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Luminosities for Vector-Boson Vector-Boson Scattering at High Energy Colliders

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Abstract

We derive exact expressions for luminosities of massive vector-boson pairs which can be used to describe the cross sections for processes in hadron collisions or e^+e^- annihilation which proceed via two-vector-boson scattering. Our approach correctly takes into account the mutual influence of the emission of one vector boson on the emission of a second one. We show that only approximately the exact luminosities can be factorized into convolutions of single-vector-boson distributions. Numerical results are given and compared to simplified approaches.

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1 Introduction

Hadron colliders will produce the electroweak vector bosons $W^{\pm}$ and Z with a high rate at large energies, and many processes, like e.g. Higgs-boson production or heavy-quark production, can proceed via vector-boson vector-boson scattering. The experimental study of these and similar processes is expected to lead to an understanding of the Higgs sector of the electroweak standard model and eventually of the electroweak symmetry breaking mechanism. In addition, vector-boson pair production at hadron colliders will provide information on the self-couplings of the $W^{\pm}$ and Z bosons and possibly play an important role in the search for new physics.

In lowest order, vector bosons can be produced by quark-antiquark annihilation in hadron collisions. However, at high energies, higher-order processes where vector bosons emitted from incoming quarks or antiquarks initiate a hard scattering process can be enhanced by logarithmic factors and thus can compete with the lowest-order production mechanism. These processes have successfully been described with the help of the effective vector boson method (EVBM) which applies the concept of partons in a hadron to the case of vector bosons: vector bosons are viewed as partons in quarks and electrons, as quarks and gluons are partons in hadrons. In analogy to the Weizsäcker-Williams approximation of QED [1] the cross section for a scattering process $a + A \to X$ at a center-of-mass energy s is factorized into probability densities $P_{pol}^{V/a}(z)$ for finding a vector boson V with polarization *pol* in the incoming fermion a, and hard vector-boson scattering cross sections at a reduced center-of-mass energy xs:

$$d\sigma(a + A \to X, s) = \int\limits_{x_{\min}}^{1} dx \sum_{V} \sum_{pol} P_{pol}^{V/a}(x) d\sigma(V_{pol} + A \to X, xs). \quad (1)$$

The basic assumptions in the effective vector boson method are that the dominant contributions for producing the final state X is due to vector-boson initiated processes and that the cross section for the scattering of an off-shell vector boson can be related to the corresponding on-shell cross section.

In the application of the method to processes with two intermediate vector bosons (see Fig. 1) it was assumed that convolutions of single-vector-boson probability densities are sufficient to obtain luminosities for vector-boson pairs,

$$\mathcal{L}_{pol_1 pol_2}^{V_1 V_2/ab}(x) = \int\limits_{z_{\min}}^{1} \frac{dz}{z} P_{pol_1}^{V_1/a}(z) P_{pol_2}^{V_2/b}(x/z), \quad (2)$$

which can be used to express the cross section for two-fermion scattering in terms of the vector-boson vector-boson scattering cross section:

$$d\sigma(a + b \to X, s) = \int\limits_{x_{\min}}^{1} dx \sum_{V_1, V_2} \sum_{pol_1, pol_2} \mathcal{L}_{pol_1 pol_2}^{V_1 V_2/ab}(x) d\sigma(V_1^{pol_1} + V_2^{pol_2} \to X, xs). \quad (3)$$

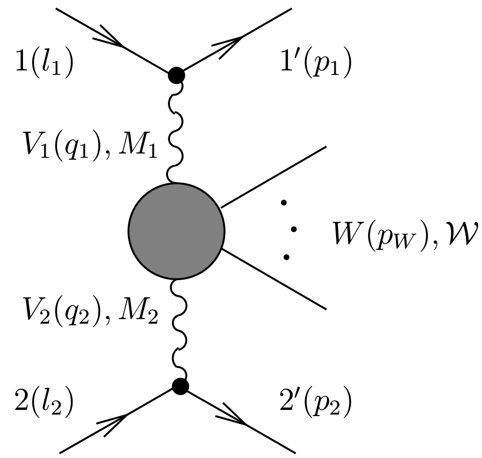

Figure 1: The vector-boson scattering diagram

The possibility to generalize the equivalent photon approximation to the case of massive vector bosons was first noted in [2] and explicitly formulated in [3, 4, 5]. Originally, the concept was invented for the description of processes at very high energies and thus included a number of approximations valid at high energies only. These approximations were partly of kinematic origin and concerned the neglect of mass terms, or of the transverse momentum of the intermediate vector bosons. According to the details of the approximation, a variety of versions for vector-boson distributions with differing numerical results can be found in the literature. The most simple of these approximations—the leading logarithmic approximation (LLA)—amounts to taking a zero-mass limit. In addition, since it was observed that for the production of a heavy Higgs particle in vector-boson scattering the cross section is dominated by longitudinal polarization [6], first applications of the method neglected contributions from transversely polarized vector bosons and the interference between amplitudes for different polarizations.

Comparisons with exact calculations have shown that the method is indeed helpful and leads to reliable results, in particular for Higgs boson production [7] and for heavy fermion production [8]. The application to vector-boson vector-boson scattering off the Higgs resonance [9] was less successful; the effective vector boson method overestimated the exact result [10]. Adding the (positive) contribution from transversely polarized vector bosons [11] could, of course, not lead to an improved agreement between the EVBM and exact calculations for vector-boson pair production.

In [12] it was shown that approximations of kinematic origin can be avoided and a set of exact vector-boson distributions was derived. There it was also shown that interference terms (i.e. non-diagonal contributions) do not appear in the case of single-vector-boson processes (see also [5]). The only remaining necessary assumption in using the EVBM for single-vector-boson processes concerned the off-shell behaviour of the hard scattering cross section.

We will show that the improvement obtained with the results of [12] is not sufficient to accurately describe two-vector-boson processes. The simple convolution of two single-vector-boson probabilities[2] as in Eq. (2) ignores the mutual influence of the emission of one boson on the probability for the emission of another. In addition, interference contributions need not vanish, as has been noticed in the specific example of Higgs boson production in [15]. This is in analogy to two-photon processes [16] where it was shown already in [17] that the extension of the Weizsäcker-Williams method from one photon to the case of two photons is not straightforward.

The main purpose of the present work is the extension of the effective vector boson method to the case of processes with two vector bosons, as needed in the study of vector-boson vector-boson scattering. It thus combines the exact treatment of the two-boson kinematics, presented for photons in [17], with the exact definition of vector-boson distributions, presented for single vector bosons in [12]. Our derivation (section 2) will not use any kinematic approximation. It turns out that non-diagonal terms are indeed needed. In section 3, we will present exact luminosity functions for vector-boson pairs in quark or electron initiated processes. One can then identify the additional approximation needed to reduce these luminosities to convolutions of the exact single-vector-boson densities of [12] (section 4) and in the high energy limit we also recover the leading logarithmic versions of vector-boson distributions as used in the literature (section 5). Finally, in section 6 we will also present numerical results for these exact luminosities and compare them with the ones of simplified approaches.

Despite of the fact that both single-vector-boson distributions and two-vector-boson luminosities can be obtained exactly without any approximation, there remains the question whether the set of Feynman diagrams that can be described with the help of the EVBM is indeed the dominating one. The answer to this problem depends on the process and has to be found in a case-by-case study. Of particular concern in this respect is the question whether the considered subset of diagrams is gauge invariant. In [18] it was observed that for off-shell vector-boson scattering there may occur strong gauge cancellations between those contributions taken into account in the EVBM and bremsstrahlung diagrams which are ignored. Motivated by this, Kunszt and Soper [19] argued that the extrapolation to off-shell masses is not always guaranteed, but for heavy Higgs-boson production they show in an axial gauge the validity of the basic assumption that the extrapolation to off-shell masses is indeed a smooth one.

Our final explicit expressions for the two-vector-boson luminosities are obtained with specific simple assumptions for the off-shell behaviour of the vector-boson scattering cross

[2]Explicit expressions for the luminosity functions derived from convolutions of single-boson distributions in the leading logarithmic approximation have been given in [13, 14] and the last but one reference of [9].

sections. However, we keep a clean separation of exactly calculable parts and model assumptions and our expressions are written in a form which allows for an easy accommodation of an improved off-shell dependence, as soon as the corresponding information would be available. Apart from these caveats, our luminosities are exact results of a calculation of a subset of Feynman diagrams. In particular, their range of validity is not restricted to large energies. Therefore we will also present some of the results for an energy of $\sqrt{s} = 500$ GeV, relevant for a next-generation e^+e^- collider. The alternative approach of using convolutions of the LLA single-vector-boson distributions is not applicable at these small energies.

2 General Formalism of the Effective Vector Boson Method

We consider the production of an arbitrary state W in the 2-fermion scattering process (see Fig. 1)

$$1(l_1) + 2(l_2) \to 1'(p_1) + 2'(p_2) + W(p_W). \tag{4}$$

The 4-momenta of the incoming and outgoing fermions are denoted by l_1, l_2 and p_1, p_2, resp., the total center-of-mass energy squared is given by $s = (l_1 + l_2)^2$. The final state W, which may contain any number of particles, has 4-momentum p_W and its invariant mass squared will be denoted by $\mathcal{W}^2 = p_W^2$. The cross section for the process (4) is given by

$$\sigma_{ff} = \frac{1}{2s}\frac{1}{(2\pi)^2}\int \frac{d^3p_1}{2p_1^0}\int \frac{d^3p_2}{2p_2^0}\int d\rho_W \overline{|\mathcal{M}_{ff}|^2}\delta^{(4)}(l_1 + l_2 - p_1 - p_2 - p_W). \tag{5}$$

In (5), $\overline{|\mathcal{M}_{ff}|^2}$ is the squared amplitude for the two-fermion initiated process, averaged and summed over helicities and $d\rho_W$ is the phase space element for the state W.

For high energies one can neglect the fermion masses. With the help of the momentum transfers $q_j = l_j - p_j$, $j = 1, 2$ and using the dimensionless variables

$$x = \frac{\mathcal{W}^2}{s}, \qquad z = \frac{M_X^2}{s}, \qquad \text{with} \;\; M_X^2 = (p_W + p_2)^2, \tag{6}$$

as well as

$$Q_2^2 = \frac{1}{1 - \frac{q_1^2}{M_X^2}} q_2^2, \tag{7}$$

one can parametrize the phase space by

$$\begin{aligned}\sigma_{ff} \;\; = \;\; & \frac{1}{32s}\int_{x_0}^{1} dx \int_{x}^{1}\frac{dz}{z}\int_{-s(1-z)}^{0} dq_1^2 \int_{-sz(1-\frac{x}{z})}^{0} dQ_2^2 \int_0^{2\pi}\frac{d\varphi_1}{2\pi}\int_0^{2\pi}\frac{d\varphi_2}{2\pi} \\ & \int d\rho_W \overline{|\mathcal{M}_{ff}|^2}\delta^{(4)}(l_1 + l_2 - p_1 - p_2 - p_W).\end{aligned} \tag{8}$$

Here, $x_0 = \mathcal{W}_{min}^2/s$ is the minimal value of the invariant mass squared of the final state W normalized to the total center-of-mass energy. In case of an n-particle final state, $\mathcal{W}_{min}$ is equal to the sum of the masses of these particles. φ_1 and φ_2 are azimuthal angles for the momenta p_1 and p_2, resp., defined in Breit systems B_1 and B_2 in which either q_1 or q_2 has only a non-vanishing z-component (see Appendix A.1).

