluctuation evolution have the same form as the Newtonian gravitational equations in an expanding space. The density fluctuation variable also has the same form inside and outside of the horizon which allows the initial conditions to be specified in a simple intuitive way. Our sample calculations demonstrate how to use these equations in cosmological models which have hot, cold, and mixed dark matter and adiabatic (isentropic) or isocurvature modes. We also give an approximation may be used to get transfer functions quickly.

*Subject Headings:* cosmology: theory - dark matter - cosmic microwave background

# 1  Introduction

Finding the correct model for the formation of structure in the universe is currently a subject of intense study. It is generally believed that cosmic structures are the result of the gravitational amplification of initially small density fluctuations in the early universe. There are candidate mechanisms for producing the primordial fluctuations, *e.g.*, inflation and topological defects, which must be tested against observations of the current universe. To test these primordial mechanisms, one has to evolve the perturbations up to the present day. This task is complicated by the fact that the amount and composition of dark matter is not known, so potentially a large number of possibilities may exist which need to be checked against observations.

The two main pieces of information required for testing theories are the predicted power spectra of the temperature and matter fluctuations. Various prescriptions for how to calculate these exist in the literature (Kodama & Sasaki, 1984: Sasaki & Gouda, 1986; Bond & Efstathiou, 1987; Mukhanov, Feldman, & Brandenberger, 1987; Durrer & Straumann, 1988; Stompor, 1994). The problem with these treatments is that either they do not span the current range of models or they that use variables which are unsatisfying in that the physical nature of the initial conditions in is not transparent.

Many of the treatments do not include equations for how to deal with the free streaming of massive collisionless relics such as few eV mass neutrinos, an element which is indicated in critical density universe models based on inflation (Schaefer & Shafi, 1994; Pogosyan & Starobinsky, 1994; Liddle & Lyth, 1994). Some gauge-invariant treatments which include a treatment of collisionless relics (Durrer & Straumann, 1988; Stompor, 1994) use a generalized density perturbation variable (there are several reasonable candidates) which has a complicated behavior outside of the horizon and thus makes it inconvenient for specifying initial conditions. Here we present a unified treatment of the evolution of matter and temperature perturbations, which include free streaming massive neutrinos, and use a density perturbation variable which is most natural to use from the point of view of matter perturbations. We also have attempted to put this into a form which can be easily used by someone to get results without the need to slog through the subtleties of the general relativistic underpinnings of the formalism.

For the sake of brevity we have only treated flat universes here. We expect to follow this paper with another which covers universes with curvature.

We begin in section 2 by constructing a gauge-invariant definition of the distribution function, in a slightly different form than that of Durrer & Straumann, (1988). We then recast the generally covariant Boltzmann equation for this distribution function into a more useful form, and this is the basis for the rest of the paper. We also derive fluid equations for the first three moments of the distribution function. In section 3 we discuss how to treat relativistic collisionless particles, *e.g.* massless neutrinos and decoupled photons, using only fluid variable equations. Section 4 explains how to derive initial conditions. In section 5 we discuss how to treat the baryon-photon fluids which are coupled early on, and then show how to calculate the present day temperature anisotropies. Section 6 gives examples of numerical implementation of these equations, and some details about the numerical computations. We end with a summary of our main results. Lastly, there is an appendix which gives a quick way to calculate transfer functions for some models.

# 2 General Gauge-Invariant Evolution Equations for Collisionless Relics

Collisionless massive relic particles cannot be adequately described in a fluid formulation of the gravitation equations, (unless the particles are "cold" on the scales of interest, *i.e.* their thermal energy is much less than the gravitational potential of a density perturbation. One must go back to the Boltzmann equation. Since we are working in a regime in which general relativistic effects are important, the best place to start is with the generally covariant Boltzmann equation. This equation has been derived and discussed in many places (see e.g., Lindquist, 1966; Ehlers, 1969; Stewart, 1971; Kodama & Sasaki, 1984; Durrer & Straumann, 1988), and we need not repeat the details here. [An interesting approach to solving this equation in power series form has also been explored (Rebhan & Schwartz, 1994).] We will only sketch the pertinent parts of the derivation of the perturbed Boltzmann equation. Small perturbations of the distribution function of the particles are not in general gauge invariant. We will remove the gauge dependent part from the distribution function. Next we expand the gauge-invariant distribution function in terms of angular moments, the first three of which are associated with gauge invariant fluid variables: the density, velocity, entropy, and anisotropic pressure perturbations. We will then discuss the correspondence of the initial values of these moments to the initial fluid variable values. We end this section with some remarks on the physical interpretation of the equations.

## 2.1 The Generally Covariant Boltzmann Equation

The non–relativistic distribution function specifies the number of particles at time $t$ with velocity between $\vec{v}$ and $\vec{v} + d\vec{v}$ and position between $\vec{x}$ and $\vec{x} + d\vec{x}$. Since we intend to describe the behavior of relativistic particles in a perturbed spacetime, we need to use a generally covariant formulation, so we must take care that our definition of the distribution function is also generally covariant. For the setup of the Boltzmann equation, we follow the treatment of Kodama & Sasaki, (1984). We will also borrow from Durrer & Straumann, (1988).

In the general relativistic version of kinetic theory, we use the invariant volume elements to define the distribution function. Here we will need the space-like covariant version of the volume element $\sigma_\alpha$:

$$\sigma_\alpha = -\frac{1}{3!} u_\alpha u^\mu \epsilon_{\mu\nu\lambda\sigma} dx^\nu dx^\lambda dx^\sigma \tag{1}$$

where $u^\mu$ is an arbitrary time like vector field and $\epsilon_{\mu\nu\lambda\sigma}$ is the totally antisymmetric tensor with $\epsilon_{0123} = \sqrt{(-\det g_{\mu\nu})}$. $g_{\mu\nu}$ is the metric. Greek indices run from 0 to 3, while latin indices i through m will run from 1 to 3. The other latin indices will be used to distinguish the different component fluids. The invariant momentum space volume element for a particle with mass $m$ is $\pi_q$

$$\pi_q = \frac{2}{4!} \theta(q^0) \delta(g_{\mu\nu} q^\mu q^\nu + m^2) \epsilon_{\mu\nu\lambda\sigma} dq^\mu dq^\nu dq^\lambda dq^\sigma. \tag{2}$$

Here $q^\mu$ is the contravariant momentum, the $\delta$ function enforces the relation among energy mass and momentum (keeps the particle momenta "on the mass shell"), and the Heavyside ($\theta$) function

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restricts us to the regime of positive energies for the particles. The number of particles crossing a unit hypersurface element $\sigma_\alpha$ within a momentum space volume $\pi_q$ is given by

$$dn = f(x^\alpha, q^\beta) q^\mu \sigma_\mu \pi_q. \tag{3}$$

Conservation of $q^\mu \sigma_\mu \pi_q$ along a particle trajectory is guaranteed by the Liouville theorem. Any change in $dn$ due to interparticle collisions can be represented by

$$C(f) q^\mu \sigma_\mu \pi_q. \tag{4}$$

The relativistic Boltzmann equation becomes

$$\mathcal{L}(f) = C(f). \tag{5}$$

where $\mathcal{L}(f)$ is the linear operator which is the total derivative of the distribution function with respect to an affine parameter $\lambda$ which traces geodesics. In non-relativistic theory $\lambda$ would be identified with the time variable. In our relativistic version, $\mathcal{L}$ is

$$\mathcal{L} = \frac{d}{d\lambda} = q^\mu \frac{d}{dx^\mu} + \frac{dq^\mu}{d\lambda} \frac{d}{dq^\mu} \tag{6}$$

where the momentum is defined, as usual, by

$$q^\mu = \frac{dx^\mu}{d\lambda}, \tag{7}$$

and the particles follow geodesics which satisfy

$$\frac{dq^\mu}{d\lambda} = -\Gamma^\mu_{\alpha\beta} q^\alpha q^\beta \tag{8}$$

The $\Gamma^\mu_{\alpha\beta}$ are the metric connection coefficients. Since we are initially interested in collisionless particles we will set $C(f) = 0$. In section 5 we will revisit the collision term for baryon–photon scattering.

Eventually, we will need to relate this microscopic distribution function to macroscopic quantities like the density and pressure. This is done by taking momentum averages. The energy-momentum tensor can be obtained from the formula

$$T^{\mu\nu} = \int q^\mu q^\nu f \pi_q \tag{9}$$

In order to evaluate these integrals and solve the Boltzmann equation, one usually introduces a tetrad (or vierbein) frame which is convenient for the purpose, usually the one which looks like flat space so that dot products are vectors contracted with the Minkowski metric. Kodama & Sasaki (1984) introduce a set of tetrads on which the momentum is physical and redshifts with the universal expansion. Bond & Szalay (1983) and Durrer & Straumann (1988) use a tetrad frame on which the momentum are comoving in flat expanding space, which has the advantage that the

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spatial momentum components are constant in time. Here we will follow the latter convention, and refer interested readers to the original works for more details about the Boltzmann equation.

One can define in a covariant way the one-particle distribution function $f$ as a scalar function of (conformal) time $\tau$, the comoving three-space coordinates $x^i$, and the three-momentum $\tilde{p}^i$ (for particles on the mass shell). We want to consider perturbations on a homogeneous and isotropic background universe, so we split the distribution function into background + perturbation. The spatial dependence of the perturbations can be expanded in terms of harmonic functions $Y(k^i, x^i)$, which for flat space (e.g., critical density universes) is like Fourier transforming the equations, as $Y = e^{ik^j x_j}$. We then split the distribution function into

$$f = f^0 + (\delta f)Y \tag{10}$$

where the perturbation $\delta f$ depends on the Fourier wavevector $k^i$, the time $\tau$ and the three momentum. We also perturb the metric $g_{\mu\nu}$ as

$$g_{\mu\nu} = a^{-2} \left[ \eta_{\mu\nu} + h_{\mu\nu} \right] \tag{11}$$

where $a$ is the time dependent "scale factor", and $\eta_{\mu\nu}$ is the Minkowski metric. The scale factor satisfies the equation

$$\left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho a^2 + \frac{\Lambda a^2}{3}, \tag{12}$$

where $G$ is Newton's gravitation constant, $\rho$ is the total density of mass in the universe, and $\Lambda$ is the cosmological constant. In the above equation and throughout the paper, the dot $(\cdot)$ over a variable denotes a derivative with respect to conformal time $\tau$. The conformal time is more convenient for doing these calculations than the real time $t$. The conformal time is, in fact, the co-moving horizon size. This property is useful for relating the scale of different effects to the scale of structures in the present universe.

