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On the Stability of Quark Solitons in QCD⋆

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ABSTRACT

We critically re-examine our earlier derivation of the effective low energy action for QCD in 4 dimensions with chiral fields transforming non-trivially under both color and flavor, using the method of anomaly integration. We find several changes with respect to our previous results, leading to much more compact expressions, and making it easier to compare with results of other approaches to the same problem. With the amended effective action, we find that there are no stable soliton solutions. In the context of the quark soliton program, we interpret this as an indication that the full low-energy effective action must include additional terms, reflecting possible modifications at short distances and/or the non-trivial structure of the gauge fields in the vacuum, such as $\langle F_{\mu\nu}^2 \rangle \neq 0$. Such terms are absent in the formalism based on anomaly integration.

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1. Introduction and Motivation

In the recent years some progress have been made towards establishing the connection between the phenomenologically successful non-relativistic constituent-quark model (NRQM), and QCD's fundamental degrees of freedom. Kaplan[1] proposed a physical picture combining some of the features of the chiral quark model[2] and the skyrmion[3−5]. It was postulated that at distances smaller than the confinement scale but large enough to allow for nonperturbative phenomena the effective dynamics of QCD is described by chiral dynamics of a bosonic field which takes values in $U(N_c \times N_f)$. This effective theory admits classical soliton solutions. Assuming that they are stable and may be quantized semiclassically, one then finds that these solitons are extended objects with spin 1/2, and that they belong to the fundamental representation of color and flavor. Their mass is of order Λ_{QCD} and radius of order $1/\Lambda_{QCD}$.[6] It is very tempting to identify them as the constituent quarks. Thus the constituent quarks in this model are "skyrmions" in color space.

It turns out that in 2 space-time dimensions this picture is exact[7]. Thanks to exact non-abelian bosonization one can rewrite the action of QCD_2 in terms of purely bosonic variables, which are chiral fields $\in U(N_c \times N_f)$. It is then straightforward to demonstrate that the only non-trivial static solutions of the classical equations of motion are those which contain either a soliton and an anti-soliton or N_c solitons. The solitons transform under both flavor and color, yet their bound states are color singlets and have the quantum numbers of baryons and mesons.

In four space-time dimensions there is no exact bosonization, and therefore any attempt at derivation of a similar picture in four dimensions must rely on certain approximations. In our previous work[8] we have derived the approximate low-energy effective chiral lagrangian with target space in $U(N_c \times N_f)$ using the approach based on integration of the anomaly equations[9−12]. Equivalent results were independently obtained in ref. 13, using an a priori different approach.

The purpose of the current work is to critically re-examine the results of ref. 8, with particular emphasis on the question whether the action we have derived can support stable, time independent classical solutions. The existence of such solutions appears to us to be a necessary condition for establishing the physical picture in which constituent quarks are solitons of a low-energy effective action of QCD in four dimensions.

We find several changes with respect to our previous results, leading to much more compact expressions, and making it easier to compare with results of other approaches to the same problem, and in particular with the action proposed by Kaplan[1].

With the amended effective action, we find that there are no stable soliton solutions. In the context of the quark soliton program, we interpret this as an indication that the full low-energy effective action must include additional terms, reflecting possible modifications at short distances and/or the non-trivial structure of the gauge fields in the vacuum, such as $\langle F_{\mu\nu}^2 \rangle \neq 0$. Such terms are absent in the formalism based on anomaly integration.

The layout of the paper is as follows. In Section 2 we rederive the low energy classical action, resulting in much more compact final expressions. In Section 3 we examine the necessary conditions for existence of stable, time independent classical solutions with finite energy, and conclude that in order for such solutions to exist, the effective action must contain additional terms which are not present in our derivation. Section 3 is devoted to discussion and interpretation of the results.

2. Derivation of the effective action

We follow the conventions and notation of ref. 8.