If the process (4) proceeds via the vector-boson fusion mechanism as shown in Fig. (1), the expression for the amplitude $\mathcal{M}_{ff}$ is given by[3]

$$\mathcal{M}_{ff} = e^2 \sum_{m,n=-1}^{1} (-1)^{m+n} \frac{j_1(l_1,p_1)\cdot\epsilon_1^*(m)}{q_1^2 - M_1^2} \frac{j_2(l_2,p_2)\cdot\epsilon_2^*(n)}{q_2^2 - M_2^2} \mathcal{M}(m,n), \tag{9}$$

where the $\epsilon_j(m)$ are polarization vectors for the vector boson V_j with mass M_j and helicity $m = 0, \pm 1$ in the center-of-mass system C of $q_1 + q_2$. Explicit expressions for them are given in App. A.2. The $j_j(l_j, p_j)$ are fermionic current 4-vectors, e is the positron charge, and $\mathcal{M}(m,n)$ is the amplitude for the production of the final state W from vector bosons V_1 and V_2 with helicities m and n, resp. The amplitudes $\mathcal{M}(m,n)$ must be evaluated at off-shell values of q_1^2 and q_2^2. The polarization vectors are normalized according to

$$\epsilon_j(m)\cdot\epsilon_j^*(m') = \delta_{m,m'}(-1)^m, \qquad j = 1, 2\ , \tag{10}$$

and satisfy the completeness relation

$$\sum_{m=-1}^{1} \epsilon_j^{\mu}(m)\epsilon_j^{*\nu}(m) = -g^{\mu\nu} + \frac{q_j^{\mu} q_j^{\nu}}{M_j^2}, \qquad \text{(no sum on } j)\ , \tag{11}$$

which corresponds to writing the vector-boson propagators in the unitary gauge.

The expression for the squared amplitude, averaged over the spin states of the initial fermions and summed over the spins of the final state fermions is

$$\overline{|\mathcal{M}_{ff}|^2} = 4\,e^4 \sum_{m,m'=-1}^{1} \sum_{n,n'=-1}^{1} (-1)^{m+m'+n+n'} \frac{\tilde{T}_1(m,m')}{(q_1^2 - M_1^2)^2} \frac{\tilde{T}_2(n,n')}{(q_2^2 - M_2^2)^2} \mathcal{M}(m,n)\mathcal{M}^*(m',n'), \tag{12}$$

with the fermionic tensors

$$\tilde{T}_j(m,m') = \frac{1}{4} \sum_{pol} j_j(l_j,p_j)\cdot\epsilon_j^*(m) j_j^*(l_j,p_j)\cdot\epsilon_j(m'). \tag{13}$$

The tensor $\tilde{T}_j(m,m')$ can be decomposed into two parts with different combinations of the vector and axial-vector coupling constants v_j and a_j of the vector bosons V_j by the relation

$$\tilde{T}_j(m,m') = (v_j^2 + a_j^2)\tilde{\mathcal{C}}_j(m,m') + 2v_j a_j \tilde{\mathcal{S}}_j(m,m'). \tag{14}$$

[3] A sum must be taken over all vector-boson pairs V_1, V_2 which can couple to the fermions and produce the final state W. We do not treat the interference terms here, but the extension of our formalism to take them into account is straightforward.

The tensors $\tilde{\mathcal{C}}_j(m,m')$ and $\tilde{\mathcal{S}}_j(m,m')$ (i.e. tensors in helicity space) are given by

$$\tilde{\mathcal{C}}_j(m,m') = p_j \cdot \epsilon_j^*(m) l_j \cdot \epsilon_j(m') + p_j \cdot \epsilon_j(m') l_j \cdot \epsilon_j^*(m) - p_j \cdot l_j \epsilon_j^*(m) \cdot \epsilon_j(m') \tag{15}$$

and

$$\tilde{\mathcal{S}}_j(m,m') = i\epsilon_{\alpha\beta\gamma\delta} p_j^\alpha \epsilon_j^{\beta *}(m) l_j^\gamma \epsilon_j^\delta(m'), \tag{16}$$

with $\epsilon_{0123} = 1$.

Factorizing the φ_2-dependence of the tensor components of $\tilde{\mathcal{C}}_j(m,m')$ and $\tilde{\mathcal{S}}_j(m,m')$ which is given in terms of simple exponential functions, we define φ_2-independent tensors $\mathcal{C}_j(m,m')$ and $\mathcal{S}_j(m,m')$:

$$\begin{aligned}
\tilde{\mathcal{C}}_1(m,m') &= \mathcal{C}_1(m,m') e^{i(m-m')\varphi_2}, \\
\tilde{\mathcal{S}}_1(m,m') &= \mathcal{S}_1(m,m') e^{i(m-m')\varphi_2}, \\
\tilde{\mathcal{C}}_2(n,n') &= \mathcal{C}_2(n,n') e^{-i(n-n')\varphi_2}, \\
\tilde{\mathcal{S}}_2(n,n') &= \mathcal{S}_2(n,n') e^{-i(n-n')\varphi_2},
\end{aligned} \tag{17}$$

for which the following relations hold:

$$\begin{aligned}
\mathcal{C}_j(m',m) &= \mathcal{C}_j^*(m,m'), \\
\mathcal{S}_j(m',m) &= \mathcal{S}_j^*(m,m'), \\
\mathcal{C}_j(-m',-m) &= (-1)^{m+m'} \mathcal{C}_j(m,m'), \\
\mathcal{S}_j(-m',-m) &= -(-1)^{m+m'} \mathcal{S}_j(m,m').
\end{aligned} \tag{18}$$

The last relation in (18) implies

$$\begin{aligned}
\mathcal{S}_j(+-) &= 0, \quad \text{and} \\
\mathcal{S}_j(00) &= 0.
\end{aligned} \tag{19}$$

Consequently, $\mathcal{C}_j(++)$, $\mathcal{C}_j(00)$, $\mathcal{C}_j(+-)$ and $\mathcal{C}_j(+0)$ can be chosen as the 2×4 independent components of $\mathcal{C}_j(m,m')$ and the $\mathcal{S}_j(m,m')$ have the 2×2 independent components $\mathcal{S}_j(++)$ and $\mathcal{S}_j(+0)$. We illustrate this situation by writing down $\tilde{\mathcal{C}}_1(m,m')$ and $\tilde{\mathcal{S}}_1(m,m')$ in matrix form:

$$\tilde{\mathcal{C}}_1(m,m') = \begin{pmatrix} \mathcal{C}_1(++) & \mathcal{C}_1^*(+0)e^{-i\varphi_2} & \mathcal{C}_1^*(+-)e^{-2i\varphi_2} \\ \mathcal{C}_1(+0)e^{i\varphi_2} & \mathcal{C}_1(00) & -\mathcal{C}_1^*(+0)e^{-i\varphi_2} \\ \mathcal{C}_1(+-)e^{2i\varphi_2} & -\mathcal{C}_1(+0)e^{i\varphi_2} & \mathcal{C}_1(++) \end{pmatrix} \tag{20}$$

and

$$\tilde{\mathcal{S}}_1(m,m') = \begin{pmatrix} \mathcal{S}_1(++) & \mathcal{S}_1^*(+0)e^{-i\varphi_2} & 0 \\ \mathcal{S}_1(+0)e^{i\varphi_2} & 0 & \mathcal{S}_1^*(+0)e^{-i\varphi_2} \\ 0 & \mathcal{S}_1(+0)e^{i\varphi_2} & -\mathcal{S}_1(++) \end{pmatrix}, \tag{21}$$

where the columns from left to right correspond to $m = +,0,-$ and the rows from top to bottom to $m' = +,0,-$. Expressions for the independent components in terms of the integration variables in (8) are given in Appendix B. Similar decompositions can be given for $\tilde{\mathcal{C}}_2(m,m')$ and $\tilde{\mathcal{S}}_2(m,m')$. The quantities $\mathcal{C}_2(m,m')$ and $\mathcal{S}_2(m,m')$ turn out to be real.

Carrying out the integration over φ_2, there remain altogether 19 terms in the m, m', n, n' helicity space, out of which nine have $h = m - m' = n - n' = 0$ (they are diagonal in the helicities of V_1 and V_2), four have $h = 1$, four have $h = -1$ and the other two have $h = 2$ and $h = -2$, resp. For the case of two-photon interactions this classification has been given in [17]. Using this decomposition, one can write the expression in Eq. (12) in the following way:

$$\frac{1}{2\pi}\int_0^{2\pi} d\varphi_2 \sum_{m,m',n,n'=-1}^{1} (-1)^{m+m'+n+n'} \tilde{T}_1(m,m')\tilde{T}_2(n,n')\mathcal{M}(m,n)\mathcal{M}^*(m',n')$$

$$\begin{aligned}
= \; & (v_1^2+a_1^2)(v_2^2+a_2^2)\,(K_{TT}M_{TT} + K_{TL}M_{TL} + K_{LT}M_{LT} + K_{LL}M_{LL} \\
& + K_{TLTL}M_{TLTL} + K_{TTTT}M_{TTTT} - K^{Im}_{TTTT}M^{Im}_{TTTT} - K^{Im}_{TLTL}M^{Im}_{TLTL}\big) \\
& + (2v_1a_1)(2v_2a_2)\left(K_{\overline{TT}}M_{\overline{TT}} + K_{\overline{TL}\overline{TL}}M_{\overline{TL}\overline{TL}} - K^{Im}_{\overline{TL}\overline{TL}}M^{Im}_{\overline{TL}\overline{TL}}\right) \\
& + (v_1^2+a_1^2)(2v_2a_2)\left(K_{T\overline{T}}M_{T\overline{T}} + K_{L\overline{T}}M_{L\overline{T}} + K_{TL\overline{T}L}M_{TL\overline{T}L} - K^{Im}_{TL\overline{T}L}M^{Im}_{TL\overline{T}L}\right) \\
& + (2v_1a_1)(v_2^2+a_2^2)\left(K_{\overline{T}T}M_{\overline{T}T} + K_{\overline{T}L}M_{\overline{T}L} + K_{\overline{T}LTL}M_{\overline{T}LTL} - K^{Im}_{\overline{T}LTL}M^{Im}_{\overline{T}LTL}\right)
\end{aligned} \tag{22}$$

$$= \sum_{pol} c_{f,pol}\, K_{pol}\, M_{pol}, \tag{23}$$

where the last line defines the notation to be used below, with *pol* being labels for the polarizations, $pol = TT$, $\overline{TT}$, etc. $c_{f,pol}$ contain the fermionic coupling constants and—depending on the index *pol*—can take the values $c_{f,pol} = (v_1^2+a_1^2)(v_2^2+a_2^2)$, $(2v_1a_1)(2v_2a_2)$, $(v_1^2+a_1^2)(2v_2a_2)$ and $(2v_1a_1)(v_2^2+a_2^2)$. The quantities K_{pol}, which are five-fold differential luminosities—they depend on $\mathcal{W}^2, q_1^2, q_2^2, M_X^2$ and φ_1—are defined by

$$\begin{aligned}
K_{TT} &= 4\mathcal{C}_1(++)\mathcal{C}_2(++), \\
K_{\overline{TT}} &= 4\mathcal{S}_1(++)\mathcal{S}_2(++), \\
K_{TL} &= 2\mathcal{C}_1(++)\mathcal{C}_2(00), \\
K_{LT} &= 2\mathcal{C}_1(00)\mathcal{C}_2(++), \\
K_{LL} &= \mathcal{C}_1(00)\mathcal{C}_2(00), \\
K_{TLTL} &= 8Re[\mathcal{C}_1(+0)]\mathcal{C}_2(+0), \\
K_{\overline{TL}\overline{TL}} &= 8Re[\mathcal{S}_1(+0)]\mathcal{S}_2(+0), \\
K_{TTTT} &= 2Re[\mathcal{C}_1(+-)]\mathcal{C}_2(+-), \\
K_{T\overline{T}} &= 4\mathcal{C}_1(++)\mathcal{S}_2(++), \\
K_{\overline{T}T} &= 4\mathcal{S}_1(++)\mathcal{C}_2(++), \\
K_{\overline{T}L} &= 2\mathcal{S}_1(++)\mathcal{C}_2(00), \\
K_{L\overline{T}} &= 2\mathcal{C}_1(00)\mathcal{S}_2(++), \\
K_{TL\overline{T}L} &= 8Re[\mathcal{C}_1(+0)]\mathcal{S}_2(+0), \\
K_{\overline{T}LTL} &= 8Re[\mathcal{S}_1(+0)]\mathcal{C}_2(+0),
\end{aligned}$$