The metric perturbation $h_{\mu\nu}$ can also be expanded in terms of harmonic functions:

$$
\begin{aligned}
h_{00} &= -2A(k,\tau)Y \\
h_{0j} &= i\frac{k_j}{k}B(k,\tau)Y \\
h_{ij} &= -2H_L(k,\tau)Y\delta_{ij} - 2H_T(k,\tau)\left[ \frac{\delta_{ij}}{3} - \frac{k_i k_j}{k^2} \right]Y.
\end{aligned}
\tag{13}
$$

The energy momentum tensor $T^{\mu\nu}$ is also perturbed. We can write the perturbations in terms of the usual (gauge dependent) fluid perturbations ($\delta$ for the density perturbation, $v_f$ for the fluid velocity perturbation, and $\pi_L$ or $\pi_T$ for the isotropic or anisotropic pressure perturbations) and $Y$:

$$
\begin{aligned}
T^0_0 &= -\rho \left[ 1 + \delta(k,\tau)Y \right] \\
T^j_0 &= -i\frac{k_j}{k}(\rho + p)v_f(k,\tau)Y \\
T^i_j &= -p \left[ \delta^i_j + \pi_L(k,\tau)Y\delta^i_j + \pi_T(k,\tau)\left( \frac{\delta^i_j}{3} - \frac{k_i k_j}{k^2} \right)Y \right].
\end{aligned}
\tag{14}
$$

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Following Durrer & Straumann (1988), we will write the Boltzmann equation (in the tetrad frame) in terms of the comoving momentum $v = \tilde{p}/T$, where $\tilde{p}$ is the magnitude of the three vector momentum and T is the temperature parameter of the particles. The Boltzmann equation with a more physical definition of the momentum which redshifts with time, such as found in (Kodama & Sasaki, 1984; Schaefer, 1991) while being more physically accurate is more complicated to solve numerically, so we avoid it. The particle energy variable $q$ (following Durrer & Straumann, 1988) associated with the comoving momentum is

$$q = \sqrt{v^2 + \frac{m^2}{T^2}} \tag{15}$$

Of course we are using units for which $c = k_B = 1$. The physical momentum $\tilde{p}$ and the temperature $T$ both decrease with time as $1/a$, so that $v$ is time independent, but the energy variable is not. In this notation the density and pressure are given by

$$\rho = T^4 \int d\Omega dv v^2 q f, \tag{16}$$

and

$$p = T^4 \frac{1}{3} \int d\Omega dv \frac{v^4}{q} f. \tag{17}$$

Since on average the universe is homogeneous and isotropic, the background distribution function must also be homogeneous and isotropic and is fixed when (and if) the particles were in thermal equilibrium in the very early universe:

$$f^0 = \frac{1}{(2\pi\hbar)^3} \frac{1}{1 \pm e^v} \tag{18}$$

where the $+$ is for fermions and the $-$ is for bosons, and $\hbar$ is Planck's constant divided by $2\pi$.

## 2.2   The Gauge–Invariant Perturbed Boltzmann Equation

The Boltzmann equation for the perturbation to $f$ in eq. (10) is then

$$(\delta f)^{\cdot} \quad + \quad ik\mu \frac{v}{q} \delta f =$$
$$\frac{df^0}{dv} \left[ ik\mu q A + v(\dot{H}_L + \frac{1}{3}\dot{H}_T) - \mu^2 v(\dot{H}_T - kB) \right]. \tag{19}$$

The perturbation variables ($\delta f$, $A$, $B$, and $H_L$) are gauge dependent, and so are not the most physically relevant variables to work with. Durrer & Straumann (1988) showed how to construct a gauge invariant version of the distribution function perturbation. The full procedure for doing this is given there. Operationally, all we need to know is that we must add something to $\delta f$ so that the appropriate integrals of $\delta f$ yield the gauge–invariant density and velocity perturbations. The construction in Durrer & Straumann (1988) was not unique, as one could subtract off the gauge dependence in other ways. In fact they chose the combination such that when one converts

the distribution function perturbation to its corresponding fluid variable density perturbation by integrating over momentum, one would get the density perturbation variable called $\Delta_g$ by Kodama & Sasaki, (1984), $\epsilon_g$ by Bardeen, (1980), and $\Delta$ by Mukhanov, Feldman, & Brandenberger, (1991).

$$\Delta_g = \delta + 3\left(1 + w\right)\left(H_L + \frac{1}{3}H_T\right),\tag{20}$$

where $w \equiv p/\rho$. Instead we would like to make another choice for a gauge–invariant distribution function in which the corresponding gauge invariant density perturbation is the variable called $\Delta$ by Kodama & Sasaki (or $\Delta_{ca}$ for a particular component–note that Kodama & Sasaki also define a component perturbation variable called $\Delta_a$, which is *not* the one we are interested in here.)

$$\Delta = \delta + 3(1 + w)\frac{\dot{a}}{a}(v_f - B)\tag{21}$$

This variable is $\epsilon_m$ in Bardeen's notation. It is the density perturbation on the spacelike hyper-surfaces which are the local matter rest frame, and hence is the most natural variable to use from the point of the matter fluctuations. Using this variable, the fluid equations most resemble their Newtonian counterparts for a fluid under the action of a gravitational perturbation in expanding coordinates. Also, this is the density perturbation which Hu & Sugiyama (1995) used in order to untangle the various effects which contribute to the cosmic microwave anisotropy. We will comment on this aspect in more detail later. We now set about rewriting the Liouville equation in a gauge-invariant way, and show explicitly how to use it to calculate the growth of density perturbations in critical density universes.

The gauge invariant generalization $\mathcal{F}$ of the distribution function perturbation $\delta f$ is given here by

$$\mathcal{F} = \delta f - \frac{df}{dv}\left[\frac{\dot{a}}{a}\frac{v}{k}(v_f - B) + i\mu\frac{q}{k}(\dot{H}_T - kB)\right].\tag{22}$$

Plugging this form into the Boltzmann equation, we get (after some algebraic manipulation)

$$\dot{\mathcal{F}} \;+\; ik\mu\frac{v}{q}\mathcal{F} + \frac{df}{dv}\left[\frac{v}{1+w}\frac{\dot{a}}{a}\left(c_s^2\Delta + w\Gamma - \frac{2}{3}w\Pi\right)\right]$$
$$+i\mu\frac{df}{dv}\left[\frac{3q}{2k}\left[\frac{\dot{a}}{a}\right]^2(\Delta + 2w\Pi) + \frac{\dot{a}}{a}\frac{v^2}{q}V\right] = 0\tag{23}$$

where all the terms are now manifestly gauge invariant. In the above $V$ is now the gauge-invariant fluid velocity perturbation, $\Gamma$ is the perturbation in the entropy of the system, and $\Pi$ is the amplitude of the anisotropic pressure perturbation. The "sound velocity" $c_s^2$ is defined as $c_s^2 \equiv \dot{p}/\dot{\rho}$. Note that there is really only one obvious choice for the gauge–invariant velocity perturbation $V = v_f - \dot{H}_T/k$ (Bardeen, 1980; Kodama & Sasaki, 1984; Lyth & Bruni, 1994). The other terms $\Gamma \equiv \pi_L - c_s^2\delta/w$ and $\Pi \equiv \pi_T$ are already gauge–invariant.

## 2.3  Angular Moments of the Distribution Function

The equation for $\mathcal{F}$ can be solved directly, and then one has to do a double integral over $\mu$ and $v$ to get quantities like the density perturbation and velocity perturbation as in (Durrer, 1989).

However, one can instead expand the angular dependence of the distribution function perturbation in orthogonal polynomials, namely Legendre polynomials $P_\ell(\mu)$ (Valdarnini & Bonometto, 1985; Schaefer, 1991) as

$$\mathcal{F} = \frac{1}{4\pi}\sum_{\ell=0}^{\infty}(2\ell+1)i^\ell P_\ell(\mu)\sigma_\ell(k,\tau) \tag{24}$$

and plugging this form into the Boltzmann equation above. Expanding the product $(2\ell+1)\mu P_\ell(\mu) = \ell P_{\ell-1} + (\ell+1)P_{\ell+1}$, resumming the series, and projecting out the coefficients of $P_\ell$, we get the following set of equations for the $\sigma_\ell$

$$
\begin{aligned}
\dot\sigma_\ell \;+\; & k\frac{v}{q}\left[\frac{\ell}{2\ell+1}\sigma_{\ell-1} - \frac{\ell+1}{2\ell+1}\sigma_{\ell+1}\right] = \\
& -\;4\pi\frac{df^0}{dv}\Big[\delta_{\ell 0}\frac{v}{1+w}\frac{\dot a}{a}(c_s^2\Delta + w\Gamma - \frac{2}{3}w\Pi) \\
& +\frac{1}{3}\delta_{\ell 1}\left(\frac{3q}{2k}\left(\frac{\dot a}{a}\right)^2(\Delta+2w\Pi) + \frac{v^2}{q}\frac{\dot a}{a}V\right)\Big]
\end{aligned}
\tag{25}
$$

where $\delta_{ij}$ is the Kronecker delta and $\sigma_{-1} = 0$. The fluid variables, shown here for component $a$, can be obtained now from integrals over only the magnitude of the momentum $v$:

$$\Delta_{ca} = \frac{T_a^4}{\rho_a}\int_0^\infty dv\, v^2 q\sigma_0 \tag{26}$$

$$V_a = -\frac{T_a^4}{\rho_a+p_a}\int_0^\infty dv\, v^3 \sigma_1 \tag{27}$$

$$w_a\Pi_a = \frac{T_a^4}{\rho_a}\int_0^\infty dv\,\frac{v^4}{q}\sigma_2 \tag{28}$$

$$w_a\Gamma_a = \frac{T_a^4}{\rho_a}\int_0^\infty dv\left[\frac{v^4}{3q} - c_a^2 v^2 q\right]\sigma_0 \tag{29}$$