The variation of the determinant under the axial transformation of the external fields, including terms up to zeroth power of the cutoff Λ, is given by

$$
\begin{aligned}
-i\frac{\delta \log Z}{\delta\lambda} = \frac{1}{4\pi^2}\Big\{ & \epsilon^{\mu\nu\lambda\sigma}\left[\frac{1}{4}F_{\mu\nu}F_{\lambda\sigma} + \frac{1}{12}H_{\mu\nu}H_{\lambda\sigma} - \frac{2i}{3}(A_\mu A_\nu F_{\lambda\sigma} + A_\mu F_{\nu\lambda}A_\sigma \right. \\
& \left. + F_{\mu\nu}A_\lambda A_\sigma) - \frac{8}{3}A_\mu A_\nu A_\lambda A_\sigma\right] + 16\Lambda^2 D_\mu A^\mu \\
& + \frac{2i}{3}\left[D^\mu F_{\mu\nu}, A^\nu\right] + \frac{i}{3}\left[F_{\mu\nu}, D^\mu A^\nu\right] \\
& + \frac{1}{3}\left\{D^\mu A_\mu, A_\nu A^\nu\right\} - 2A_\mu D_\nu A^\nu A^\mu - \\
& - \frac{2}{3}\left\{D^\mu A^\nu, \{A_\mu, A_\nu\}\right\} + \frac{1}{3}(D^\mu D_\mu D^\nu A_\nu)\Big\} + \mathcal{O}(\Lambda^{-2})
\end{aligned} \tag{1}
$$

where $F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu + i\left[V_\mu, V_\nu\right] + i\left[A_\mu, A_\nu\right] = \frac{1}{2}(L_{\mu\nu} + R_{\mu\nu})$, $D_\mu A_\nu = \partial_\mu A_\nu + i\left[V_\mu, A_\nu\right]$ and $H_{\mu\nu} = \frac{1}{2}(R_{\mu\nu} - L_{\mu\nu}) = (D_\mu A_\nu - D_\nu A_\mu)$. (cf. eq. (4.1) of ref. 8). V_μ is the external source coupled to the vector current of the fermions, while A_μ is the external source coupled to the axial current of the fermions.

An action which has equation (1) as its variation is given by

$$
\begin{aligned}
-i\log Z_1 = & -\frac{2\Lambda^2}{\pi^2}\int \mathrm{Tr}(A_\mu A^\mu) + S^5_{CS}\left[R\right] - S^5_{CS}\left[L\right] \\
& - \frac{i}{48\pi^2}\int \epsilon^{\mu\nu\lambda\sigma}\,\mathrm{Tr}\left[\frac{i}{2}(R_{\mu\nu} + L_{\mu\nu})\left\{L_\lambda, R_\sigma\right\} + (L_\mu L_\nu - \frac{1}{2}L_\mu R_\nu + R_\mu R_\nu)L_\lambda R_\sigma\right] \\
& - \frac{1}{12\pi^2}\int \mathrm{Tr}\left\{\frac{1}{4}(F_{\mu\nu})^2 - iF^{\mu\nu}\left[A_\mu, A_\nu\right] - \frac{1}{2}(D_\mu A^\mu)^2 - (A_\mu A_\nu)^2\right\}
\end{aligned} \tag{2}
$$

where S^5_{CS} is the five dimensional Chern-Simons action

$$S^5_{CS}\left[R\right] = \frac{1}{24\pi^2}\int \epsilon^{\mu\nu\lambda\sigma\rho}\,\mathrm{Tr}(R_\mu\partial_\nu R_\lambda\partial_\sigma R_\rho + \frac{3}{2}iR_\mu R_\nu R_\lambda\partial_\sigma R_\rho - \frac{3}{5}R_\mu R_\nu R_\lambda R_\sigma R_\rho). \tag{3}$$

To derive equation (2), it is useful to employ

$$\begin{aligned}\delta S^5_{CS}\left[R\right] = &\frac{1}{8\pi^2}\int d^5x\epsilon^{\mu\nu\lambda\sigma\rho}\,\mathrm{Tr}\left[\delta R_\mu\left(\partial_\nu R_\lambda\partial_\sigma R_\rho + i\left\{R_\nu R_\lambda, \partial_\sigma R_\rho\right\} - R_\nu R_\lambda R_\sigma R_\rho\right)\right]\\ &+ \frac{1}{24\pi^2}\int d^4x\epsilon^{\mu\nu\lambda\sigma}\,\mathrm{Tr}\left[\delta R_\mu\left(\left\{R_\nu, \partial_\lambda R_\sigma\right\} + \frac{3}{2}iR_\nu R_\lambda R_\sigma\right)\right].\end{aligned} \tag{4}$$