$$
\begin{aligned}
K^{Im}_{TL\overline{T}L} &= 8Im[\mathcal{C}_1(+0)]\mathcal{S}_2(+0), \\
K^{Im}_{\overline{T}LTL} &= 8Im[\mathcal{S}_1(+0)]\mathcal{C}_2(+0), \\
K^{Im}_{TLTL} &= 8Im[\mathcal{C}_1(+0)]\mathcal{C}_2(+0), \\
K^{Im}_{\overline{T}L\overline{T}L} &= 8Im[\mathcal{S}_1(+0)]\mathcal{S}_2(+0), \\
K^{Im}_{TTTT} &= 2Im[\mathcal{C}_1(+-)]\mathcal{C}_2(+-),
\end{aligned} \tag{24}
$$

with $\mathcal{C}_j(m,m')$ and $\mathcal{S}_j(m,m')$ from (54) and (55) (see App. B). The averaged sums of products of amplitudes for the vector-boson scattering processes, M_{pol}, to be simply called squared amplitudes in what follows, are defined through

$$
\begin{aligned}
M_{TT} &= \frac{1}{4}(|\mathcal{M}(++)|^2 + |\mathcal{M}(--)|^2 + |\mathcal{M}(+-)|^2 + |\mathcal{M}(-+)|^2), \\
M_{\overline{TT}} &= \frac{1}{4}(|\mathcal{M}(++)|^2 + |\mathcal{M}(--)|^2 - |\mathcal{M}(+-)|^2 - |\mathcal{M}(-+)|^2), \\
M_{TL} &= \frac{1}{2}(|\mathcal{M}(+0)|^2 + |\mathcal{M}(-0)|^2), \\
M_{LT} &= \frac{1}{2}(|\mathcal{M}(0+)|^2 + |\mathcal{M}(0-)|^2), \\
M_{LL} &= |\mathcal{M}(00)|^2, \\
M_{TLTL} &= \frac{1}{4}Re[\mathcal{M}(++)\mathcal{M}^*(00) + \mathcal{M}(--)\mathcal{M}^*(00) - \mathcal{M}(+0)\mathcal{M}^*(0-) - \mathcal{M}(-0)\mathcal{M}^*(0+)], \\
M_{\overline{T}L\overline{T}L} &= \frac{1}{4}Re[\mathcal{M}(++)\mathcal{M}^*(00) + \mathcal{M}(--)\mathcal{M}^*(00) + \mathcal{M}(+0)\mathcal{M}^*(0-) + \mathcal{M}(-0)\mathcal{M}^*(0+)], \\
M_{TTTT} &= Re[\mathcal{M}(++)\mathcal{M}^*(--)], \\
M_{T\overline{T}} &= \frac{1}{4}(|\mathcal{M}(++)|^2 - |\mathcal{M}(--)|^2 - |\mathcal{M}(+-)|^2 + |\mathcal{M}(-+)|^2), \\
M_{\overline{T}T} &= \frac{1}{4}(|\mathcal{M}(++)|^2 - |\mathcal{M}(--)|^2 + |\mathcal{M}(+-)|^2 - |\mathcal{M}(-+)|^2), \\
M_{\overline{T}L} &= \frac{1}{2}(|\mathcal{M}(+0)|^2 - |\mathcal{M}(-0)|^2), \\
M_{L\overline{T}} &= \frac{1}{2}(|\mathcal{M}(0+)|^2 - |\mathcal{M}(0-)|^2), \\
M_{TL\overline{T}L} &= \frac{1}{4}Re[\mathcal{M}(++)\mathcal{M}^*(00) - \mathcal{M}(--)\mathcal{M}^*(00) + \mathcal{M}(+0)\mathcal{M}^*(0-) - \mathcal{M}(-0)\mathcal{M}^*(0+)], \\
M_{\overline{T}LTL} &= \frac{1}{4}Re[\mathcal{M}(++)\mathcal{M}^*(00) - \mathcal{M}(--)\mathcal{M}^*(00) - \mathcal{M}(+0)\mathcal{M}^*(0-) + \mathcal{M}(-0)\mathcal{M}^*(0+)], \\
M^{Im}_{TL\overline{T}L} &= \frac{1}{4}Im[\mathcal{M}(++)\mathcal{M}^*(00) + \mathcal{M}(--)\mathcal{M}^*(00) + \mathcal{M}(+0)\mathcal{M}^*(0-) + \mathcal{M}(-0)\mathcal{M}^*(0+)], \\
M^{Im}_{\overline{T}LTL} &= \frac{1}{4}Im[\mathcal{M}(++)\mathcal{M}^*(00) + \mathcal{M}(--)\mathcal{M}^*(00) - \mathcal{M}(+0)\mathcal{M}^*(0-) - \mathcal{M}(-0)\mathcal{M}^*(0+)], \\
M^{Im}_{TLTL} &= \frac{1}{4}Im[\mathcal{M}(++)\mathcal{M}^*(00) - \mathcal{M}(--)\mathcal{M}^*(00) - \mathcal{M}(+0)\mathcal{M}^*(0-) + \mathcal{M}(-0)\mathcal{M}^*(0+)], \\
M^{Im}_{\overline{T}L\overline{T}L} &= \frac{1}{4}Im[\mathcal{M}(++)\mathcal{M}^*(00) - \mathcal{M}(--)\mathcal{M}^*(00) + \mathcal{M}(+0)\mathcal{M}^*(0-) - \mathcal{M}(-0)\mathcal{M}^*(0+)], \\
M^{Im}_{TTTT} &= Im[\mathcal{M}(++)\mathcal{M}^*(--)].
\end{aligned} \tag{25}
$$

The squared amplitudes

$$
M_{T\overline{T}}, M_{\overline{T}T}, M_{\overline{T}L}, M_{L\overline{T}}, M_{TL\overline{T}L}, M_{\overline{T}LTL}, M^{Im}_{\overline{T}L\overline{T}L}, M^{Im}_{TLTL} \tag{26}
$$

vanish if both the interaction responsible for the transition $V_1V_2 \to W$ is parity conserving, i.e. if $\mathcal{M}(m,n) = \mathcal{M}(-m,-n)$, and a summation over the polarization of the final state

W is performed. The luminosities K^{Im}_{pol} vanish after integrating over the azimuthal angle φ_1. We also note, that the squared amplitudes M^{Im}_{pol} are zero if all amplitudes $\mathcal{M}(m,n)$ can be chosen real. Therefore we restrict the following discussion to the remaining eight luminosities

$$K_{TT}, K_{\overline{TT}}, K_{TL}, K_{LT}, K_{LL}, K_{TLTL}, K_{\overline{TL}\overline{TL}}, K_{TTTT}. \tag{27}$$

The expression Eq. (23) shows explicitly the trivial factorization of the cross section into parts describing the vector-boson emission from the incoming fermions and parts pertaining to the vector-boson vector-boson scattering. These latter pieces, combined with the phase space integral for the final state W, can be interpreted as cross sections and correlations for virtual vector-boson scattering processes:

$$\sigma_{pol}(q_1^2, q_2^2) = (2\pi)^4 \frac{1}{2\kappa} \int d\rho_W M_{pol} \delta^{(4)}(q_1 + q_2 - p_W). \tag{28}$$

In Eq. (28) we included the 'flux factor' $1/2\kappa$ with

$$\kappa = \sqrt{\mathcal{W}^4 + q_1^4 + q_2^4 - 2\mathcal{W}^2 q_1^2 - 2\mathcal{W}^2 q_2^2 - 2q_1^2 q_2^2}, \tag{29}$$

$\mathcal{W}^4 \equiv (\mathcal{W}^2)^2$ and $q_j^4 \equiv (q_j^2)^2$, so that (28) leads to the correct expression for real vector-boson scattering in the limit $q_j^2 \to M_j^2$.

In terms of the cross sections (28) for virtual vector-boson scattering, the cross section (8) for the two-fermion initiated process is given by

$$\begin{aligned}\sigma_{ff} \quad = \quad & \left(\frac{\alpha}{2\pi}\right)^2 \int\limits_{x_0}^{1} dx \int\limits_{x}^{1} \frac{dz}{z} \int\limits_{-s(1-z)}^{0} dq_1^2 \int\limits_{-zs(1-\frac{x}{z})}^{0} dQ_2^2 \frac{1}{(q_1^2 - M_1^2)^2} \frac{1}{(q_2^2 - M_2^2)^2} \\ & \int\limits_{0}^{2\pi} \frac{d\varphi_1}{2\pi} \kappa \sum_{pol} c_{f,pol} K_{pol} \, \sigma_{pol}(q_1^2, q_2^2),\end{aligned} \tag{30}$$

where α is the fine structure constant.

Up to this point, the calculation has been exact without any approximation. The basic assumption of the equivalent vector boson method concerns the dependence of the off-shell cross sections $\sigma_{pol}(q_1^2, q_2^2)$ on the off-shell masses q_i^2. For transverse polarization it is certainly a good approximation to identify $\sigma_{TT}(q_1^2, q_2^2)$ with its on-shell value $\sigma_{TT}(M_1^2, M_2^2)$. However, for longitudinal polarizations, $\sigma_{pol}(q_1^2, q_2^2)$ contains kinematic singularities at $q_1^2 = 0$ and $q_2^2 = 0$, as can be seen from the explicit form of the polarization vectors (Eq. (51) in Appendix A.2). Therefore, for longitudinal polarization, the resulting factors M_i^2/q_i^2 should be taken into account explicitly. Apart from this, there are good arguments from dispersion relation techniques to believe that the extrapolation to off-shell masses is a smooth one.

We therefore make the assumption that the extrapolation to off-shell masses can be described by simple proportionality factors $f_{pol}(q_1^2, q_2^2)$ with $f_{pol}(M_1^2, M_2^2) = 1$. Taking also the q_i^2-dependence of the flux factor κ into account, we write

$$\kappa \, \sigma_{pol}(q_1^2, q_2^2) = \tilde{\kappa}_0 \, f_{pol}(q_1^2, q_2^2) \, \sigma_{pol}(M_1^2, M_2^2), \tag{31}$$

where $\tilde{\kappa}_0$ is a flux factor for on-shell vector-boson scattering to be specified below and $\sigma_{pol}(M_1^2, M_2^2)$ are the cross-sections for on-shell vector-boson scattering evaluated at the rescaled energy squared $(q_1+q_2)^2 = xs$ of the vector-boson vector-boson scattering process.

To describe the q_j^2-dependence of the off-shell cross sections, we will consider the following specific forms of the proportionality factors f_{pol} which take into account the q_j^2-dependence of the longitudinal polarization vectors $\epsilon_j(0)$:

$$\begin{aligned}
f_{TT} = f_{\overline{TT}} = f_{TTTT} &= 1, \\
f_{TL} &= \frac{M_2^2}{-q_2^2}, \\
f_{LT} &= \frac{M_1^2}{-q_1^2}, \\
f_{LL} &= \frac{M_1^2}{-q_1^2}\frac{M_2^2}{-q_2^2}, \\
f_{TLTL} = f_{\overline{TL}\overline{TL}} &= \frac{M_1}{\sqrt{-q_1^2}}\frac{M_2}{\sqrt{-q_2^2}}.
\end{aligned} \tag{32}$$

We now introduce luminosities $\mathcal{L}_{pol}(x)$ which are differential in the variable x, writing the differential cross section in the form

$$\frac{d\sigma_{ff}}{dx} = \sum_{pol} \mathcal{L}_{pol}(x)\, \sigma_{pol}(M_1^2, M_2^2), \tag{33}$$

with the luminosities

$$\begin{aligned}
\mathcal{L}_{pol}(x) \quad = \quad & \left(\frac{\alpha}{2\pi}\right)^2 \frac{\tilde{\kappa}_0}{s} c_{f,pol} \int_x^1 \frac{dz}{z} \int_{-s(1-z)}^0 dq_1^2 \int_{-sz(1-\frac{x}{z})}^0 dQ_2^2 \frac{1}{(q_1^2 - M_1^2)^2} \frac{1}{(q_2^2 - M_1^2)^2} \\
& \int_0^{2\pi} \frac{d\varphi_1}{2\pi}\, f_{pol}(q_1^2, q_2^2)\, K_{pol}.
\end{aligned} \tag{34}$$

The luminosities $\mathcal{L}_{pol}(x)$ depend on x and, since they are dimensionless, on the masses of the vector bosons via the ratios M_1^2/s and M_2^2/s. As well, we have included the coupling constants of the vector bosons in the definition. $\mathcal{L}_{pol}(x)\, dx$ can be interpreted as the probability that the vector-boson pair V_1, V_2 with the specified polarization and with center-of-mass energy in the interval $[xs, (x+dx)s]$ will be emitted from the fermion pair 1 and 2.