The density perturbation is $\Delta_{ca}$, the velocity perturbation is $V_a$, $\Pi_a$ is the anisotropic pressure perturbation, and $\Gamma_a$ is the internal entropy perturbation, which becomes non-zero when density and pressure perturbations get out of phase. One can derive from the above equations the values for the total perturbations for a multi–component system, finding

$$\Delta \;=\; \rho^{-1}\sum\rho_a\Delta_{ca} \tag{30}$$

$$V \;=\; (\rho+p)^{-1}\sum(\rho_a+p_a)V_a \tag{31}$$

$$\Pi \;=\; p^{-1}\sum p_a\Pi_a \tag{32}$$

$$\Gamma \;=\; p^{-1}\sum\left[p_a\Gamma_a + (c_a^2 - c_s^2)\rho_a\Delta_{ca}\right]. \tag{33}$$

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Using the above relations, the expression in equations (23 & 25) which is common to all the component fluid equations can be simply expressed as

$$c_s^2 \Delta + w\Gamma - \frac{2}{3}w\Pi = \rho^{-1} \sum \rho_a (c_a^2 \Delta_{ca} + w_a \Gamma_a - \frac{2}{3}w_a \Pi_a). \tag{34}$$

One disadvantage of this formalism is that one must now solve a possibly large number of coupled $\sigma_\ell$ equations. However, one can regulate the number of $\sigma_\ell$ calculated to get an accurate answer, which can be estimated ahead of time. Specifically, one would like to ensure that the logarithmic derivative $d\log\sigma_\ell/d\log\tau$ is small. Roughly, this condition is met when $k\tau(T/m)/\ell \ll 1$ for a massive particle (if the particle is massless, just ignore the $T/m$ term) over the entire range of integration. For a particle which becomes non-relativistic prior to the epoch of matter/radiation equality, one can easily show that the constraint is most stringent when evaluated at equality, thus one should satisfy the condition $k\tau(T/m)/\ell|_{eq} \ll 1$ to ensure an accurate integration. Should one be interested in an extremely fast integration in which lower accuracy can be tolerated, one can use the abbreviated fluid version of these equations as described in the Appendix.

One can then integrate the moments [using the procedure as given, for example in Schaefer (1991)] to recover the fluid equations given in other work (Bardeen, 1980; Kodama & Sasaki, 1984). For example, multiplying the equation for $\sigma_0$ by $T_a^4 v^2 q/\rho_a$ and integrating over $v$ we get

$$
\begin{aligned}
\dot{\Delta}_{ca} \quad + \quad & 3\frac{\dot{a}}{a}(c_a^2 - w_a)\Delta_{ca} = -3\frac{\dot{a}}{a}w_a\Gamma_a \\
& -k(1+w_a)V_a + 3\frac{\dot{a}}{a}\frac{1+w_a}{1+w}\left[c_s^2\Delta + w\Gamma - \frac{2}{3}w\Pi\right].
\end{aligned}
\tag{35}
$$

Similarly we obtain from the equation for $\sigma_1$

$$
\begin{aligned}
\dot{V}_a \quad + \quad & \frac{\dot{a}}{a}V_a = 3\frac{\dot{a}}{a}c_a^2(V_a - V) - \frac{3}{2k}\left(\frac{\dot{a}}{a}\right)^2(\Delta + 2w\Pi) \\
& + \frac{k}{(1+w_a)}\left[c_a^2\Delta_{ca} + w_a\Gamma_a - \frac{2}{3}w_a\Pi_a\right].
\end{aligned}
\tag{36}
$$

This equation can also be expressed in the following useful form:

$$
\begin{aligned}
(\dot{V}_a - \dot{V}) \quad + \quad & \frac{\dot{a}}{a}(1 - 3c_a^2)(V_a - V) = +\frac{k}{(1+w_a)}\left[c_a^2\Delta_{ca} + w_a\Gamma_a - \frac{2}{3}w_a\Pi_a\right] \\
& -\frac{k}{(1+w)}\left[c_s^2\Delta + w\Gamma - \frac{2}{3}w\Pi\right].
\end{aligned}
\tag{37}
$$

For completeness, we also give the equations for the total density and velocity perturbations which can be derived using eqs. (30-33, 35, and 36):

$$\dot{\Delta} - 3w\frac{\dot{a}}{a}\Delta = -k(1+w)V - 2\frac{\dot{a}}{a}w\Pi, \tag{38}$$

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and

$$\dot{V} + \frac{\dot{a}}{a}V = -\frac{3}{2k}\left(\frac{\dot{a}}{a}\right)^2 (\Delta + 2w\Pi)$$

$$+ \frac{k}{(1+w)}\left[c_s^2\Delta + w\Gamma - \frac{2}{3}w\Pi\right]. \tag{39}$$

## 2.4 Initial Values For $\sigma_\ell$

One can see from the equation (25) for the $\sigma_\ell$ that only the $\ell = 0$ and $\ell = 1$ modes are driven by gravitational coupling to the matter fluctuations in the universe. If we look at the state of the free streaming particles when the perturbation is well outside the horizon ($k\tau \ll 1$), we can see that the higher moment equations (25) are solved by

$$\sigma_\ell \sim (k\tau)^{\ell-1}\sigma_1, \tag{40}$$

in this regime. However, as we will discuss in section 4 on initial conditions, $\sigma_0 \sim k\tau\sigma_1$, so in fact the $\ell = 2$ mode $\sigma_2 \sim \sigma_0$ and must be included in the definition of initial conditions. We can neglect all $\ell \geq 3$ moments, however.

If we start the evolution at $\tau_i$ when all scales of interest are outside the horizon $k\tau_i \ll 1$, then we only need to specify the first three moments. From equation (23) for $\mathcal{F}$, we see $\mathcal{F}(\tau_i)$ ought to have an initial form like (see also Peebles, 1973)

$$\mathcal{F} = \frac{df^0}{dv}\left[-\frac{v}{3}A + iq\mu B + \frac{v}{3}C\frac{1}{2}(3\mu^2 - 1)\right] \tag{41}$$

where the factors $v/3$ and $q$ have been chosen for the simplest interpretation of $A$, $B$, and $C$.

From the definitions of $\Delta_{ca}$, $V_a$, and $\Pi_a$ we can see that

$$A = \frac{\Delta_{ca}}{1 + w_a}(\tau_i) \tag{42}$$

$$B = V_a(\tau_i) \tag{43}$$

and

$$C = \frac{5}{3}\frac{w_a}{c_a^2}\frac{\Pi_a(\tau_i)}{1 + w_a} \tag{44}$$

Using the definition of $\mathcal{F}$ in eq. (24) we can now make our identification of the starting values for the $\sigma_\ell$:

$$\sigma_0 = -\frac{4\pi}{3}v\frac{df}{dv}\frac{\Delta_{ca}}{1 + w_a} \tag{45}$$

$$\sigma_1 = \frac{4\pi}{3}q\frac{df}{dv}V_a \tag{46}$$

$$\sigma_2 = -\frac{4\pi}{45}v\frac{df}{dv}\frac{w_a}{c_a^2}\frac{\Pi_a}{1 + w_a}, \tag{47}$$

$$\sigma_\ell = 0 \quad \text{for } \ell \geq 3 \tag{48}$$

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where all variables are to be evaluated at the initial time $\tau_i$.

The initial conditions needed for setting up the fluid variables $\Delta_{ca}$, $V_a$, and $\Pi_a$ will be discussed in section 4. We now have all the ingredients necessary for solving the coupled set of equations for an arbitrary composition of different matter components for the universe.

## 2.5 Why "Quasi-Newtonian", and the Physical Interpretation of the Equations

The main reason for the calling our formalism "quasi-Newtonian" is that the equation for the gravitational potential has exactly the Newtonian form. The gauge–invariant gravitational potential $\Phi$ and our density perturbation variable are related via Poisson's equation, which in Fourier space is:

$$\frac{k^2}{a^2}\Phi = 4\pi G\rho\Delta \tag{49}$$

With other choices for a gauge-invariant density perturbation, this equation only holds on sub-horizon scales. Since the initial conditions are usually specified when the perturbation scales are larger than the horizon, it is convenient for the variables to behave in a simple intuitive manner outside the horizon. The equations in these variables then become easily interpretable in terms of physics (of the sub-horizon behavior) in this formulation.

The Newtonian resemblance of this formalism goes deeper than the Poisson relation. The equations themselves have the characteristic that these are very nearly the equations Newton would have written for an expanding gas acting under the influence of its own gravity. Note that in an expanding gas (in $\Omega = 1$ models), one can redefine the Newtonian potential in such a way that equation (49) holds (see, e.g., Peebles, 1980; section II.7). The ubiquitous expression

$$\frac{k}{a(1+w)}\left[c_s^2\Delta + w\Gamma - \frac{2}{3}w\Pi\right] \tag{50}$$

can be interpreted in a simple Newtonian way. The first two terms are simply the acceleration (in the rest frame of the matter) due to the pressure gradient force, and the last term is the acceleration due to the divergence of the anisotropic part of the stress tensor. In the case of a universe filled with cold matter, the equations for $\dot{V}$ and $\dot{\Delta}$ have exactly the same form as the Newtonian equations in expanding coordinates. This is why we refer to our treatment as the "quasi–Newtonian" formulation.

This formulation is closest to working in the gauge known as the "velocity–orthogonal isotropic gauge" (Kodama & Sasaki, 1984) in which there is no residual gauge freedom. We will elaborate more on the physical meaning of these variables and the nature of this gauge in a future publication (Lyth & Schaefer, in preparation). In both the $\Delta_{ca}$ and $V_a - V$ equations the term (50) appears. This term is in turn equal to

$$\frac{1}{(1+w)}\frac{\dot{a}}{a}\left[c_s^2\Delta + w\Gamma - \frac{2}{3}w\Pi\right] = -\left(\Phi - \frac{1}{k}\frac{\dot{a}}{a}V\right)^{\cdot} \tag{51}$$

which is effectively zero in a cold matter dominated universe. The quantity $\Phi - \frac{1}{k}\frac{\dot{a}}{a}V$ is the perturbation of the expansion rate due to the matter perturbations.