For the special case where $\delta R_\mu = \partial_\mu\delta\omega + i\left[R_\mu, \delta\omega\right]$ we have

$$\delta S^5_{CS}\left[R\right] = \frac{1}{24\pi^2}\int d^4x\epsilon^{\mu\nu\lambda\sigma}\,\mathrm{Tr}\left[\delta\omega\left(\partial_\mu R_\nu\partial_\lambda R_\sigma + \frac{1}{2}i\left\{R_\mu R_\nu, \partial_\lambda R_\sigma\right\} - \frac{1}{2}iR_\mu\partial_\nu R_\lambda R_\sigma\right)\right]. \tag{5}$$

The resulting effective action takes the form

$$\begin{aligned}S_{\mathit{eff}} \;=\; &\frac{\Lambda^2}{2\pi^2}\int \mathrm{Tr}\left[D_\mu U D^\mu U^{-1} - (L_\mu - R_\mu)^2\right]\\
&- \frac{1}{240\pi^2}\int \epsilon^{\mu\nu\lambda\sigma\rho}\,\mathrm{Tr}(J_\mu J_\nu J_\lambda J_\sigma J_\rho)\\
&+ \frac{i}{48\pi^2}\int \epsilon^{\mu\nu\lambda\sigma}\,\mathrm{Tr}\Big\{\frac{i}{2}R_{\mu\nu}\left\{R_\lambda, \bar{J}_\sigma\right\} + R_\mu R_\nu R_\lambda \bar{J}_\sigma\\
&+ \frac{1}{2}R_\mu\bar{J}_\nu R_\lambda\bar{J}_\sigma - \bar{J}_\mu\bar{J}_\nu\bar{J}_\lambda R_\sigma\Big\}\\
&- \frac{i}{48\pi^2}\int \epsilon^{\mu\nu\lambda\sigma}\,\mathrm{Tr}\left\{\frac{i}{2}(R_{\mu\nu} + L_{\mu\nu})\left\{L_\lambda, R_\sigma\right\} + (L_\mu L_\nu - \frac{1}{2}L_\mu R_\nu + R_\mu R_\nu)L_\lambda R_\sigma\right\}\\
&+ \frac{i}{48\pi^2}\int \epsilon^{\mu\nu\lambda\sigma}\,\mathrm{Tr}\left\{\frac{i}{2}(U^{-1}R_{\mu\nu}U + L_{\mu\nu})\left\{L_\lambda, R^U_\sigma\right\} + (L_\mu L_\nu - \frac{1}{2}L_\mu R^U_\nu + R^U_\mu R^U_\nu)L_\lambda R^U_\sigma\right\}\\
&+ \frac{1}{192\pi^2}\int \mathrm{Tr}\Big\{[2(D_\mu(U^{-1}D^\mu U))^2 - (D_\mu U^{-1}D_\nu U)^2\\
&- 4i(L_{\mu\nu}D^\mu U^{-1}D^\nu U + R_{\mu\nu}D^\mu U D^\nu U^{-1}) + 2(U^{-1}R_{\mu\nu}UL^{\mu\nu} - L_{\mu\nu}R^{\mu\nu})]\\
&+ 16[iF^{\mu\nu}\left[A_\mu, A_\nu\right] + \frac{1}{2}(D_\mu A^\mu)^2 + (A_\mu A_\nu)^2)]\Big\}\end{aligned} \tag{6}$$

where $D_\mu U = \partial_\mu U + iR_\mu U - iUL_\mu$, $D_\mu U^{-1} = \partial_\mu U^{-1} + iL_\mu U^{-1} - iU^{-1}R_\mu$, $J_\mu = U^{-1}i\partial_\mu U$ and $\bar{J}_\mu = -UJ_\mu U^{-1}$.