3 The exact Luminosities

We evaluate the expressions (34) adopting the forms (32) for the behavior of the virtual cross-sections. No other assumptions are made. The flux factor $\tilde{\kappa}_0$ is evaluated at $q_j^2 = M_j^2, j = 1, 2$, so that we have

$$\tilde{\kappa}_0 = \kappa_0 = \sqrt{\mathcal{W}^4 + M_1^4 + M_2^4 - 2\mathcal{W}^2 M_1^2 - 2\mathcal{W}^2 M_2^2 - 2M_1^2 M_2^2}. \tag{35}$$

We rewrite the phase space integral in (34) in the following way,

$$\int_x^1 \frac{dz}{z} \int_{-s(1-z)}^0 dq_1^2 \int_{-sz(1-\frac{x}{z})}^0 dQ_2^2 \int_0^{2\pi} \frac{d\varphi_1}{2\pi} = \int_{-s+\mathcal{W}_1^2}^0 dq_1^2 \int_{-s+\mathcal{W}_2^2}^0 dq_2^2 \int_{\hat{x}s}^s \frac{d\mu_X}{\mu_X} \int_0^{2\pi} \frac{d\varphi_1}{2\pi}, \tag{36}$$

where we have introduced the variables $\hat{x} = \dfrac{\nu + K\mathcal{W}}{s}$, with $\nu = q_1 \cdot q_2 = \frac{1}{2}(\mathcal{W}^2 - q_1^2 - q_2^2)$, $\mathcal{W} = \sqrt{\mathcal{W}^2}$ and $K = \frac{\kappa}{2\mathcal{W}}$, K being the magnitude of the three-momentum of the vector-bosons V_1, V_2 in their center-of-mass frame, and $\mu_X = M_X^2 - q_1^2$. The integration limits for q_1^2 and q_2^2 in (36), following from $(q_1^2+s)(q_2^2+s) > \mathcal{W}^2 s$ (with $q_1^2 < 0$ and $q_2^2 < 0$), are written with the help of $\mathcal{W}_1^2 = \mathcal{W}^2$ and $\mathcal{W}_2^2 = \mathcal{W}^2 \dfrac{s}{s+q_1^2}$. The luminosities vanish for $x < \dfrac{(M_1+M_2)^2}{s}$.

Using Eq. (36), the expressions (34) for the luminosities become

$$\mathcal{L}_{pol}(x) = \left(\frac{\alpha}{2\pi}\right)^2 \frac{\kappa_0}{s} c_{f,pol} \int_{-s+\mathcal{W}_1^2}^0 dq_1^2 \int_{-s+\mathcal{W}_2^2}^0 dq_2^2 \frac{q_1^2}{(q_1^2-M_1^2)^2} \frac{q_2^2}{(q_2^2-M_2^2)^2} f_{pol}(q_1^2, q_2^2)\, J_{pol}, \tag{37}$$

with the triple-differential luminosities—they are functions of x, q_1^2 and q_2^2—

$$J_{pol} = \frac{1}{q_1^2 q_2^2} \int_{\hat{x}s}^s \frac{d\mu_X}{\mu_X} \int_0^{2\pi} \frac{d\varphi_1}{2\pi} K_{pol}, \tag{38}$$

and K_{pol} were defined in (24). The integrations over z and φ_1 in (38) can be performed analytically and the results are given in (41). We will discuss later which limiting cases will lead to results already obtained in the literature.

The singularities of the integrands in Eq. (37) at $q_j^2 = M_j^2$ lead, after integration, to mass singular terms. In the high-energy limit $s \gg M_j^2$ they appear either as familiar logarithms $\ln(s/M_j^2)$, or as a pole singularity $1/M_j^2$. The latter happens, e.g., in both masses for the LL-term, or in one of the masses for the TL and LT-luminosities. Since we will evaluate the two-dimensional integration over q_1^2 and q_2^2 in (37) numerically, specific care has to be taken of these singularities. This is done by introducing new integration variables x_j, y_j, and z_j ($j = 1, 2$) depending on the type of the singularity. The new variables are chosen such that the integration region becomes the unit cube in two dimensions. Their relations to q_j^2 are given by:

$$\begin{aligned} q_j^2 &= M_j^2 \left[1 - \left(\frac{M_j^2 + s - \mathcal{W}_j^2}{M_j^2}\right)^{x_j}\right] \\ &= M_j^2 \left[1 - \frac{M_j^2 + s - \mathcal{W}_j^2}{(s-\mathcal{W}_j^2)(1-y_j) + M_j^2}\right] \\ &= (-s + \mathcal{W}_j^2)\, z_j, \qquad j = 1, 2\ , \end{aligned} \tag{39}$$

and the luminosities (37) take the final form

$$
\begin{aligned}
\mathcal{L}_{TT}(x) &= \left(\frac{\alpha}{2\pi}\right)^2 (v_1^2+a_1^2)(v_2^2+a_2^2)\frac{\kappa_0}{s}\ln\left(\frac{M_1^2+s-\mathcal{W}_1^2}{M_1^2}\right)\int_0^1 dx_1\int_0^1 dx_2 \\
&\quad \ln\left(\frac{M_2^2+s-\mathcal{W}_2^2}{M_2^2}\right)\frac{q_1^2}{q_1^2-M_1^2}\frac{q_2^2}{q_2^2-M_2^2}J_{TT}, \\
\mathcal{L}_{\overline{TT}}(x) &= \left(\frac{\alpha}{2\pi}\right)^2 (2v_1a_1)(2v_2a_2)\frac{\kappa_0}{s}\ln\left(\frac{M_1^2+s-\mathcal{W}_1^2}{M_1^2}\right)\int_0^1 dx_1\int_0^1 dx_2 \\
&\quad \ln\left(\frac{M_2^2+s-\mathcal{W}_2^2}{M_2^2}\right)\frac{q_1^2}{q_1^2-M_1^2}\frac{q_2^2}{q_2^2-M_2^2}J_{\overline{TT}}, \\
\mathcal{L}_{TL}(x) &= \left(\frac{\alpha}{2\pi}\right)^2 (v_1^2+a_1^2)(v_2^2+a_2^2)\frac{\kappa_0}{s}\ln\left(\frac{M_1^2+s-\mathcal{W}_1^2}{M_1^2}\right)\int_0^1 dx_1\int_0^1 dy_2 \\
&\quad \left(1-\frac{M_2^2}{M_2^2+s-\mathcal{W}_2^2}\right)\frac{q_1^2}{q_1^2-M_1^2}J_{TL}, \\
\mathcal{L}_{LT}(x) &= \left(\frac{\alpha}{2\pi}\right)^2 (v_1^2+a_1^2)(v_2^2+a_2^2)\frac{\kappa_0}{s}\left(1-\frac{M_1^2}{M_1^2+s-\mathcal{W}_1^2}\right)\int_0^1 dy_1\int_0^1 dx_2 \\
&\quad \ln\left(\frac{M_2^2+s-\mathcal{W}_2^2}{M_2^2}\right)\frac{q_2^2}{q_2^2-M_2^2}J_{LT}, \\
\mathcal{L}_{LL}(x) &= \left(\frac{\alpha}{2\pi}\right)^2 (v_1^2+a_1^2)(v_2^2+a_2^2)\frac{\kappa_0}{s}\left(1-\frac{M_1^2}{M_1^2+s-\mathcal{W}_1^2}\right)\int_0^1 dy_1\int_0^1 dy_2 \\
&\quad \left(1-\frac{M_2^2}{M_2^2+s-\mathcal{W}_2^2}\right)J_{LL}, \\
\mathcal{L}_{TLTL}(x) &= \left(\frac{\alpha}{2\pi}\right)^2 (v_1^2+a_1^2)(v_2^2+a_2^2)\frac{\kappa_0}{s}\,M_1\,M_2\,\ln\left(\frac{M_1^2+s-\mathcal{W}_1^2}{M_1^2}\right)\int_0^1 dx_1\int_0^1 dx_2 \\
&\quad \ln\left(\frac{M_2^2+s-\mathcal{W}_2^2}{M_2^2}\right)\frac{q_1^2}{q_1^2-M_1^2}\frac{q_2^2}{q_2^2-M_2^2}\frac{J_{TLTL}}{\sqrt{-q_1^2}\sqrt{-q_2^2}}, \\
\mathcal{L}_{\overline{TL}\overline{TL}}(x) &= \left(\frac{\alpha}{2\pi}\right)^2 (2v_1a_1)(2v_2a_2)\frac{\kappa_0}{s}\,M_1\,M_2\,\ln\left(\frac{M_1^2+s-\mathcal{W}_1^2}{M_1^2}\right)\int_0^1 dx_1\int_0^1 dx_2 \\
&\quad \ln\left(\frac{M_2^2+s-\mathcal{W}_2^2}{M_2^2}\right)\frac{q_1^2}{q_1^2-M_1^2}\frac{q_2^2}{q_2^2-M_2^2}\frac{J_{\overline{TL}\overline{TL}}}{\sqrt{-q_1^2}\sqrt{-q_2^2}}, \\
\mathcal{L}_{TTTT}(x) &= \left(\frac{\alpha}{2\pi}\right)^2 (v_1^2+a_1^2)(v_2^2+a_2^2)\frac{\kappa_0}{s}(s-\mathcal{W}_1^2)\int_0^1 dz_1\int_0^1 dz_2 \\
&\quad (s-\mathcal{W}_2^2)\frac{q_1^4}{(q_1^2-M_1^2)^2}\frac{q_2^4}{(q_2^2-M_2^2)^2}\frac{J_{TTTT}}{q_1^2\,q_2^2}, \qquad (40)
\end{aligned}
$$

and the J_{pol} are given by

$$
\begin{aligned}
J_{TT} &= \frac{8}{\kappa^4}\Big[(2\nu^2(s+\nu)^2+q_1^2q_2^2(s^2+8s\nu+q_1^2q_2^2))\ln\left(\frac{1}{\hat{x}}\right)-6s^2\nu^2-4s\nu^3+2\nu^4 \\
&\quad +q_1^2q_2^2(-3s^2+4s\nu+6\nu^2+q_1^2q_2^2)+K\mathcal{W}(3s^2\nu+8s\nu^2+2\nu^3+q_1^2q_2^2(4s+\nu))\Big],
\end{aligned}
$$