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Note that all of our evolution equations depend only on the perturbations in the distribution of particles. We do not need an explicit solution of the behavior of the metric perturbation as in the gauge dependent approach (Bond & Szalay, 1983; Ma & Bertschinger, 1995). We also do not need an explicit solution of the potential equation (for $\Phi$) as in Durrer & Straumann, (1988) or Stompor, (1994). All of the background cosmological information is carried in the equation for the expansion rate. To solve these equations then, the only remaining piece of information we need is the initial conditions. Before getting to these, we would like to derive a set of fluid equations for purely relativistic particles, as this would represent a great simplification over the equations in this section (for this component).

## 3 Equations for Collisionless Relativistic Particles

If the particles are effectively massless over the range of time of interest in the universe, so that $v = q$ is at least a good approximation, we can integrate the equations for the $\sigma_\ell$ to get equations for fluid variables. In this case $\Gamma_a$ will be zero.

We multiply the equations for $\sigma_\ell$ by $v^3$, set $q = v$ and then integrate over $v$ to get the fluid variables. The first three fluid variables $\Delta_{ca}$, $V_a$, and $\Pi_a$ have been defined previously. We repeat them again here for the case of relativistic particles for completeness. Using the subscript $r$ for relativistic fluids:

$$\Delta_{cr} = \frac{T_r^4}{\rho_r} \int_0^\infty dv v^3 \sigma_0 \tag{52}$$

$$V_r = -\frac{3T_r^4}{4\rho_r} \int_0^\infty dv v^3 \sigma_1 \tag{53}$$

$$w_r \Pi_r = \frac{T_r^4}{\rho_r} \int_0^\infty dv v^3 \sigma_2 \tag{54}$$

$$\Gamma_r = 0 \tag{55}$$

and we also have higher moments[1] which we call $\Pi^{(\ell)}$ corresponding to $\sigma_\ell$ for $\ell \geq 3$:

$$w_r \Pi_r^{(\ell)} = \frac{T_r^4}{\rho_r} \int_0^\infty dv v^3 \sigma_\ell; \quad l \geq 3 \tag{56}$$

If one were observing the radiation pattern from the origin of this coordinate system at time $\tau$, the $\Pi^{(\ell)}$ would be the amplitudes of the coefficients of a multipole moment expansion of that pattern.

The lowest moments equations are coupled by gravity to the other matter in the universe. The equations for these are

$$\dot{\Delta}_{cr} + \frac{4}{3}kV_r - \frac{4}{1+w}\frac{\dot{a}}{a}\left[c_s^2\Delta + w\Gamma - \frac{2}{3}w\Pi\right] = 0 \tag{57}$$

---

[1]Note that although the same symbol is used in Schaefer (1991), these $\Pi^{(\ell)}$ are defined to be a factor of 3 larger than in Schaefer (1991).

$$\dot{V}_r - \frac{k}{4}(\Delta_{cr} - \frac{2}{3}\Pi_r) + \frac{3}{2k}\left(\frac{\dot{a}}{a}\right)^2 [\Delta + 2w\Pi] + \frac{\dot{a}}{a}V = 0 \tag{58}$$

$$\dot{\Pi}_r - \frac{k}{5}(8V_r + 3\Pi_r^{(3)}) = 0 \tag{59}$$

$$\dot{\Pi}_r^{(\ell)} + \frac{k}{2\ell+1}\left[\ell\Pi_r^{(\ell-1)} - (\ell+1)\Pi_r^{(\ell+1)}\right] = 0 \tag{60}$$

These moments are used for describing the massless neutrino and decoupled photon fluctuations in the universe.

# 4  Initial Conditions: Solutions for Super–Horizon Perturbations In the Radiation Dominated Epoch

To numerically integrate structure formation models, one requires an accurate set of solutions which may be used as initial conditions. To that end, we consider solutions from both of the "independent" types of perturbations, adiabatic and isocurvature, on super-horizon scales. Adiabatic fluctuations typically arise from inflationary models and isocurvature modes often result from phase transitions in the early universe. In each type we will solve for the behavior of the growing mode of the perturbations well outside the horizon, deep within the radiation dominated era. These solutions will be the most relevant for starting numerical calculations. We will start first with the simpler adiabatic fluctuations.

## 4.1  Adiabatic Perturbations

In an adiabatic perturbation, the density fluctuations $\delta\rho_a$ in in the different components ($a$, $b$, *etc.*) have amplitudes which are related by

$$\frac{\delta\rho_a}{(1+w_a)\rho_a} = \frac{\delta\rho_b}{(1+w_b)\rho_b} = \cdots = \frac{\delta\rho}{(1+w)\rho} \tag{61}$$

in order that the entropy per particle does not change. With our choice of gauge invariant density fluctuation variable, the same condition applies *even when we are outside the horizon*. Thus the adiabatic initial conditions are exactly as expected intuitively:

$$\frac{\Delta_{ca}}{(1+w_a)} = \frac{\Delta_{cb}}{(1+w_b)} = \cdots = \frac{\Delta}{(1+w)}, \tag{62}$$

for all components. We now proceed to identify the growing mode solutions since these are the only ones that will survive long after their formation. It will be assumed that we are beginning our simulation deep enough in the radiation dominated epoch that the temperature is much larger

than the mass of any hot components. Look first at eq. (37). Because we are considering super–horizon scales, we can ignore all terms of order $(k\tau)\Delta$, an assumption which may be validated self–consistently. It is obvious that $V_a = V$ is a consistent solution for all components:

$$V_a = V_b = \cdots = V. \tag{63}$$

Now consider eq. (35), the equation for the growth of $\Delta_{ca}$. For each component, $\Gamma_a$ and $(c_a^2 - w_a)$ are both negligible, and, since the temperature is much larger than the mass of any hot component, the factor $1 + w_a$ is a constant for each component as well. So, eq. (35) shows that the evolution of $\Delta_{ca}/(1 + w_a)$ is the same for each component, preserving the adiabatic initial conditions. We are left with the greatly simplified problem of solving only one set of component equations. One can see from the equations for the evolution of the collisionless particles that we will need to retain the first three moments ($\Delta, V$ and $\Pi$). For the photons, due to their strong interaction with the baryons, and for the cold components, we only retain the first two moments. The total anisotropic pressure, $\Pi$, which appears in the evolution equations for $\Delta$ and $V$ can be found by solving for the anisotropic pressure of the collisionless components, $\Pi_{cr}$. The total $\Pi$ is related to the the collisionless components by the ratio of the collisionless to total pressure, *i.e.* $\Pi = (p_{cr}/p)\Pi_{cr}$ where $p_{cr}$ is the pressure contribution from the collisionless components and $p$ is the total pressure. It is left as a exercise for the reader to verify that the solutions for curvature perturbations have the following dependence on $\tau$:

$$\Delta(k, \tau) = \Delta_H \hat{\alpha}(\vec{k}) k^2 \tau^2 \tag{64}$$

$$V(k, \tau) = -\frac{3}{4} \frac{\Delta(k, \tau)}{k\tau} \left[1 + \frac{2p_{cr}}{5p}\right]^{-1} \tag{65}$$

$$\Pi(k, \tau) = -\frac{3}{5} \Delta(k, \tau) \left[1 + \frac{2p_{cr}}{5p}\right]^{-1}, \tag{66}$$

where $\hat{\alpha}(\vec{k})$ is a stochastic variable which carries information about the probability and spectrum of the initial fluctuations, and $\Delta_H$ is the amplitude of density fluctuation at *Horizon crossing*, *i.e.*, when $k\tau = 1$. $\hat{\alpha}(\vec{k})$ is usually assumed to be a Gaussian random variable [*i.e.* with random phase – see Bardeen, *et al.*, 1986], and was inspired by the formalism of quantum field theoretic description of the quantum fluctuations during inflation (Guth & Pi, 1982; Abbott & Wise, 1984). When averaged over an ensemble of universes (or equivalently over many horizon volumes), $\hat{\alpha}(\vec{k})$ has a mean of zero and a variance defined by

$$\langle \hat{\alpha}^{\star}(\vec{k}) \hat{\alpha}(\vec{k'}) \rangle = (2\pi)^6 k^{n-4} \delta^3(\vec{k} - \vec{k'}) \tag{67}$$

where the brackets denote the ensemble average and the $\star$ signifies the complex conjugate. $n$ is the usual spectral index, and $n = 1$ defines the scale free (Harrison-Zeldovich) spectrum. The factors of $2\pi$, which are not in Abbott & Wise, (1984), appear here because we have associated the $1/(2\pi)^3$ factor with the wavenumber integral of the Fourier transform, instead of with the space integral transform.

Note that $\Delta_H$ above differs from the more common specification of the amplitude at *Hubble crossing* when the wavenumber is equal to the Hubble length ($ka/\dot{a} = 1$). Thus the amplitude

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$\Delta_H$ is a factor of 4 larger than the parameter $\epsilon_H$ of Abbott & Wise (1984). We have chosen to use the Horizon crossing specification so that $\Delta_H$ has the same definition in the radiation and matter dominated epochs. Using the more common Hubble crossing definition leads to the use of different amplitudes for radiation and matter dominated epochs (see *e.g.* Bardeen, Steinhardt, & Turner, 1983). In the strongly radiation dominated phase $\tau$ is related to the scale factor by

$$\tau \simeq \frac{a}{H_0} \sqrt{\frac{\rho_m^0}{\rho_r^0}} = 4.66 \times 10^5 a, \tag{68}$$

where $H_0$ is the present Hubble constant, and $\rho_m^0$ and $\rho_r^0$ are the present day matter and radiation (as if there were 3 types of relativistic neutrinos) energy densities. The above number assumes the COBE central value for the current photon temperature $T = 2.726$ K (Mather, 1994).

The power spectrum $P(k)$ of adiabatic perturbations is the Fourier transform of the two point density correlation function. Using our notation it can be simply computed from the following definition:

$$\langle \Delta^\star(\vec{k}, \tau) \Delta(\vec{k}', \tau) \rangle \equiv (2\pi)^3 P(k, \tau) \delta^3(\vec{k} - \vec{k}'). \tag{69}$$

Our initial power spectrum at $\tau_i$ is then

$$P(k, \tau_i) = (2\pi)^3 \Delta_H^2 k^n \tau_i^4 \tag{70}$$

The evolution equations in the previous sections can then be used to get the present day $\Delta(k, \tau_0)$ and *via* eq (69) the present day power spectrum.