In order to make contact with the usual formulation of QCD, we write down the effective action for $L_\mu = R_\mu = G_\mu$ in eq. (6), to obtain

$$\begin{aligned}
S_{\mathit{eff}} &= \frac{\Lambda^2}{2\pi^2}\int \mathrm{Tr}(D_\mu U D^\mu U^{-1}) - \frac{1}{240\pi^2}\int \epsilon^{\mu\nu\lambda\sigma\rho}\,\mathrm{Tr}(J_\mu J_\nu J_\lambda J_\sigma J_\rho) \\
&+ \frac{i}{48\pi^2}\int \epsilon^{\mu\nu\lambda\sigma}\,\mathrm{Tr}\left\{\frac{i}{2}G_{\mu\nu}\left\{G_\lambda, \bar{J}_\sigma\right\} + (G_\mu G_\nu + \frac{1}{2}G_\mu \bar{J}_\nu + \bar{J}_\mu \bar{J}_\nu)G_\lambda \bar{J}_\sigma\right\} \\
+\frac{i}{48\pi^2}\int \epsilon^{\mu\nu\lambda\sigma}\,&\mathrm{Tr}\left[\frac{i}{2}(U^{-1}G_{\mu\nu}U + G_{\mu\nu})\left\{G_\lambda, G^U_\sigma\right\} + (G_\mu G_\nu - \frac{1}{2}G_\mu G^U_\nu + G^U_\mu G^U_\nu)G_\lambda G^U_\sigma\right] \\
&+ \frac{1}{192\pi^2}\int \mathrm{Tr}\left\{2\left[D_\mu(U^{-1}D^\mu U)\right]^2 - (D_\mu U^{-1} D_\nu U)^2\right. \\
&\left. -4iG_{\mu\nu}\left[D^\mu U^{-1}, D^\nu U\right] + 2(U^{-1}G_{\mu\nu}UG^{\mu\nu} - G_{\mu\nu}G^{\mu\nu})\right\},
\end{aligned} \tag{7}$$

where $G^U_\mu = U^{-1}G_\mu U - U^{-1}i\partial_\mu U = U^{-1}(G_\mu + \bar{J}_\mu)U$.

3. Stability analysis of classical solutions

Our initial hope was to find nontrivial stable minima of the action (7). In the process of looking for such solutions, we observed some numerical instabilities, which caused us to suspect that the action (7) might be unbounded from below.

In order to demonstrate that this is indeed the case, it is sufficient to show this for one trial set of functions, $U_{trial}(x)$ and $G_{trial}(x)$. In order to simplify the stability analysis, we will therefore begin with classical action without gauge fields, $G_{trial}(x) = 0$.

Had such an action lead to stable soliton solutions, it would amount to an (approximate) bosonization of free quarks in four dimensions, which would have been an interesting result in its own right. As the following shows, this has not been attained in the present formalism. We shall comment later on what we believe might be the possible reasons for this.

We therefore set the gauge field to zero, resulting in

$$\begin{aligned} S_{\mathit{eff}}\ [0,0,U] = \frac{\Lambda^2}{2\pi^2}\int \mathrm{Tr}(J_\mu J^\mu) - \frac{1}{240\pi^2}\int \epsilon^{\mu\nu\lambda\sigma\rho}\,\mathrm{Tr}(J_\mu J_\nu J_\lambda J_\sigma J_\rho) \\ - \frac{1}{192\pi^2}\int \mathrm{Tr}[2(\partial_\mu J^\mu)^2 + (J_\mu J_\nu)^2] \end{aligned} \tag{8}$$

Motivated by the Skyrme model, we are looking for a radially symmetric hedgehog solution. We choose the classical solution to be a field of the form

$$U_c = e^{if(r)\vec{\tau}\cdot\hat{r}}$$

where $\vec{\tau}$ are pauli matrices, the generators of some $SU(2)$ subgroup of $U(N_c N_f)$ and $f(r)$ is a radial shape function with boundary conditions $f(0) = \pi$, $f(\infty) = 0$. The choice of the embedding will become relevant only if stable solutions exists, and then it should be discussed together with quantization of the collective coordinates. The Wess-Zumino term vanishes and the rest of the terms are given by

$$\begin{aligned} -(J^c_\mu)^2 &= (f')^2 + \frac{2\sin^2 f}{r^2} \\ (J^c_\mu J^c_\nu)^2 &= (f')^4 - \frac{4(f')^2\sin^2 f}{r^2} \\ (\partial_\mu J^{c\mu})^2 &= \left(f'' + \frac{2f'}{r} - \frac{\sin^2 f}{r^2}\right)^2 \end{aligned} \tag{9}$$

Next, we define an effective potential V_{eff} as minus the action divided by $N_c N_f$ and integrated over space only,

$$
\begin{aligned}
V_{\mathit{eff}}\,(f) = \frac{2\Lambda^2}{\pi} \int\limits_0^\infty dr\,[r^2 (f')^2 + 2\sin^2 f] \\
+ \frac{1}{48\pi} \int\limits_0^\infty dr \left[2\left(r f'' + 2f' - \frac{\sin(2f)}{r} \right)^2 + [r^2 (f')^4 - 4(f')^2 \sin^2 f] \right]
\end{aligned}
\tag{10}
$$

Finding stable solutions of the action (8) is now reduced to functional minimization of the effective potential (10).