$$
\begin{aligned}
J_{\overline{TT}} &= \frac{4}{\kappa^2}\left[(2s\nu+\nu^2+q_1^2q_2^2)\ln\left(\frac{1}{\hat{x}}\right)-4s\nu+2\nu^2+2q_1^2q_2^2+2K\mathcal{W}(s+\nu)\right], \\
J_{TL}=J_{LT} &= \frac{4}{\kappa^4}\Big[(4s^2\nu^2+8s\nu^3+2q_1^2q_2^2(s^2+8s\nu+3\nu^2))\ln\left(\frac{1}{\hat{x}}\right)-13s^2\nu^2-4s\nu^3 \\
&\quad +2\nu^4+q_1^2q_2^2(-5s^2+4s\nu+13\nu^2+3q_1^2q_2^2) \\
&\quad +2K\mathcal{W}(3\nu^2+8s\nu^2+\nu^3+2q_1^2q_2^2(2s+\nu))\Big], \\
J_{LL} &= \frac{8}{\kappa^4}\Big[(2s^2\nu^2+4s\nu^3+q_1^2q_2^2(s^2+8s\nu+2\nu^2+q_1^2q_2^2))\ln\left(\frac{1}{\hat{x}}\right)-7s^2\nu^2 \\
&\quad +q_1^2q_2^2(-2s^2+7\nu^2+2q_1^2q_2^2)+K\mathcal{W}(3s^2\nu+8s\nu^2+q_1^2q_2^2(4s+3\nu))\Big], \\
J_{TLTL} &= \frac{32}{\kappa^4}\sqrt{-q_1^2}\sqrt{-q_2^2}\Big[(3s^2\nu+9s\nu^2+\nu^3+q_1^2q_2^2(3s+2\nu))\ln\left(\frac{1}{\hat{x}}\right)-\frac{\nu^2(s+\nu)}{\hat{x}} \\
&\quad -8s^2\nu+3\nu^3+q_1^2q_2^2(s+6\nu)+K\mathcal{W}(2s^2+11s\nu+2\nu^2+q_1^2q_2^2)\Big], \\
J_{\overline{TL}\overline{TL}} &= \frac{8}{\kappa^2}\sqrt{-q_1^2}\sqrt{-q_2^2}\left[(s+\nu)\ln\left(\frac{1}{\hat{x}}\right)-\frac{\nu}{\hat{x}}+2\nu-s+K\mathcal{W}\right], \\
J_{TTTT} &= \frac{4}{\kappa^4}q_1^2q_2^2\Big[(3s^2+12s\nu+2\nu^2+q_1^2q_2^2)\ln\left(\frac{1}{\hat{x}}\right)+\frac{\nu^2}{\hat{x}^2}-\frac{5s\nu+4\nu^2}{\hat{x}} \\
&\quad -5s^2+4s\nu+6\nu^2+3q_1^2q_2^2+K\mathcal{W}(8s+3\nu)\Big].
\end{aligned} \tag{41}
$$

For the case of two-photon processes initiated by electron-electron scattering, analogous expressions have been derived in [17]. Our results are related to the corresponding $\hat{J}_{pol}$ from [17] by $J_{pol}=\frac{1}{x^2}\hat{J}_{pol}$ for $pol=TT$, TL and LL, $J_{TTTT}=\frac{1}{x^2}\hat{J}^{ex}_{TT}$ and $J_{TLTL}=\frac{2}{x^2}\hat{J}^{ex}_{LT}$ (note that we have neglected the fermion masses). We finally remark that, for $M_1=M_2$, we have $\mathcal{L}_{LT}(x)=\mathcal{L}_{TL}(x)$.

The integrals in Eq. (40) are well-suited for numerical evaluation. Their integrands contain no singularities; instead, the poles of order one show up as logarithms of the form $\ln\left((M_j^2+s-\mathcal{W}_j^2)/M_j^2\right)$, while the poles of order two (which would by themselves lead to a factor M_j^{-2}) have been canceled by corresponding factors M_j^2 included in our assumptions for the behaviour of the $f_{pol}(q_1^2,q_2^2)$, Eq. (32). Since the expressions Eq. (40) involve two-dimensional numerical integrations of the momentum transfers q_1^2 and q_2^2, it would be straightforward to replace the model assumptions Eq. (32) by better ones if required. The contribution from the leading singularities would not change then; however, subleading terms (non-logarithmic contributions for transverse polarization, logarithmic contributions for longitudinal polarization) are model-dependent. For the cases of Higgs production and heavy quark production, modifications of single-W boson distributions following from the exact off-shell behaviour of the corresponding hard cross sections have been studied in [20].

4 Convolutions of single-vector-boson distributions

Since helicities of massive particles are not Lorentz-invariant, the polarization vectors have to be defined in a definite reference frame, which we chose to be the center-of-mass system of the two vector bosons. Therefore, the $\mathcal{C}_i$ and $\mathcal{S}_i$ depend on both momentum transfers q_1^2 and q_2^2 at the same time. This means that the emission of a vector boson V_1 with definite helicity from fermion 1 is not independent from the off-shell mass of the second vector boson V_2, and the two-boson luminosities do not factorize into single-boson densities. However, since at high energies the process is dominated by small momentum transfers, it seems justified to neglect this mutual dependence on q_i^2. Then the expressions (34) for the two-vector-boson luminosities reduce to convolutions of single-vector-boson densities. These single-vector-boson distributions have been reported in [12].

To be specific, we consider the following simplifications:

1. Set $q_2^2 = 0$ in $\mathcal{C}_1(m, m')$ and $\mathcal{S}_1(m, m')$;
2. Set $q_1^2 = 0$ in $\mathcal{C}_2(n, n')$ and $\mathcal{S}_2(n, n')$;
3. Set $Q_2^2 = q_2^2$ in Eq. (34), i.e. omit the factor $(1 - q_1^2/M_x^2)^{-1}$ in the definition Eq. (7) of Q_2^2.

In addition, we evaluate the flux factor $\tilde{\kappa}_0$ at $q_1^2 = 0$ and $q_2^2 = 0$, i.e. we choose $\tilde{\kappa}_0 = \mathcal{W}^2$. Note that with the simplifications 1 and 2, the luminosities for the non-diagonal squared amplitudes, $\mathcal{L}_{TLTL}(x), \mathcal{L}_{\overline{T}L\overline{T}L}(x)$ and $\mathcal{L}_{TTTT}(x)$ vanish.

With these simplifications the integrals over q_1^2 and Q_2^2 in (34) can be carried out independently and the luminosities (34) take the factorized form

$$\mathcal{L}_{kl}(x) = \int_x^1 \frac{dz}{z} P_k^1(z, M_1^2)\, P_l^2\left(\frac{x}{z}, \frac{M_2^2}{z}\right), \tag{42}$$

where $k, l = T, \overline{T}, L$ and the functions P_T^i, $P_{\overline{T}}^i$ and P_L^i are the single-vector-boson distributions of [12], explicit forms of which are

$$\begin{aligned}
P_T^j(z, M^2) &= \frac{\alpha}{2\pi}(v_j^2 + a_j^2)\frac{z}{2} \int_{-s(1-z)}^{0} \frac{d(q^2)\,(-q^2)(c_0^2+1)}{(q^2 - M^2)^2}, \\
P_{\overline{T}}^j(z, M^2) &= \frac{\alpha}{2\pi}(2v_j a_j)\, z \int_{-s(1-z)}^{0} \frac{d(q^2)\,(-q^2)\, c_0}{(q^2 - M^2)^2}, \\
P_L^j(z, M^2) &= \frac{\alpha}{2\pi}(v_j^2 + a_j^2)\, M^2 \frac{z}{2} \int_{-s(1-z)}^{0} \frac{d(q^2)\, s_0^2}{(q^2 - M^2)^2},
\end{aligned} \tag{43}$$

with

$$c_0 = \frac{2 - z + \frac{q^2}{s}}{z - \frac{q^2}{s}} \qquad \text{and} \qquad s_0 = 2\frac{\sqrt{1 - z + \frac{q^2}{s}}}{z - \frac{q^2}{s}}. \tag{44}$$

The integrals in (43) can be performed analytically and the results have been given in [12][4]. The quantities $P^j_k(z, M^2)$ are the probability densities for the emission of a vector boson with mass M from a fermion j with couplings v_j and a_j. The scaling variable z describes the invariant mass squared remaining after the emission of the vector boson V_1 from fermion 1. Since q_2^2 has been neglected in describing the emission, the center-of-mass system C of the two vector bosons is related to the center-of-mass system of vector boson V_1 and fermion 2 by a boost in the direction of the fermion 2. Therefore, the helicities of the vector boson V_1, originally defined in the center-of-mass system C, agree in the two reference systems. The same line of thought applies to the emission of vector boson V_2 from fermion 2 with the scaling variable z being replaced by x/z.

In summary, the luminosities (34) can be written as convolutions (42) of single-vector-boson distributions (43) if one neglects the mutual effects of the variation of the off-shellness of one of the vector bosons on the probability for the emission of the other vector boson. The luminosities for the off-diagonal squared amplitudes vanish in this case.

5 Leading Logarithmic Approximation

Further approximations in Eqs. (40) allow to derive simplified expressions which have often been used in the literature and are referred to as the leading logarithmic approximation (LLA). The approximation consists in neglecting the off-shell masses q_i^2 in J_{pol} and performing a high-energy limit, $s \gg M_j^2$. To be precise, with the following substitutions in (40),

$$\begin{aligned}
\ln\left(\frac{M_j^2 + s - \mathcal{W}_j^2}{M_j^2}\right) &\rightarrow \ln\left(\frac{s}{M_j^2}\right), \\
\left(1 - \frac{M_j^2}{M_j^2 + s - \mathcal{W}_j^2}\right) &\rightarrow 1, \\
\frac{q_j^2}{q_j^2 - M_j^2} &\rightarrow 1, \\
\kappa_0 &\rightarrow \mathcal{W}^2,
\end{aligned} \qquad (45)$$

one obtains

$$\begin{aligned}
\mathcal{L}_{TT}(x) \rightarrow & \left(\frac{\alpha}{2\pi}\right)^2 (v_1^2 + a_1^2)(v_2^2 + a_2^2)\frac{1}{x}\left[(2+x)^2 \ln\left(\frac{1}{x}\right) - 2(1-x)(3+x)\right] \cdot \\
& \ln\left(\frac{s}{M_1^2}\right) \ln\left(\frac{s}{M_2^2}\right),
\end{aligned}$$

[4] Also the distributions of [5] are exact for processes with only one internal vector boson which couples to the amplitude for the hard scattering subprocess like a fermion. This specific assumption in [5] is the only difference between the distributions of [5] and [12].

$$
\begin{aligned}
\mathcal{L}_{\overline{TT}}(x) &\to \left(\frac{\alpha}{2\pi}\right)^2 (2v_1a_1)(2v_2a_2)\left[(4+x)\ln\left(\frac{1}{x}\right)-4(1-x)\right]\ln\left(\frac{s}{M_1^2}\right)\ln\left(\frac{s}{M_2^2}\right),\\
\mathcal{L}_{TL}(x) &\to \left(\frac{\alpha}{2\pi}\right)^2 (v_1^2+a_1^2)(v_2^2+a_2^2)\frac{1}{x}\left[4(1+x)\ln\left(\frac{1}{x}\right)-(1-x)(7+x)\right]\ln\left(\frac{s}{M_1^2}\right),\\
\mathcal{L}_{LT}(x) &\to \left(\frac{\alpha}{2\pi}\right)^2 (v_1^2+a_1^2)(v_2^2+a_2^2)\frac{1}{x}\left[4(1+x)\ln\left(\frac{1}{x}\right)-(1-x)(7+x)\right]\ln\left(\frac{s}{M_2^2}\right),\\
\mathcal{L}_{LL}(x) &\to \left(\frac{\alpha}{2\pi}\right)^2 (v_1^2+a_1^2)(v_2^2+a_2^2)\frac{4}{x}\left[(1+x)\ln\left(\frac{1}{x}\right)-2(1-x)\right]. \qquad (46)
\end{aligned}
$$

Expressions for $\mathcal{L}_{TT}$, $\mathcal{L}_{TL}$ and $\mathcal{L}_{LL}$ have been given already in [13] and the complete set of luminosities including $\mathcal{L}_{\overline{TT}}$, $\mathcal{L}_{\overline{T}L}$ and $\mathcal{L}_{L\overline{T}}$ can be found in [14]. In a similar way, LLA expressions for single-vector-boson distributions can be obtained from the exact ones, Eq. (43). Their convolutions lead again to Eq. (46).