## 4.2   Isocurvature Perturbations

Isocurvature perturbations are distinctly different from the curvature type in that the variation is not in local energy density, but rather in local content. We specify the initial conditions as perturbations which violate eq. (62), in such a way as the total perturbation vanishes ($\Delta = 0$). The relevant variable for parameterizing an isocurvature perturbation is then naturally

$$S_{ab} = \frac{\Delta_{ca}}{1 + w_a} - \frac{\Delta_{cb}}{1 + w_b} \tag{71}$$

which measures the degree of non–adiabaticity. Note that in the adiabatic case there was only one set of initial conditions; here we see that isocurvature intial conditions allow for many different possibilities. Rather than discuss the system in general, it is more instructive to consider a specific type of isocurvature perturbation and see how to set up the isocurvature perturbations. It is typical to consider the type of perturbation where the perturbation in the matter is compensated by a perturbation in the radiation. For simplicity we will consider a case where the matter is baryons and the radiation is only photons. In this case the density perturbations will have the following form:

$$\Delta_{cb} = \frac{S_{br}}{1 + 3\rho_b/(4\rho_r)} \tag{72}$$

$$\Delta_{cr} = -\frac{\rho_b}{\rho_r} \frac{S_{br}}{1 + 3\rho_b/(4\rho_r)}. \tag{73}$$

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However, we are considering the initial conditions deep in the radiation dominated era, when we take $(\rho_b/\rho_r) \ll 1$. When solving the equations to find the growing mode, one must take care to keep only those terms which are of the same order in small quantities. For example, if we keep only the zeroeth order quantities in $k\tau$ and $(\rho_b/\rho_r)$ in a universe with only baryons and radiation, the solution would be

$$
\begin{align}
\Delta_{cb} &\approx S_{br} \tag{74} \\
\Delta_{cr} &\approx 0 \approx \Delta \tag{75} \\
V &\approx 0 \approx V_r \approx V_b. \tag{76}
\end{align}
$$

This is not accurate enough for numerical work, so we will find the leading order dependence to insure that roundoff error does not induce an unwanted adiabatic perturbation. In the specific case we are considering a small adiabatic perturbation is actually induced because the two components do not have the same sound speeds. In fact, it is generally true that adiabatic and isocurvature modes cannot be completely separated when different components have different sound speeds. (See, *e.g.*, Kodama & Sasaki, 1984).

We now solve for the leading order terms in the initial conditions where we simply assume that $\Delta \ll \Delta_{ca}$. To make our universe more realistic, we will also include neutrinos which we assume to be distributed in the same manner as the photons $(S_{br} = S_{b\nu})$.

$$
\begin{align}
\Delta_{cr} &= \Delta_{c\nu} = -\frac{\rho_b}{\rho_r + \rho_\nu}\Delta_{cb} \tag{77} \\
&\approx -\frac{\rho_b}{\rho_r + \rho_\nu}S_{br}.
\end{align}
$$

Because $\Delta$ is small, the term which drives the evolution of the fluctuations is the entropy perturbation, $w\Gamma$, given in the relativistic epoch by

$$
\Gamma \approx -\frac{\rho_b}{\rho_r + \rho_\nu}S_{br}, \tag{78}
$$

The $\Gamma$ terms in equations (38) and (39) then drive $\Delta$ and $V$. As was the case for adiabatic perturbations, only the first three moments are significant. One can easily verify that the solutions for isocurvature initial conditions are

$$
\begin{align}
V(k,\tau) &= \frac{1}{8}k\tau\Gamma(k,\tau)\left[1 + \frac{p_{cr}}{p}\frac{2}{15}\right]^{-1} \tag{79} \\
\Pi(k,\tau) &= \frac{8}{15}\frac{p_{cr}}{p}k\tau V(k,\tau) \tag{80} \\
\Delta(k,\tau) &= -\frac{2}{3}k\tau V(k,\tau) - \frac{1}{3}\Pi(k,\tau) \tag{81}
\end{align}
$$

Furthermore, it can be shown using eq.'s (35 & 36) that

$$
\begin{align}
\Delta_{cr} &= \Delta_{c\nu} = \Gamma \tag{82} \\
V_r &= V_\nu = V_b = V.
\end{align}
$$

As we will discuss in the next section, $V_b$ and $V_r$ are kept to be the same by the strong scattering of photons and baryons (*via* electrons). We can now clearly see from eq.'s (79–81) that our assumption that $\Gamma$ induces total density perturbations $\Delta$ has been verified self consistently.

We have avoided introducing a random variable in the isocurvature case because the fluctuations tend to come from non-gaussian sources. The probability distribution function is then specific to the particular source of the perturbations, and the accompanying formalism is perhaps best left unspecified. In the case of Gaussian intial fluctuations, the initial entropy perturbation $S_{ab}$ can be expressed using the stochastic variables used in the adiabatic case: $S_{ab} = S\hat{\alpha}(\vec{k})$, where $S$ is a constant amplitude of the intial perturbation. The present day power spectrum can be directly computed using eq. (69).

# 5 Equations for Photons: Baryon Coupling and Temperature Anisotropies

Observations of fluctuations in the temperature of the relic photons are an important diagnostic tool for studying theories of structure formation. The techniques for relating the measured cosmic microwave anisotropies to theoretical models often use the predicted multipole moment spectrum. We now want to specify how these moments are calculated using our formalism to facilitate comparisons of temperature anisotropies with estimates of the matter fluctuation amplitudes. Before doing this, however, we need to cover one more topic which is necessary for calculating the evolution of the photon distribution.

Up to now we have only considered collisionless particles. When matter is ionized in the early universe, there is strong scattering between the photons and electrons. We discuss the treatment of this scattering in the next sub–section.

## 5.1 Photons Coupled to Baryons

The equations in section 3 work well for photons when they are sufficiently decoupled from the baryons that they are collisionless. However in the early universe the temperature was high enough to keep normal (baryonic) matter fully ionized. The free electrons have a large cross section for interacting with the photons and the baryons, effectively coupling the baryons and photons tightly. As time progresses, the temperature cools off to the point where the electrons can combine with the protons (and nuclei) to form neutral atoms, which have a small cross section. We need equations to describe the coupling of the photons and baryons. The equations for this have been derived elsewhere (Kodama & Sasaki, 1984 - cf. Appendix E). Here we will just give the answer using the present formalism for the sake of completeness. For a more detailed explanation of some of the physical effects of on the cosmic photon distribution, we refer the reader to Hu & Sugiyama (1995), who use the same density perturbation variables as in this work.

The basic Liouville equation now has a collision term $C(f)$, which depends on the distribution function $f$. The equation becomes

$$\mathcal{L}(f) = C(f). \tag{83}$$

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where $C(f)$ is the scattering functional which must be worked out for the electromagnetic scattering of photons by electrons. To get the fluid equations, we must then integrate $C(f)$ over momentum in the same way that we integrated the Boltzmann operator $\mathcal{L}$. One can see that the equations are just the same as the collisionless equations, but now with an extra term added to describe the effect of scattering on the distribution.

The equations for the photons are as follows. First, the equation for $\Delta_{cr}$ is unchanged:

$$\dot{\Delta}_{cr} + \frac{4}{3}kV_r - \frac{4}{1+w}\frac{\dot{a}}{a}\left[c_s^2\Delta + w\Gamma - \frac{2}{3}w\Pi\right] = 0 \tag{84}$$

The fluid velocity equation (as well as the higher moment equations), get scattering terms.

$$\dot{V}_r - \frac{k}{4}(\Delta_{cr} - \frac{2}{3}\Pi_r) + \frac{3}{2k}\left(\frac{\dot{a}}{a}\right)^2[\Delta + 2w\Pi] + \frac{\dot{a}}{a}V = an_e\sigma_T(V_b - V_r) \tag{85}$$

where $n_e$ is the electron number density and $\sigma_T = 6.6524 \times 10^{-25}$ cm$^2$ is the Thompson scattering cross section. The product $n_e\sigma_T$ is the photon collisional frequency.

The higher moments go as

$$\dot{\Pi}_r - \frac{k}{5}(8V_r + 3\Pi_r^{(3)}) = -\frac{9}{10}an_e\sigma_T\Pi_r \tag{86}$$

and

$$\dot{\Pi}_r^{(\ell)} + \frac{k}{2\ell+1}\left[\ell\Pi_r^{(\ell-1)} - (\ell+1)\Pi_r^{(\ell+1)}\right] = -an_e\sigma_T\Pi_r^{(\ell)} \tag{87}$$

We have included the $\cos^2\theta$ angular dependence in the differential scattering cross section. This is not explicitly evaluated in the formula in Appendix E of Kodama & Sasaki, but is included in their later work (Kodama & Sasaki, 1986; Gouda & Sasaki, 1986).