Consider a family of trial functions

$$
f(r) = 2\tan^{-1}(\frac{a}{r}) \tag{11}
$$

where a is a variational parameter which also determines the soliton size. The functions (11) satisfy the boundary conditions at both $r = 0$ and $r \to \infty$:

$$
\begin{aligned}
f(r = 0) = \pi \\
f(r \to \infty) = 0
\end{aligned}
\tag{12}
$$

The first condition is needed to ensure that the solution carries one unit of winding number, which in our normalization corresponds to one quark.

For this family of trial functions,

$$
V_{\mathit{eff}}\,(a) = 6\Lambda^2 a - \frac{1}{96a} \tag{13}
$$

Hence for $a \to 0$ the potential is unbounded below. Thus, when attempting to solve the equations of motion, we will find that the soliton profile will be “squeezed” to zero width around the origin. This shrinking of the classical solution to zero size occurs despite the presence of both two- and four-derivative terms in the action (8).

A similar effect occurs already in the σ model[14]. Also there, the approximate action with up to four derivatives on the σ and $\vec{\pi}$ fields, constrained to $\sigma^2 + \vec{\pi}^2 = f^2$ has its classical solutions "squeezed" to zero width, and with energy tending to $-\infty$.

In two dimensions, stabilization of the soliton is provided by the mass term. In four dimensions, one may add a mass term

$$2m_Q^4\left[1 - \cos(f)\right] \tag{14}$$

where m_Q is some scale related to the original quark mass in the QCD Lagrangian, (not quite the bare mass itself, as there is normal ordering to be performed; see Appendix of ref. [7]). Such a mass term does not provide the desired stabilization, however. For the trial function above, this term will have a divergent contribution to the potential coming from the integration over large r. Since the stabilization problem occurs at small distances, this large-r divergence due to the mass term cannot cure the problem. In order to isolate the large-r divergence, we will treat the problem of large r by putting a cut-off R. We expect that eventually such a cut-off will actually be provided by the confinement in QCD.

Now the contribution of the mass term to the potential will be

$$16\pi m_Q^4 a^3\left[\frac{R}{a} - \tan^{-1}(\frac{R}{a})\right] \tag{15}$$

which tends to zero as "a" tends to zero, thus not changing the fact that the potential is unbounded below for small scales a.

4. Discussion and Interpretations

There are various options to overcome these difficulties. The first is by choosing a different regularization scheme. This may change the coefficients of the dimension four operators which appear in the Lagrangian. In particular it may change the terms in such a way that we will have a commutator squared as in the Skyrme model. This probably is the only known action which produces a positive Hamiltonian. A second way out of this problem is to refer to non-perturbative corrections which will change the form of the coefficients in such a way as to get a Skyrme like action.

We should remark, however, that in general we would not expect a scheme change to influence physical results, like the emergence of constituent quarks. It may happen, however, that due to the approximations made, we may be able to derive certain quantities in one scheme and not in another.

Recall that in two dimensions, the scheme was completely fixed by requiring vector conservation and that the axial be the dual of the vector. The latter requirement was a result of our wish to have the bosonic version correspond to the fermionic one, and in the latter the axial is indeed the dual of the vector (see our work, ref. [8] for details). We do not have an analogous requirement in four dimensions as yet.

Let us also remark that our classical configurations tend to be "squeezed" to zero size, and with energy tending to $-\infty$. The troublesome part is at short distances. But this is precisely the regime of high momenta, where our approximations are inadequate, as we have neglected terms with six derivatives or more. So we either have to find a better approximation, or maybe exclude some short distance region.

A final comment. We expect the effective action (7), after integrating out the gauge fields and taking trace over color, to yield an effective action in flavor space. But due to the non-positive nature of the potential that we discovered above, we do not expect, within the present approximation, to get the Skyrme model with the assumed standard positive-definite stabilizing four-derivative term[3,4].

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