These formulae are obtained from the exact ones by taking into account only the contributions from the singularities at $q_j^2 \to 0$ to the q_j^2-integrals and neglecting the contribution from other regions in the q_1^2, q_2^2 integration. The choice of s in the arguments of the logarithms is arguable; many other choices are also acceptable in the leading logarithmic approximation and have been used in the literature. For example, xs as argument instead of s has been advocated in [5, 13], since the quantity $s - \mathcal{W}_2^2$ varies in the whole interval $[0, s]$ as q_1^2 varies within its limits. We have checked numerically that the LLA with this choice deviates less from the exact calculation. The deviation for $x \to 1$ can be improved by choosing $x(1-x)s$ instead of xs in the argument of the logarithms. This choice is motivated by interpreting the approximation as resulting from a zero-mass limit and noting that $s - \mathcal{W}^2 = (1-x)s$. We will use this form in our numerical examples.

Related to the different possible choices of the argument of the logarithm is the interpretation of the scaling variable x. In [3, 4, 14, 21], the scaling variable x was defined as the ratio of the vector-boson energy and the energy of the fermion from which it is emitted. With this definition, the relation $\hat{s} = xs$ between the fermion scattering energy and the subprocess energy only holds strictly if the vector-boson is emitted in the forward direction. These versions thus imply a small angle approximation. In addition, the mentioned distributions differ by various additional approximations. The distributions of [4] neglect terms of the order $\mathcal{O}(M_i^2/s)$. In [3, 21][5], the calculation was performed using a longitudinal polarization vector for on-shell vector bosons, whereas in [14] $\epsilon^\mu(0)$ was defined taking into account that the vector bosons have off-shell masses $-q_j^2$. This and a more sophisticated assumption concerning the off-shell behaviour of the hard scattering cross section in [14] is the reason for the difference between the distribution functions for longitudinal polarization in [3, 21] and [14]. The distribution for transversely polarized vector bosons in [3, 21] and [14] agree with each other (after correcting misprints in the latter reference). Of course, all distributions agree in the leading logarithmic approximation.

[5]The distributions of [21] supplement those of [3] by the distribution function $P_{\overline{T}}$ (see Eq. (43)).

6 Numerical Results

In presenting numerical results for luminosities of vector-boson pairs, we restrict ourselves to the representative case of e^+e^- annihilation. In our examples for the numerical evaluation we used $\alpha = 1/137$, $M_W = 80.2$ GeV, $M_Z = 91.2$ GeV and the fermion vector-boson couplings are determined using the weak mixing angle as given by $\cos\theta_W = M_W/M_Z$. In Figs. 2 and 3 we show the exact luminosities (40) for finding a W^+W^- pair in an e^+e^- pair of $\sqrt{s} = 2$ TeV. The luminosity $\mathcal{L}_{TT}$ for transversely polarized $W^\pm$ is the biggest one, followed by $\mathcal{L}_{TL}$ and $\mathcal{L}_{LL}$. From Fig. 3 one concludes that the non-diagonal luminosities $\mathcal{L}_{TLTL}$ and $\mathcal{L}_{TTTT}$ are comparable in size with the diagonal ones and thus can not be neglected. The parity violating luminosity $\mathcal{L}_{\overline{TT}}$ varies comparatively little with x at not too high x, and at higher x it becomes equal to the TT luminosity.

In order to estimate the improvement obtained by using the exact luminosities as compared to former simpler approaches, we show in the following series of figures ratios of the exact results and the convolutions of the exact single-vector-boson distributions from [12] as well as their LLA versions. The ratio of the convolutions (42) and the exact luminosities is shown in Fig. 4 for a W^+W^- pair in a 2 TeV e^+e^- pair. The discrepancy grows with decreasing x and is largest for transverse polarizations in which case it reaches a factor of 2.4 at $x = 0.01$. At higher energies the agreement between the two versions is better as seen in Fig. 5 where the same ratio is shown for a value $\sqrt{s} = 4$ TeV, which is a typical $q\bar{q}$ sub-process energy in pp collisions at 14 TeV. However, the ratio of the TT luminosities for $x = 0.01$ is still 1.6 (this corresponds to the production of a final state W of 400 GeV).

Fig. 6 shows the ratio of the LLA version of the luminosities, Eq. (46), and the exact formulae for W^+W^- in e^+e^- at 2 TeV. The LLA versions always overestimate the exact results by far and only for the LL luminosity at not too small values of x the LLA might be useful. We note that the disagreement at $x \to 1$ would have been larger if we had used xs instead of $x(1-x)s$ in the argument of the logarithms.

We also present some results relevant for a 500 GeV e^+e^- collider. Figs. 7 and 8 show the luminosities for a W^+W^- pair as a function of the W^+W^- pair invariant mass $\mathcal{W}$ related to x by $\mathcal{W}^2 = xs$. The luminosities reach their highest value not far from threshold. The behavior of the different polarizations with varying x is as described for the 2 TeV case. There is a resemblance between the pairs TT, $\overline{TT}$ and $TLTL$, $\overline{T}L\overline{T}L$. In both cases, the luminosity proportional to the product of vector and axial-vector coupling is smaller than its partner at low x but then joins it at high x. Finally, Figs. 9 and 10 show the luminosities for a ZZ pair. The major changes as compared to the W^+W^- case are due to the change in the vector-boson couplings, while the changes due to the different vector-boson masses are small. The ZZ luminosities are more than an order of magnitude smaller than the W^+W^- luminosities. Owing to the small vector coupling of the Z, the luminosities which are proportional to the product of vector and axial-vector coupling are negligible.

In summary, only the luminosities for longitudinally polarized vector boson pairs in regions of high $\sqrt{s}$ and x might be described by the convolutions or the LLA. For luminosities involving transverse polarizations, neither of these two approximations reproduces the exact calculations with a reasonable accuracy. The disagreement becomes worse with decreasing x and decreasing $\sqrt{s}$.

To obtain luminosities relevant for deep-inelastic lepton nucleon scattering or for processes at hadron colliders, one would have to adjust the factors in (40) containing the vector and axial-vector coupling constants and, in addition, to fold the luminosities with quark distribution functions. This would result in luminosities for vector-boson pairs in an ep, pp or $p\bar{p}$ initial state.

7 Conclusion

We have derived exact distribution functions for a pair of vector bosons inside a pair of fermions. In contrast to previously used approximations, our distributions take into account the mutual influence of the emission of one boson on the emission of the other. The commonly used leading logarithmic approximation and a convolution of exact distribution functions for single vector bosons inside fermions are obtained if one neglects regions in phase space in which the virtual vector bosons have four-momenta squared much larger than their squared masses. We have shown that for transverse polarizations of the vector bosons, these approximations do not reproduce the exact calculation with a reasonable accuracy.

Our results are obtained from an exact calculation of a subset of Feynman diagrams without the need to introduce any approximation except specific assumptions for the off-shell behaviour of vector-boson scattering cross sections. A different off-shell behaviour could be taken into account in our formalism without additional complications. Of course, in order to obtain complete predictions for cross sections of vector-boson production in e^+e^- or hadron colliders, one would have to add contributions from Feynman diagrams which are not of the type as shown in Fig. 1, as for example $q\bar{q}$ annihilation or bremsstrahlung processes. These additional contributions might become particularly important at smaller energies.

Finally one should note that we did not attempt to take into account any kind of experimental cuts on kinematical variables for final state particles, like transverse momenta or rapidities. These cuts would, first of all, enter in the expressions for the vector-boson scattering cross sections. As far as experimental cuts on final state momenta imply restrictions also for the momentum transfers q_i^2, or the scale variable x, it would be straightforward to modify our expressions for the luminosities accordingly.

A Breit-Systems and Polarization Vectors

A.1 Definition of Reference Frames

The four-momenta in the center-of-mass system C of V_1 and V_2 are

$$\left(q_1^C\right)^\mu = (k_0; 0, 0, K), \qquad \left(q_2^C\right)^\mu = (q_0; 0, 0, -K), \tag{47}$$

with $k_0 = (\mathcal{W}^2 + q_1^2 - q_2^2)/2\mathcal{W}$ and $q_0 = (\mathcal{W}^2 - q_1^2 + q_2^2)/2\mathcal{W}$. For simplicity, we assume that the final state W produced via the 2-boson process allows to specify the x- and y-axes of a coordinate system. If the state W decays into n particles with momenta k_i, we choose this system such that the y-component of one specific four-momentum, say k_s, of the set of k_i vanishes and its x-component is non-negative.

We define two Breit systems, a system B_1 in which q_1 has only a non-vanishing z-component and $\vec{l}_2$ points in the negative z-direction, and a system B_2 in which q_2 has only a non-zero z-component and $\vec{q}_1$ points in the negative z-direction. The four-momenta in B_1 are

$$\begin{aligned}
\left(l_1^{B_1}\right)^\mu &= \frac{\sqrt{-q_1^2}}{2}(c_h; -s_h\cos\varphi_1, -s_h\sin\varphi_1, 1), \\
\left(p_1^{B_1}\right)^\mu &= \frac{\sqrt{-q_1^2}}{2}(c_h; -s_h\cos\varphi_1, -s_h\sin\varphi_1, -1), \\
\left(q_1^{B_1}\right)^\mu &= (0; 0, 0, \sqrt{-q_1^2}), \\
\left(l_2^{B_1}\right)^\mu &= \frac{\mu_X}{2\sqrt{-q_1^2}}(1; 0, 0, -1), \\
p'^\mu \equiv p_W^\mu + p_2^\mu &= \frac{1}{2\sqrt{-q_1^2}}(\mu_X; 0, 0, -M_X^2 - q_1^2),
\end{aligned} \tag{48}$$

with $c_h = \frac{2s}{\mu_X} - 1$, $s_h = \sqrt{c_h^2 - 1} = 2\frac{\sqrt{s}}{\mu_X}\sqrt{s - \mu_X}$, and $\mu_X = M_X^2 - q_1^2$. The overall azimuth of the system is defined by choosing the y-component of $q_2^{B_1}$ to be zero and its x-component non-negative, so that

$$\left(q_2^{B_1}\right)^\mu = \left(q_0'; \frac{\sqrt{-q_2^2}\beta}{\mu_X}, 0, -\frac{\nu}{\sqrt{-q_1^2}}\right), \tag{49}$$

with $q_0' = \frac{1}{\sqrt{-q_1^2}}\left(\nu - \frac{q_1^2 q_2^2}{\mu_X}\right)$ and $\beta = \sqrt{\mu_X^2 - 2\nu\mu_X + q_1^2 q_2^2}$.

The four-momenta in B_2 are

$$\begin{aligned}
\left(l_2^{B_2}\right)^\mu &= \frac{\sqrt{-q_2^2}}{2}(c_h'; -s_h'\cos\varphi_2, -s_h'\sin\varphi_2, 1), \\
\left(p_2^{B_2}\right)^\mu &= \frac{\sqrt{-q_2^2}}{2}(c_h'; -s_h'\cos\varphi_2, -s_h'\sin\varphi_2, -1),
\end{aligned}$$

$$
\begin{aligned}
\left(q_2^{B_2}\right)^\mu &= \left(0; 0, 0, \sqrt{-q_2^2}\right), \\
\left(q_1^{B_2}\right)^\mu &= \frac{1}{2\sqrt{-q_2^2}}(\kappa; 0, 0, -2\nu), \\
\left(p_W^{B_2}\right)^\mu &= \frac{1}{2\sqrt{-q_2^2}}(\kappa; 0, 0, -2\mathcal{W}\, q_0),
\end{aligned}
\tag{50}
$$

with $c_h' = \frac{2}{\kappa}(\mu_X - \nu)$ and $s_h' = \sqrt{(c_h')^2 - 1} = \frac{2\beta}{\kappa}$. The overall azimuth of the system B_2 is defined by choosing the y-component of the same four-momentum k_s as employed in defining the system C equal to zero and its x-component non-negative.