The equations for the baryons can be obtained by knowing that the total energy momentum tensor is conserved. This means that the baryons behave like cold matter with a scattering term on the right hand side of the velocity equation.

$$[\dot{V}_b - \dot{V}] + \frac{\dot{a}}{a}[V_b - V] = -\frac{k}{1+w}[c_s^2\Delta + w\Gamma - \frac{2}{3}w\Pi] + \frac{4}{3}\frac{\rho_r}{\rho_b}an_e\sigma_T(V_r - V_b) \tag{88}$$

while the density perturbation equation remains unchanged:

$$\dot{\Delta}_{cb} = -kV_b + \frac{3}{1+w}\frac{\dot{a}}{a}[c_s^2\Delta + w\Gamma - \frac{2}{3}w\Pi] \tag{89}$$

To see the effect of the coupling induced by the scattering terms it is instructive to look at the equations for the differences between the photon and baryon fluid variables. This is done most conveniently in terms of the variables $S_{br}$ and $V_{br}$ (Kodama & Sasaki, 1984), where $S_{br}$ is the relative entropy perturbation as in eq. (71)

$$S_{br} = \frac{\Delta_{cb}}{1+w_b} - \frac{\Delta_{cr}}{1+w_r}, \tag{90}$$

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and
$$V_{br} = V_b - V_r. \tag{91}$$

After a little algebra we arrive at the equations
$$\dot{S}_{br} = -kV_{br} \tag{92}$$

and
$$\dot{V}_{br} = -\left[\frac{\dot{a}}{a} + an_e\sigma_T\left(1 + \frac{4}{3}\frac{\rho_r}{\rho_b}\right)\right]V_{br} - \frac{\dot{a}}{a}(V_r - V) - \frac{k}{4}\left(\Delta_{cr} - \frac{2}{3}\Pi_r\right) \tag{93}$$

The key to understanding the behavior of the photon baryon system is contained in the first term on the right hand side of eq. (93). If the collision frequency $(n_e\sigma_T)$ is much larger than the expansion rate $(\dot{a}/a^2)$, then the two fluids are tightly coupled. Note that in this regime small differences between the baryon and photon fluid velocities will be strongly damped out. As long as $V_{br}$ is very small, then the baryon density perturbation and the photon perturbation will be kept proportional [in adiabatic perturbation cases, $\Delta_{cb} = (3/4)\Delta_{cr}$].

Through decoupling, it is numerically easier to integrate the equations for the photons and $S_{br}$ and $V_{br}$. Well before decoupling we can obtain values for $V_{br}$ and $\Pi_r$ by assuming "equilibrium" conditions, that is, by using the value of $V_{br}$ and $\Pi_r$ for which $\dot{V}_{br} = 0$ and $\dot{\Pi}_r = 0$ in equations (93 & 86), respectively. We have found that the tight coupling limit (with these equilibrium solutions) is valid down to a temperature of about 6000 K below which one should integrate the full equations. After decoupling, we switch from integrating equations for $S_{br}$ and $V_{br}$ in favor of those for the baryon perturbations $\Delta_{cb}$ and $V_b$. $\Delta_{cb}$ ($V_b$) can be recovered using a linear combination of $S_{br}$ ($V_{br}$) and $\Delta_{cr}$ ($V_r$).

## 5.2   Evolution of Photon Perturbations in the Matter Dominated Epoch

The equations introduced in the previous section are appropriate for determining the evolution of the photon perturbations through the epoch of recombination. However, to accurately track the photons until the present epoch requires the retention of an inordinate number of terms in the multipole expansion and is therefore numerically undesirable. Fortunately, following recombination, it is possible to calculate an analytic solution by making a few reasonable approximations. Most significantly, the interactions between the photons and baryons can be ignored, and therefore the collisionless Boltzmann equation may be solved. In the next sub–section, we will estimate the errors made by using this simplification.

Since we are primarily interested in calculating the temperature fluctuation in the microwave background, it is convenient notationally to write our equations in the variable $\mathcal{M} = \Delta T/T$. Given $\Delta T/T = (1/4)\Delta\rho/\rho$, $\mathcal{M}$ is related to the photon phase space distribution $\mathcal{F}$ by
$$\mathcal{M} = \frac{1}{4}\frac{T_r^4}{\rho_r}\int_0^\infty 4\pi dv v^3 \mathcal{F}, \tag{94}$$

where the multipole moments $\mathcal{M}_\ell$ are defined similar to the $\sigma_\ell$ of $\mathcal{F}$ in eq. (24), *i.e.*
$$\mathcal{M} = \frac{1}{4\pi}\sum_{\ell=0}^\infty (2\ell + 1)i^\ell P_\ell(\mu)\mathcal{M}_\ell. \tag{95}$$

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The variable $\mathcal{M}$ is more convenient to work with here, and it is related to the variables of section 3 as

$$(4\pi)^{-1}\mathcal{M}_0 = \frac{1}{4}\Delta_{cr} \tag{96}$$

$$(4\pi)^{-1}\mathcal{M}_1 = -\frac{1}{3}V_r \tag{97}$$

$$(4\pi)^{-1}\mathcal{M}_2 = \frac{1}{12}\Pi_r \tag{98}$$

$$(4\pi)^{-1}\mathcal{M}_\ell = \frac{1}{12}\Pi_r^{(\ell)} \tag{99}$$

Cosmic microwave anisotropy observations can be expressed in terms of multipole moments derived from the spatially averaged temperature autocorrelation function (looking along two direction vectors $\hat{x}_1$ and $\hat{x}_2$),

$$\left\langle \frac{\Delta T}{T}(\hat{x}_1)\frac{\Delta T}{T}(\hat{x}_2) \right\rangle = \sum Q_\ell^2 P_\ell(\hat{x}_1 \cdot \hat{x}_2). \tag{100}$$

One also sees this series in terms of the Abbott & Wise (1984) multipole moment coefficients $a_\ell$ which are related to the multipole moments $Q_\ell$ by

$$a_\ell^2 = 4\pi Q_\ell^2. \tag{101}$$

Note that the $a_\ell^2$ used in Gorski, *et al.*, (1994) are a factor of $1/(2\ell+1)$ times $a_\ell^2$ as defined above. The theoretical multipole moments, $Q_\ell^2$, are found by integrating over all the Fourier components of $\mathcal{M}_\ell$, *i.e.*

$$Q_\ell^2 = \frac{1}{(2\pi)^3}\int_0^\infty dk k^2 \frac{2\ell+1}{4\pi}\left|\mathcal{M}_\ell\right|^2. \tag{102}$$

However, while the $\mathcal{M}_\ell$'s are directly related to the observed quantities, it is more convenient for solving the Boltzmann equation to consider an alternate gauge invariant variable $\Theta$ (Kodama & Sasaki, 1986), defined by

$$\Theta = \mathcal{M} - \left[\frac{1}{k}\frac{\dot{a}}{a}V - \Psi\right], \tag{103}$$

where $\Phi$ and $\Psi$ are the gauge-invariant potentials. $\Psi$ is given by

$$\Psi + \Phi = -3\left(\frac{\dot{a}}{ak}\right)^2 w\Pi \tag{104}$$

and $\Phi$ is defined in eq. (49). Using this definition of $\Theta$ along with the relation for the derivative of $\Phi$ expressed in eq. (51), the collisionless Boltzmann equation for the photons may be recast into

$$\dot{\Theta} + ik\mu\Theta = \left[\dot{\Psi} - \dot{\Phi}\right]. \tag{105}$$

This equation can be solved by utilizing the integrating factor $\exp(ik\mu\tau)$ producing the result

$$\Theta = e^{-ik\mu(\tau-\tau_i)}\Theta^i(\tau_i) + \int_{\tau_i}^\tau d\tau' e^{-ik\mu(\tau-\tau')}\left[\dot{\Psi} - \dot{\Phi}\right]. \tag{106}$$

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We designate the first term in this solution as the intrinsic term since it contains the intrinsic fluctuations as well as the Sachs–Wolfe effect. The second term is called the integrated Sachs–Wolfe (ISW) effect, and we will discuss it first. For the case of adiabatic perturbations in the matter dominated epoch, the potentials $\Phi$ and $\Psi$ are effectively constant over the length scales of interest. One then can reasonably approximate the ISW effect by ignoring the oscillatory part of the integral because most of the change in the derivative occurs when $k(\tau - \tau') \ll 1$, *i.e.* the change occurs before the oscillations become significant. When this condition is violated, the SW effect is small relative to the intrinsic fluctuations and will not affect the approximation. The resulting ISW effect is then given by

$$[(\Phi - \Psi) - (\Phi - \Psi)_i] \, e^{-ik\mu(\Delta\tau)}, \tag{107}$$

with $\Delta\tau = \tau - \tau_i$. The contribution to each multipole can be found by utilizing the Legendre expansion of the exponential

$$e^{ik\mu\Delta\tau} = \sum_0^\infty (2\ell + 1)(-i)^\ell j_\ell(k\Delta\tau) P_\ell(\mu), \tag{108}$$

where $j_\ell$ is the spherical Bessel function. When considering isocurvature perturbations, the Sachs–Wolfe effect is small in proportion to the intrinsic fluctuations, so one can ignore the ISW effect altogether.

Calculating the contribution from the intrinsic term is more difficult, as it involves finding the moments of products of more than 2 Legendre polynomials. These have been calculated by Bond & Efstathiou (1987) who showed that

$$e^{-i\mu k\Delta\tau} \Theta_l^i = \sum_{l'} \sum_{m=0}^{\min(l,l')} \Theta_{l'}^i (2l' + 1) C_{l,l',m} (-1)^{l-m} j_{l+l'-2m}(k\Delta\tau), \tag{109}$$

where the constant $C$ is given by

$$C_{l,l',m} = \frac{c_{l'-m} c_m c_{l-m}}{c_{l+l'-m}} \frac{2l + 2l' - 4m + 1}{2l + 2l' - 2m + 1}, \tag{110}$$

with

$$c_n = \frac{(2n - 1)!!}{n!}. \tag{111}$$

The advantage to using this formulation should now be clear. One need only retain the number of terms significant after decoupling, which will be many fewer than are needed for an accurate solution today, and calculate a single summation rather than solve a large set of coupled differential equations.

## 5.3  Adiabatic Sachs-Wolfe Multipole Moments

A useful approximation of the multipole moments $Q_\ell^2$ which can be evaluated analytically is the case of the Sachs-Wolfe effect for adiabatic perturbations. The simplest versions of inflation

predict Gaussian adiabatic fluctuations so this case is of particular interest theoretically (Abbott & Wise, 1984, Abbott & Schaefer, 1986; Fabbri, Lucchin, & Matarrese, 1987). For this reason this approximation has been used extensively in the analysis of COBE data (Wright, *et al.*, 1992; Smoot, *et al.* 1992, Gorski, *et al.*, 1994; Bennett *et al.*, 1994).