A.2 Polarization Vectors

The polarization vectors for the helicity eigenstates of the vector bosons V_j in the system C using the Jacob and Wick phase conventions are

$$
\begin{aligned}
\left(\epsilon_1^C\right)^\mu(\pm) &= \frac{1}{\sqrt{2}}(0; \mp 1, -i, 0), \\
\left(\epsilon_1^C\right)^\mu(0) &= \frac{1}{\sqrt{-q_1^2}}(K; 0, 0, k_0), \\
\left(\epsilon_2^C\right)^\mu(\pm) &= \frac{1}{\sqrt{2}}(0; \pm 1, -i, 0), \\
\left(\epsilon_2^C\right)^\mu(0) &= \frac{1}{\sqrt{-q_2^2}}(-K; 0, 0, q_0).
\end{aligned}
\tag{51}
$$

By applying an appropriate coordinate transformation, the polarization vectors for V_1 in the system B_1 are found to be

$$
\begin{aligned}
\left(\epsilon_1^{B_1}\right)^\mu(\pm) &= \frac{1}{\sqrt{2}} e^{\mp i\varphi_2}\left(\mp\tilde{\sigma} s_y; \mp\frac{\sqrt{-q_1^2}q_0'}{K\mathcal{W}}, -i, 0\right), \\
\left(\epsilon_1^{B_1}\right)^\mu(0) &= \frac{\sqrt{-q_1^2}}{K\mathcal{W}}\left(q_0'; \frac{\sqrt{-q_2^2}\beta}{\mu_X}, 0, 0\right),
\end{aligned}
\tag{52}
$$

with $\tilde{\sigma} s_y = \sqrt{q_1^2 q_2^2}\beta/(\mu_X K\mathcal{W})$. Likewise, the polarization vectors for V_2 in B_2 are found to be

$$
\begin{aligned}
\left(\epsilon_2^{B_2}\right)^\mu(\pm) &= \frac{1}{\sqrt{2}}(0; \pm 1, i, 0), \\
\left(\epsilon_2^{B_2}\right)^\mu(0) &= (-1; 0, 0, 0).
\end{aligned}
\tag{53}
$$

B Five-Fold Differential Luminosities

Here we give explicit expressions needed to determine the five-fold differential luminosities K_{pol} of Eq. (24). The helicity tensors $\mathcal{C}_j(m, m')$ and $\mathcal{S}_j(m, m')$, defined in Eqs. (15) and

(16), are evaluated most easily in their respective Breit systems B_j using the expressions (48) and (50) for the four-momenta and the expressions (52) and (53) for the polarization vectors. The results are:

$$
\begin{aligned}
\mathcal{C}_1(++) &= -\frac{q_1^2}{4}\left[c_h^2 + 1 + \frac{4 q_1^2 q_2^2\,\beta^2}{\mu_X^2\kappa^2}(c_h^2 + s_h^2\cos^2\varphi_1)\right.\\
&\quad\left. + \frac{8\, c_h\, s_h \sqrt{-q_1^2}\sqrt{-q_2^2}\,\beta}{\mu_X^2\kappa^2}(\nu\mu_X - q_1^2 q_2^2)\cos\varphi_1\right],\\
\mathcal{C}_1(00) &= -\frac{q_1^2}{2}\left[s_h^2 + \frac{4 q_1^2 q_2^2\,\beta^2}{\mu_X^2\kappa^2}(c_h^2 + s_h^2\cos^2\varphi_1)\right.\\
&\quad\left. - \frac{8\, c_h\, s_h \sqrt{-q_1^2}\sqrt{-q_2^2}\,\beta}{\mu_X^2\kappa^2}(\nu\mu_X - q_1^2 q_2^2)\cos\varphi_1\right],\\
\mathcal{C}_1(+-) &= \frac{q_1^2}{2}\left[\frac{2 q_1^2 q_2^2\,\beta^2}{\mu_X^2\kappa^2}(c_h^2 + s_h^2\cos^2\varphi_1) + s_h^2\left(\cos^2\varphi_1 - \frac{1}{2}\right)\right.\\
&\quad + \frac{4\, c_h\, s_h \sqrt{-q_1^2}\sqrt{-q_2^2}\,\beta}{\mu_X^2\kappa^2}(\nu\mu_X - q_1^2 q_2^2)\cos\varphi_1\\
&\quad\left. - 2\,i\frac{s_h}{\mu_X\kappa}(c_h\sqrt{-q_1^2}\sqrt{-q_2^2}\,\beta + s_h\,(\nu\mu_X - q_1^2 q_2^2)\cos\varphi_1)\sin\varphi_1\right],\\
\mathcal{C}_1(+0) &= \frac{q_1^2}{\sqrt{2}}\left[\frac{2\sqrt{-q_1^2}\sqrt{-q_2^2}\,\beta}{\mu_X^2\kappa^2}(\nu\mu_X - q_1^2 q_2^2)(c_h^2 + s_h^2\cos^2\varphi_1)\right.\\
&\quad + \frac{2\, c_h\, s_h}{\mu_X^2\kappa^2}((\nu\mu_X - q_1^2 q_2^2)^2 + q_1^2 q_2^2\,\beta^2)\cos\varphi_1\\
&\quad\left. - i\frac{s_h}{\mu_X\kappa}(c_h(\nu\mu_X - q_1^2 q_2^2) + s_h\,\sqrt{-q_1^2}\sqrt{-q_2^2}\,\beta\cos\varphi_1)\sin\varphi_1\right],\\
\mathcal{S}_1(++) &= -\frac{q_1^2}{2}\left[2\,\frac{\nu\mu_X - q_1^2 q_2^2}{\mu_X\kappa}\,c_h + 2\,\frac{s_h\,\sqrt{-q_1^2}\sqrt{-q_2^2}\,\beta}{\mu_X\kappa}\cos\varphi_1\right],\\
\mathcal{S}_1(+0) &= \frac{q_1^2}{\sqrt{2}}\left[\frac{c_h\,\sqrt{-q_1^2}\sqrt{q_2^2}\,\beta}{\mu_X\kappa} + \frac{s_h}{\mu_x\kappa}(\nu\mu_X - q_1^2 q_2^2)\cos\varphi_1 - i\frac{s_h}{2}\sin\varphi_1\right];
\end{aligned}
\tag{54}
$$

$$
\begin{aligned}
\mathcal{C}_2(++) &= -\frac{q_2^2}{4}\left((c_h')^2 + 1\right),\\
\mathcal{C}_2(00) &= -\frac{q_2^2}{2}(s_h')^2,\\
\mathcal{C}_2(+-) &= \frac{q_2^2}{4}(s_h')^2,\\
\mathcal{C}_2(+0) &= \frac{q_2^2}{2\sqrt{2}}\,c_h'\,s_h',\\
\mathcal{S}_2(++) &= -\frac{q_2^2}{2}c_h',\\
\mathcal{S}_2(+0) &= \frac{q_2^2}{2\sqrt{2}}s_h'.
\end{aligned}
\tag{55}
$$

References

[1] E. Fermi, Z. Phys. 29 (1924) 315;
C. Weizsäcker, Z. Phys. 88 (1934) 612;
E. Williams, Phys. Rev. 45 (1934) 729.

[2] G. L. Kane, Proc. "Physics of the XXIst century", Tucson, Arizona, Dec. 1983.

[3] S. Dawson, Nucl. Phys. B249 (1985) 42.

[4] G. L. Kane, W. W. Repko, and W. B. Rolnick, Phys. Lett. 148B (1984) 367.

[5] J. Lindfors, Z. Phys. C28 (1985) 427.

[6] R. Cahn and S. Dawson, Phys. Lett. 136B (1984) 196; Err. ibid. 138B (1984) 464.

[7] R. Cahn, Nucl. Phys. B255 (1985) 341; Err. ibid. B262 (1985) 744;
D. A. Dicus, S. Willenbrock, Phys. Rev. D32 (1985) 1642;
M. J. Duncan, G. L. Kane and W. W. Repko, Nucl. Phys. B272 (1986) 517;
G. Altarelli, B. Mele, F. Pitalli, Nucl. Phys. B287 (1987) 205;
M. C. Bento and C.-H. Llewellyn-Smith, Nucl. Phys. B289 (1987) 36;
D. A. Dicus, K. J. Kallianpur, and S. D. Willenbrock, Phys. Lett. B200 (1988) 187.

[8] S. Willenbrock and D. A. Dicus, Phys. Rev. D34 (1986) 155;
J. Lindfors, Z. Phys. C33 (1987) 385;
S. Dawson, G. L. Kane, C. P. Yuan, and S. Willenbrock, Proc. of the 1986 Summer Study on Physics of the Superconducting Super Collider, Snowmass, CO, Jun 23 – Jul 11, 1986, p. 235;
S. Dawson and S. Willenbrock, Nucl. Phys. B284 (1987) 449;
R. P. Kauffman, Phys. Rev. D41 (1990) 3343.

[9] M. Chanowitz and M. K. Gaillard, Phys. Lett. 142B (1984) 85;
M. Chanowitz and M. K. Gaillard, Nucl. Phys. B261 (1985) 379;
J. F. Gunion, J. Kalinowski, A. Tofighi-Niaki, A. Abbasabadi, and W. Repko, Proc. of the 1986 Summer Study on Physics of the Superconducting Super Collider, Snowmass, CO, Jun 23 – Jul 11, 1986, p. 156;
J. F. Gunion, J. Kalinowski, and A. Tofighi-Niaki, Phys. Rev. Lett. 57 (1986) 2351;
B. Mele, in Proc. of La Thuile Workshop on Physics at Future Accelerators, La Thuile, Italy, Jan 7 – 13, 1987;
A. Abbasabadi, W. W. Repko, D. A. Dicus, and R. Vega, Phys. Rev. D38 (1988) 2770;
D. A. Dicus, S. L. Wilson, and R. Vega, Phys. Lett. B192 (1987) 231;
M. Kuroda, F. M. Renard and D. Schildknecht, Z. Phys. C40 (1988) 575;
G. J. Gounaris and F. M. Renard, Z. Phys. C59 (1993) 143.

[10] D. A. Dicus and R. Vega, Phys. Rev. Lett. 57 (1986) 1110.

[11] A. Abbasabadi and W. W. Repko, Proc. of the 1986 Summer Study on Physics of the Superconducting Super Collider, Snowmass, CO, Jun 23 – Jul 11, 1986, p. 154;
J. P. Ralston and F. Olness, Proc. of the 1986 Summer Study on Physics of the Superconducting Super Collider, Snowmass, CO, Jun 23 – Jul 11, 1986, p. 191;
A. Abbasabadi and W. W. Repko, Phys. Rev. D36 (1987) 289;
A. Abbasabadi and W. W. Repko, Phys. Rev. D50 (1994) 5704;
W. W. Repko and W.-K. Tung, Proc. of the 1986 Summer Study on Physics of the Superconducting Super Collider, Snowmass, CO, Jun 23 – Jul 11, 1986, p. 159.

[12] P. W. Johnson, F. I. Olness, and Wu-Ki Tung, Proc. of the 1986 Summer Study on Physics of the Superconducting Super Collider, Snowmass, CO, Jun 23 – Jul 11, 1986, p. 164;
P. W. Johnson, F. I. Olness, and Wu-Ki Tung, Phys. Rev. D36 (1987) 291.

[13] J. Lindfors, Z. Phys. C35 (1987) 355.

[14] M. Capdequi Peyranére, J. Layssac, H. Leveque, G. Moultaka and F. M. Renard, Z. Phys. C41 (1988) 99.

[15] A. Dobrovolskaia and V. Novikov, preprint LPTHE 93/14 (April 1993).

[16] V. M. Budnev, I. F. Ginzburg, G. V. Meledin, and V. G. Serbo, Phys. Rep. 15C (1974) 183;
N. S. Craigie, K. Hidaka, M. Jacob and F. M. Renard, Phys. Rep. 99 (1983) 69.