Our goal is to calculate the anisotropy in the photon distribution $\mathcal{M}$ at the present time $\tau_0$. From eq. (103),

$$\mathcal{M}|_{\tau_0} = \Theta|_{\tau_0} - \left[\Psi - \frac{1}{k}\frac{\dot{a}}{a}V\right]|_{\tau_0}. \tag{112}$$

During the matter dominated era $a \approx \tau^2/\tau_0^2$ and

$$\dot{\Phi} = 0 = \dot{\Psi}, \tag{113}$$

which is true even for if the matter is made of free streaming particles like neutrinos as long as we stick to scales larger than the Jeans length $k \ll 0.02(m_\nu/1 \text{ eV})^{1/2} h \text{ Mpc}^{-1}$ (see *e.g.* Babu, *et al.*, 1995), where $m_\nu$ is the neutrino mass. In this case the solution of the Boltzmann equation in eq. (106) takes a particularly simple form:

$$\Theta|_{\tau_0} = e^{-ik\mu\Delta\tau}\Theta|_{\tau_d}, \tag{114}$$

where $\Delta\tau = \tau_0 - \tau_d$, and $\tau_d$ is the time when the photons decouple from the baryons.

On the largest scales, we can make the approximation $\mathcal{M}|_{\tau_d} \approx 0$. This is justified as follows. $\mathcal{M}$ is a linear combination of fluid variables $\Delta_{cr}$, $V_r$, $\Pi_r$, and $\Pi_r^{(\ell)}$. If $k \ll \tau_d^{-1}$ then to leading order, $\mathcal{M}|_{\tau_d} \propto (k\tau_d)^1$. On the other hand, the potential $\Psi \propto (k\tau_d)^0$, so $\mathcal{M}|_{\tau_d} \ll \Psi$. This approximation describes the redshifting of photons by the gravitational potentials of matter perturbations, which is often identified as the Sachs-Wolfe effect. Using this approximation,

$$\mathcal{M}|_{\tau_0} = \left[\Psi - \frac{1}{k}\frac{\dot{a}}{a}V\right]|_{\tau_d}e^{-ik\mu\Delta\tau} - \left[\Psi - \frac{1}{k}\frac{\dot{a}}{a}V\right]|_{\tau_0}. \tag{115}$$

During the matter dominated epoch,

$$\Psi = -6\Delta_H\hat{\alpha}(\vec{k}), \text{ and } \frac{1}{k}\frac{\dot{a}}{a}V = -4\Delta_H\hat{\alpha}(\vec{k}). \tag{116}$$

Using eq. (95), then we find for $\ell \geq 1$:

$$\mathcal{M}_\ell|_{\tau_0} = (-1)^{\ell+1}8\pi\Delta_H\hat{\alpha}(\vec{k})j_\ell(k\Delta\tau). \tag{117}$$

For the multipole moment predictions we need the rms values of $\mathcal{M}_\ell$.

$$\langle\mathcal{M}_\ell^\star(\vec{k})\mathcal{M}_\ell(\vec{k}'')\rangle = (2\pi)^3|\mathcal{M}_\ell|^2\delta(\vec{k} - \vec{k}') \tag{118}$$

Using equation (67) and plugging the resulting value of $|\mathcal{M}_\ell|^2$ in eq. (102) we get

$$Q_\ell^2 = 4\pi^{3/2}\Delta_H^2(2\ell+1)(\Delta\tau)^{1-n}\frac{\Gamma[3/2-n/2]}{\Gamma[2-n/2]}\frac{\Gamma[\ell+n/2-1/2]}{\Gamma[\ell-n/2+5/2]} \tag{119}$$

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In the special case of the scale free Harrison-Zeldovich spectrum ($n = 1$), this reduces to

$$Q_\ell^2 = 8\pi\Delta_H^2 \frac{(2\ell + 1)}{\ell(\ell + 1)} \tag{120}$$

The above equation shows why the quantity $\ell(\ell+1)Q_\ell^2/(2\ell+1)$, [called $\ell(\ell+1)C_\ell/(4\pi)$ by some authors (*e.g.* Bond & Efstathiou, 1987; Crittenden, *et al.*, 1994; Hu & Sugiyama, 1995; White, Scott, & Silk, 1994)] is often plotted versus $\ell$. If the Sachs Wolfe effect were the only source of anisotropy, this quantity would correspond to a horizontal line. Deviations from a straight line then illustrate the contributions from $\mathcal{M}|_{\tau_d}$ and the integrated Sachs Wolfe effect. We will plot the results of a numerical calculation in section 6.

## 5.4 The Effects of Relic Electrons and Reionization

We have so far ignored the effects of scattering on our results for the evolution of photon perturbations in the decoupled epoch, but it may have significant consequences. Recombination is not complete and there is always present a small relic ionization. The observations of Gunn & Peterson (1965) indicate reionization has occurred prior to a redshift of $z \approx 5$ (see Giallongo *et al.* 1995 for recent results). We would now like to consider these possibilities in our solutions and estimate their effects.

The Boltzmann equation with interaction terms can be written in the following form:

$$\dot{\Theta} + ik\mu\Theta + R_c\frac{\dot{a}}{a}\Theta = \left[\dot{\Psi} - \dot{\Phi}\right] + R_c\frac{\dot{a}}{a}\frac{\Theta_0}{4\pi} - R_c\frac{\dot{a}}{a}V_b i\mu - R_c\frac{\dot{a}}{a}\frac{\Theta_2}{8\pi}P_2(\mu), \tag{121}$$

where $R_c = a^2 n_e \sigma_T/\dot{a}$ is the ratio of the interaction rate to the Hubble expansion rate and $\Theta_\ell$ is related to $\Theta$ the same way as $\mathcal{M}_\ell$ is related to $\mathcal{M}$, *i.e.*, through eq. (95). This leads to a modified integration factor which includes a real component, $\exp\left(ik\mu\tau + \int_{a_i}^a da\frac{R_c}{a}\right)$, that tends to wipe out fluctuations on scales smaller than the scattering mean free path. Note that throughout this section we implicitly assume that $a_0 = 1$. Outside the horizon, as can be seen from eq.'s (92 & 93), the photon perturbations grow like the dominant matter with scattering only affecting the growth of the $\Pi$ term; scattering effects become important on the sub–horizon scales. Well inside the horizon, the oscillating part of the integrating factor tends to eliminate the effects of the inhomogeneous terms in the Boltzmann equation, those appearing on the right–hand side of eq. (121). As a first approximation, it is reasonable merely to consider the homogeneous solution, that is to multiply the collisionless solution by the damping factor $\exp\left(-\int_{a^*}^a da\frac{R_c}{a}\right)$, where $a^*$ is defined as the greater of $a_i$ and $a_{hc}$, defined as the scale factor for which $Ha_{hc}/k = 1$.

We now can use this approximate solution to estimate the effects of the relic ionization on our collisionless results. In the matter dominated epoch following recombination, we can write an expression for the damping factor $R_c = .069(1 - Y_p)\Omega_b h\chi_e a^{-3/2}$, where $Y_p = .23$ is the Helium mass fraction and $\chi_e$ is the Hydrogen ionization fraction. As a particular example, consider a universe with $h = 1/2$ and $\Omega_b = .05$ which leaves a relic ionization of approximately $\chi_e \approx 6 \times 10^{-4}$. Recombination occurs at a redshift of $z_{rec} \approx 1250$, so if one uses the collisionless solution beginning

at $z = 300$ the damping factor is .997, meaning that at most the error is less than .3%. One makes only a negligible error by neglecting the relic ionization.

If the universe is reionized, we can also estimate the damping effect. Using the same model as the previous calculation, we assume that $\chi_e = 1$ for redshifts less than $z \approx 5$, consistent with Giallongo *et al.*(1995). We find a damping factor of .988 or a 1.2% reduction.

# 6   Numerical Solution of the Equations

We have implemented the equations presented here in two separate codes which agree very well with each other and with other calculations (Holtzman, 1989; Ma & Bertschinger, 1995). The first code is a fixed stepsize predictor corrector Haming type integrator which is extremely accurate when integrating oscillatory equations. The second is an adaptive stepsize Gear's method implicit differentiation solver which offers a good balance between speed and accuracy. In the predictor corrector method, the stepsize is chosen for each specific $k$ value, as the photon-baryon fluid oscillates with period $2\pi/k$ in conformal time. This integrator was reasonably fast. In the second method, we actually recast the equations here as differential equations where the variable was the scale factor $a$. During the matter dominated epoch $a$ is not a linear function of $\tau$, so the stepsize needs to be changed continuously. This approach should of course only be taken using with a method with an adaptive stepsize. If we solve the perturbation equations as a function of $a$ we need only specify the initial $a$, whereas with the fixed stepsize method we also have to calculate the initial value of $\tau$ from eq. (68). Apart from the differences mentioned above the two codes are otherwise identical.

The equations for evolution of the fluids and the collisionless particles are integrated explicitly to follow the evolution of density perturbations in the early universe. For the collisionless particles we integrate the $\sigma_\ell$ at 15 momenta values and then use Gauss-Laguerre integration to get the associated fluid variables. We found the value of 15 momenta gave reasonably accurate results (error in $\Delta_{c\nu}$ was around 1 part in $10^6$). The universe we have modeled contains 1 flavor of massive neutrinos (collisionless particles), two flavors of massless neutrinos, photons, baryons, and, in the adiabatic case, cold dark matter. We treated the total fluid perturbations as independent variables from the separate components which was especially useful when dealing with isocurvature perturbations. The baryons and photons are treated using a tight coupling approximation until near recombination. The ionization fraction evolution is then approximated by interpolation to a table of separate numerical calculations (see Peebles, 1993). After $z = 300$, we no longer track the evolution of the photons and neutrinos as individual components. Keeping them requires the numerical evolution program to follow an exceedingly large number of oscillations of the relativistic fluids after that time when the wavelength is small ($\sim 1$ Mpc). The net effect is to lose the influence of the anisotropic pressure in the equations governing the evolution of the total perturbations. However, since the contribution goes like $w\Pi$, it is not significant in the matter dominated epoch.

We now present some results from the calculations for illustration. First we present the evolution of perturbations in an adiabatic cold plus hot dark matter model (C+HDM) model with 5% baryons and 25 % hot dark matter (massive neutrinos) and a Hubble parameter $h = 0.5$. This

model was noted for its attractive large scale structure properties long before the COBE results were known (Schaefer, Shafi & Stecker, 1989; see also Holtzman, 1989). In figure 1 we show the evolution of the gauge–invariant fluid variables with time $\tau$ for $k = 0.01$ $h/\text{Mpc}$. In the three panels a), b) and c), we present the (component) perturbations in density, velocity, anisotropic pressure, and neutrino entropy. We also present the total density and velocity perturbations. The growth on this large length scale is not very exciting, as the massive neutrinos ($m_\nu = 5.9$ eV) are effectively cold on this scale. The adiabatic conditions are preserved until the perturbation crosses the horizon at $\tau = 1/k$. The photon–baryon fluid begins to oscillate, but soon experience decoupling at $\tau \sim 220$ Mpc. The photons then, like the massless neutrinos, simply free stream after that time. The entropy in the neutrino distribution grows but never becomes significant, especially when we consider that it only enters the through the combination $w_\nu \Gamma_\nu$, where $w_\nu$ decreases rapidly after the neutrinos become non–relativistic. We stop following the evolution of the relativistic components (photons and massless neutrinos) at $\tau \sim 700$ Mpc, which corresponds to a redshift of $z \sim 300$.

In figure 2 we present similar results for $k = 1$ $h/\text{Mpc}$. Here we start with the same initial amplitude as in figure 1: $\Delta(\tau_i = 0.06 \text{ Mpc}) = 0.01$. The evolution is more interesting. First of all we see in panel a) that the CDM component experiences growth which is damped (relative to figure 1) by the effects of photons and neutrinos during the radiation dominated epoch and by the free streaming of the massive neutrinos in the matter dominated epoch. The time of radiation matter equality is $\tau_{eq} \approx 60$ Mpc. The photon–baryon system undergoes acoustic oscillations maintaining the adiabatic condition $\Delta_{cb} = (3/4)\Delta_{cr}$ until decoupling. Note that the velocity and density perturbations are $90°$ out of phase, as the particles flow in and out of the perturbation, the density contrast must lag the velocity. As the photons and baryons begin to decouple, they undergo viscous (Silk) damping. After decoupling, the baryons fall into the potential wells of the CDM component. The massive neutrinos free– stream, but slowly settle into the CDM potential wells as the neutrino momenta redshift.

We next consider a case with isocurvature perturbations, the isocurvature hot dark matter (IHDM) model. We begin the model with the initial conditions as specified in eqs. (77 - 82). Again we plot the results in figure 3 for $k = 0.01$ $h \text{ Mpc}^{-1}$ and in figure 4 for $k = 1$ $h \text{ Mpc}^{-1}$. Here there is no CDM component and with $h = 0.5$ we have $\Omega_\nu = 1$ with a neutrino of mass $m_\nu = 23$ eV. In panel c) we plot the total entropy $\Gamma$, because of its role in driving the density perturbation growth. In figures 3 and 4 we see that $\Delta_{cb}$ is approximately constant until the induced adiabatic curvature perturbation $\Delta$ becomes of the same order. Outside the horizon the neutrino and photon perturbations scale with the ratio of the matter density over the radiation density, while the total perturbation is down by a factor $k^2\tau^2$ as in eq. 79. The velocities and anisotropic pressure perturbations scale with the amplitude of the induced adiabatic perturbations.

In figure 4, again we see that $\Delta_{cb}$ remains constant until the induced adiabatic perturbation amplitude becomes comparable. Now we also see that despite the different amplitudes of $\Delta_{cb}$ and $\Delta_{cr}$ the velocity perturbations oscillate together in the same way as in an adiabatic perturbation. We also see the viscous damping effect, and as in figure 2 the massive neutrinos gradually fall into the potential wells set up by the baryons.

As an example of using our formalism for calculating temperature anisotropies, we also calcu-

late the multipole moments $Q_\ell^2$ for Gaussian $n = 1$ spectra in an adiabatic CDM (figure 5) and an isocurvature HDM universe (figure 6). The multipole moments in figure 5 show nearly a horizontal line for the low $\ell$, as expected from the Sachs-Wolfe approximation of section 5.2. We have normalized the moments to COBE, using the quadrupole ($Q_2$) with the value $Q_{rms-PS} = 20\ \mu$ K.

# 7   Conclusions

We have presented a simple formalism for following the evolution of density perturbations in the early universe. Our treatment includes the important case of collisionless relic particles like massive neutrinos, which cannot be accurately described as a fluid. The density perturbations variables have properties which make their evolution appear to be Newtonian with an expanding space. This makes it much easier to understand the physics of the evolution, and we find this formalism to be very attractive for this feature alone. We have demonstrated the use of these equations for transfer functions and temperature anisotropies in both adiabatic and isocurvature models.

We have shown examples here which demonstrate how the formalism works. The equations here can be easily modified to incorporate other cases, as we have shown elsewhere (de Laix, Scherrer, & Schaefer, 1995). The initial conditions are particularly easy to specify here (especially in isocurvature models). We have presented this new formalism as a working man's set of equations which do not require a deep understanding of general relativity to comprehend the behavior the perturbations under the action of gravity.

Lastly, if one is interested in approximate transfer functions we have presented a modified set of equations (in the Appendix) which can be integrated extremely quickly on a computer.

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## 9  Figure Captions

1. A plot of temporal evolution (in conformal time $\tau$) of perturbations in a) density b) velocity and c) anisotropic pressure for the various components and neutrino entropy in an adiabatic cold + hot dark matter model (with $\Omega_\nu = 0.25$). The perturbation wavenumber is $k = 0.01\,h$ Mpc, and we are using $h = 0.5$. We stop following the relativistic neutrino and photon perturbations after a redshift of $\sim 300$.

2. Similar to figure 1, using the same model, but for $k = 1\,h$ Mpc. Here we can see the effects of viscous damping of the photon–baryon fluid, and the gradual fall–in of the neutrinos (and the baryons after decoupling) into the CDM potential wells.

3. Same as figure 1, but here we consider an isocurvature HDM model, which has no CDM component. The baryon density perturbation is compensated for by the perturbation in the relativistic (photons and neutrinos) components. The rapid rise of the induced adiabatic perturbation $\Delta$ can be seen.

4. Same as figure 3, but for $k = 1\,h$ Mpc. As in figure 2 we can see the various subhorizon evolutionary processes taking place.

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5. Multipole moments of the temperature anisotropy as a function of $\ell$ in an $n = 1$, $h = 0.5$, CDM universe, normalized to $Q_2 = 20 \ \mu \ \mathrm{K}/T_0$. If the anisotropy were only due to the Sachs-Wolfe effect, the multipole moments would fit a horizontal line up to the $\ell$ corresponding the to horizon size at matter domination.

6. Same as figure 5, but for the isocurvature HDM universe. Here initial perturbations in the photon distribution dominate the anisotropy. The moments are normalized to $Q_2 = 20 \ \mu$ $\mathrm{K}/T_0$.

7. Comparison of transfer functions calculated using the full treatment solid lines and the approximation described in the appendix dotted lines for $\Omega_\nu = 1 - \Omega_{CDM} - \Omega_b = 0$, 0.1, 0.2, 0.3, 0.4 and 0.5, using $h = 0.5$ and adiabatic initial conditions.

# A   A Quick Method for Getting Transfer Functions

Setting up and running the codes to integrate the equations presented here can be time consuming in terms of man-hours and CPU time. If one only needs an accuracy of about 10% in the mass fluctuation, the procedure described above can be replaced with a very simple set of equations. These equations have been discussed previously (Schaefer, 1991). Here we will summarize that method and compare it to the exact calculations of the current work.

The method, based on the "Grad" approximation of kinetic theory relies on 2 simplifications. First, for the collisionless particle components of the universe, we ignore all the higher moments where ($\ell \geq 3$). This means we only consider $\Delta_{cr}$, $V_r$, and $\Pi_r$, and their equations, and set $\Pi_r^{(\ell)} = 0$, for $\ell \geq 3$. This simplification, when used for the massless relics, is well known and has been exploited before (*e.g.,* Kodama & Sasaki, 1986). However, in order to use it effectively for the massive neutrinos, one needs a second approximation.

The reason for this is that the equation for the massive neutrino $\Pi_\nu$ is problematic. As described in Schaefer, (1991), one can derive the fluid equations from the sigma equations. However, the equations involve two different momentum space integrals of each $\sigma_\ell$ (see eq. 25). The corresponding fluid variables for these two momentum space integrals for $\sigma_0$ are the density and pressure perturbations. The pressure perturbation shows up in the $\Gamma$ term in the equations presented here. If the particles are relativistic so that the energy and momentum are equal, these two integrals are identical, and the problem disappears. Schaefer (1991) advocated approximating the two integrals as

$$c_\nu^2 \int dv q v^2 \sigma_\ell \approx \int dv \frac{v^4}{q} \sigma_\ell, \quad (\ell = 0, 2). \tag{122}$$

Using this approximation, we find $\Gamma = 0$ and we can derive an equation for $\Pi$:

$$\left( \frac{w_\nu \Pi_\nu}{c_\nu^2} \right)^{\cdot} = -3 \frac{\dot{a}}{a} (c_\nu^2 - w_\nu) \left( \frac{w_\nu \Pi_\nu}{c_\nu^2} \right) + 3k(1 + w_\nu) \frac{2}{5} V_\nu. \tag{123}$$

The other two equations are

$$\left( \frac{\Delta_{c\nu}}{1 + w_\nu} \right)^{\cdot} = -kV_\nu + \frac{3}{1 + w} \frac{\dot{a}}{a} (c_s^2 \Delta + w\Gamma - \frac{2}{3} w\Pi), \tag{124}$$

$$
\begin{aligned}
(V_\nu - V)^{\cdot} \;=\; & -\frac{k}{1+w}(c_s^2 \Delta + w\Gamma - \frac{2}{3}w\Pi) \\
& +\frac{k}{1+w_\nu}(c_\nu^2 \Delta_{c\nu} - \frac{2}{3}w_\nu \Pi_\nu).
\end{aligned}
\tag{125}
$$

These equations are certainly not adequate for calculating the temperature anisotropy, because we have made a gross error in treating the photons. However, the density transfer functions are reasonably accurate. In Figure (7) we show a comparison of transfer functions using the approximations in this appendix and the full equations. Over the interesting range $k = 0 -$ 1.0 $h/\mathrm{Mpc}$, the error is less than 10%.