[17] G. Bonneau, M. Gourdin, and F. Martin, Nucl. Phys. B54 (1973) 573;
G. Bonneau and F. Martin, Nucl. Phys. B68 (1974) 367.

[18] R. Kleiss and J. W. Stirling, Phys. Lett. B182 (1986) 75.

[19] Z. Kunszt and D. E. Soper, Nucl. Phys. B296 (1988) 253.

[20] S. Cortese and R. Petronzio, Phys. Lett. B276 (1992) 203.

[21] R. M. Godbole and F. Olness, Int. J. Mod. Phys. A2 (1987) 1025;
R. M. Godbole and S. D. Rindani, Phys. Lett. B190 (1987) 192; Z. Phys. C36 (1987) 395.

Figure Caption

Fig. 2: Luminosities $\mathcal{L}_{TT}(x)$, $\mathcal{L}_{\overline{TT}}(x)$, $\mathcal{L}_{LT}(x)$, and $\mathcal{L}_{LL}(x)$ for a W^+W^- pair in e^+e^- collisions at $\sqrt{s} = 2$ TeV.

Fig. 3: Luminosities $\mathcal{L}_{TLTL}(x)$, $\mathcal{L}_{\overline{T}L\overline{T}L}(x)$, and $\mathcal{L}_{TTTT}(x)$ for a W^+W^- pair in e^+e^- collisions at $\sqrt{s} = 2$ TeV.

Fig. 4: Ratios of the convolutions of single-vector-boson distributions Eq. (42) and the exact luminosities for $pol = TT$, $\overline{TT}$, LT and LL for a W^+W^- pair in e^+e^- collisions at $\sqrt{s} = 2$ TeV.

Fig. 5: Ratios of the convolutions of single-vector-boson distributions Eq. (42) and the exact luminosities for $pol = TT$, $\overline{TT}$, LT and LL for a W^+W^- pair in e^+e^- collisions at $\sqrt{s} = 4$ TeV.

Fig. 6: Ratios of the leading logarithmic approximation for vector-boson pair luminosities Eq. (46) and the exact luminosities for $pol = TT$, $\overline{TT}$, LT and LL for a W^+W^- pair in e^+e^- collisions at $\sqrt{s} = 2$ TeV.

Fig. 7: Luminosities $\mathcal{L}_{TT}(\mathcal{W})$, $\mathcal{L}_{\overline{TT}}(\mathcal{W})$, $\mathcal{L}_{LT}(\mathcal{W})$, and $\mathcal{L}_{LL}(\mathcal{W})$ for a W^+W^- pair in e^+e^- collisions at $\sqrt{s} = 500$ GeV.

Fig. 8: Luminosities $\mathcal{L}_{TLTL}(\mathcal{W})$, $\mathcal{L}_{\overline{T}L\overline{T}L}(\mathcal{W})$, and $\mathcal{L}_{TTTT}(\mathcal{W})$ for a W^+W^- pair in e^+e^- collisions at $\sqrt{s} = 500$ GeV.

Fig. 9: Luminosities $\mathcal{L}_{TT}(\mathcal{W})$, $\mathcal{L}_{\overline{TT}}(\mathcal{W})$, $\mathcal{L}_{LT}(\mathcal{W})$, and $\mathcal{L}_{LL}(\mathcal{W})$ for a ZZ pair in e^+e^- collisions at $\sqrt{s} = 500$ GeV.

Fig. 10: Luminosities $\mathcal{L}_{TLTL}(\mathcal{W})$, $\mathcal{L}_{\overline{T}L\overline{T}L}(\mathcal{W})$, and $\mathcal{L}_{TTTT}(\mathcal{W})$ for a ZZ pair in e^+e^- collisions at $\sqrt{s} = 500$ GeV.

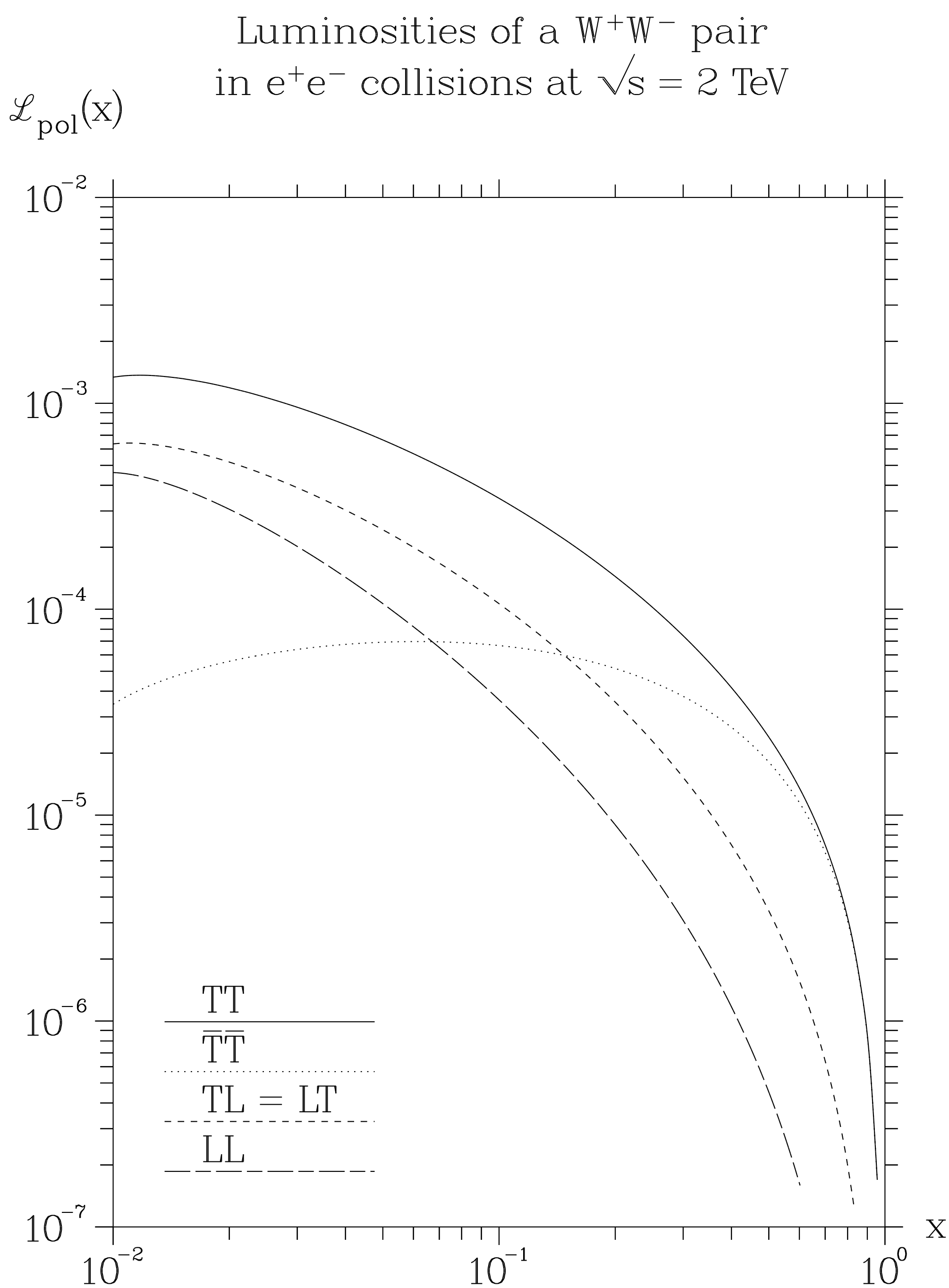

Figure 2: Luminosities $\mathcal{L}_{TT}(x)$, $\mathcal{L}_{\overline{TT}}(x)$, $\mathcal{L}_{LT}(x)$, and $\mathcal{L}_{LL}(x)$ for a W^+W^- pair in e^+e^- collisions at $\sqrt{s} = 2$ TeV.

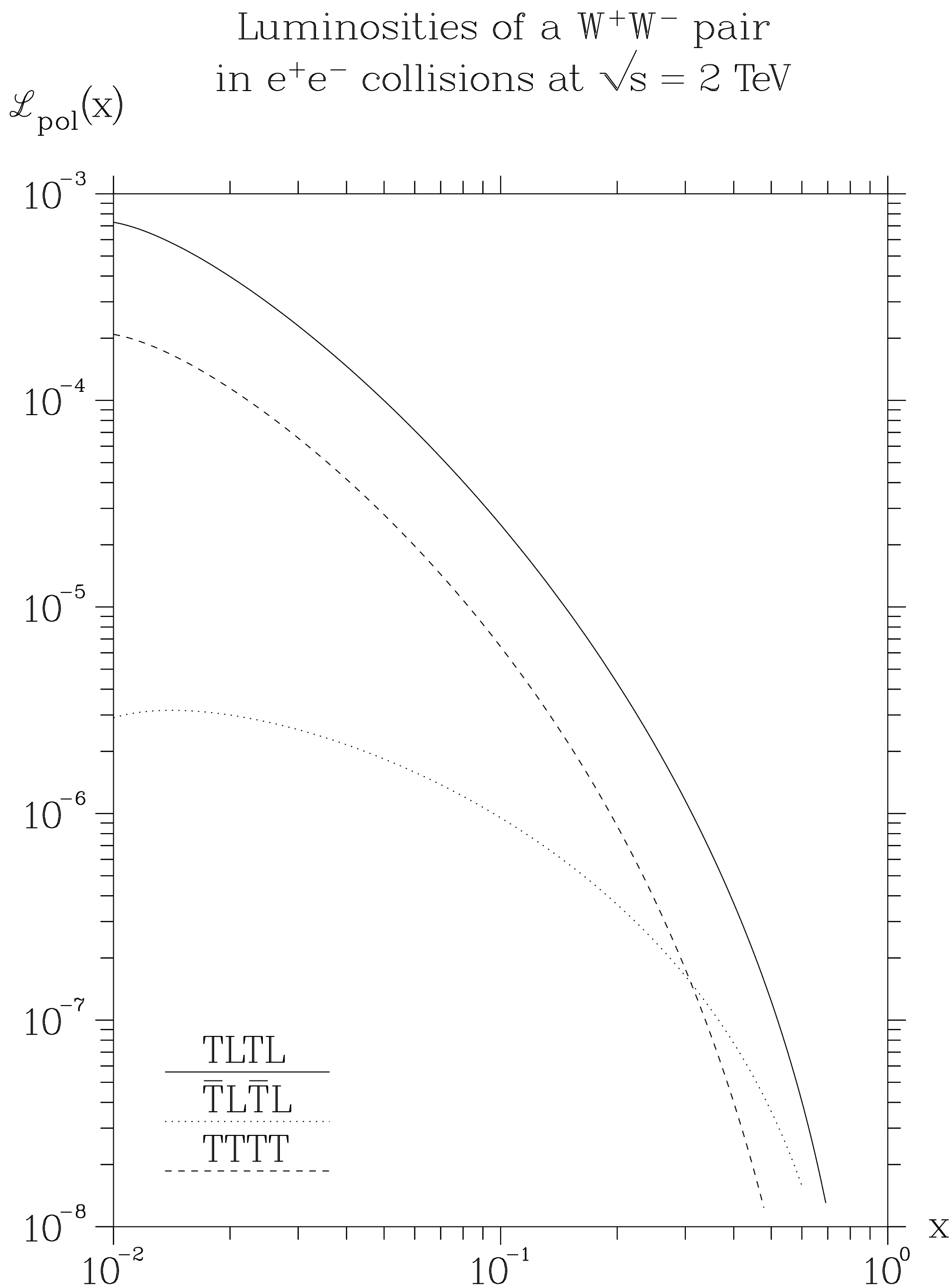

Figure 3: Luminosities $\mathcal{L}_{TLTL}(x)$, $\mathcal{L}_{\overline{T}L\overline{T}L}(x)$, and $\mathcal{L}_{TTTT}(x)$ for a W^+W^- pair in e^+e^- collisions at $\sqrt{s} = 2$ TeV.

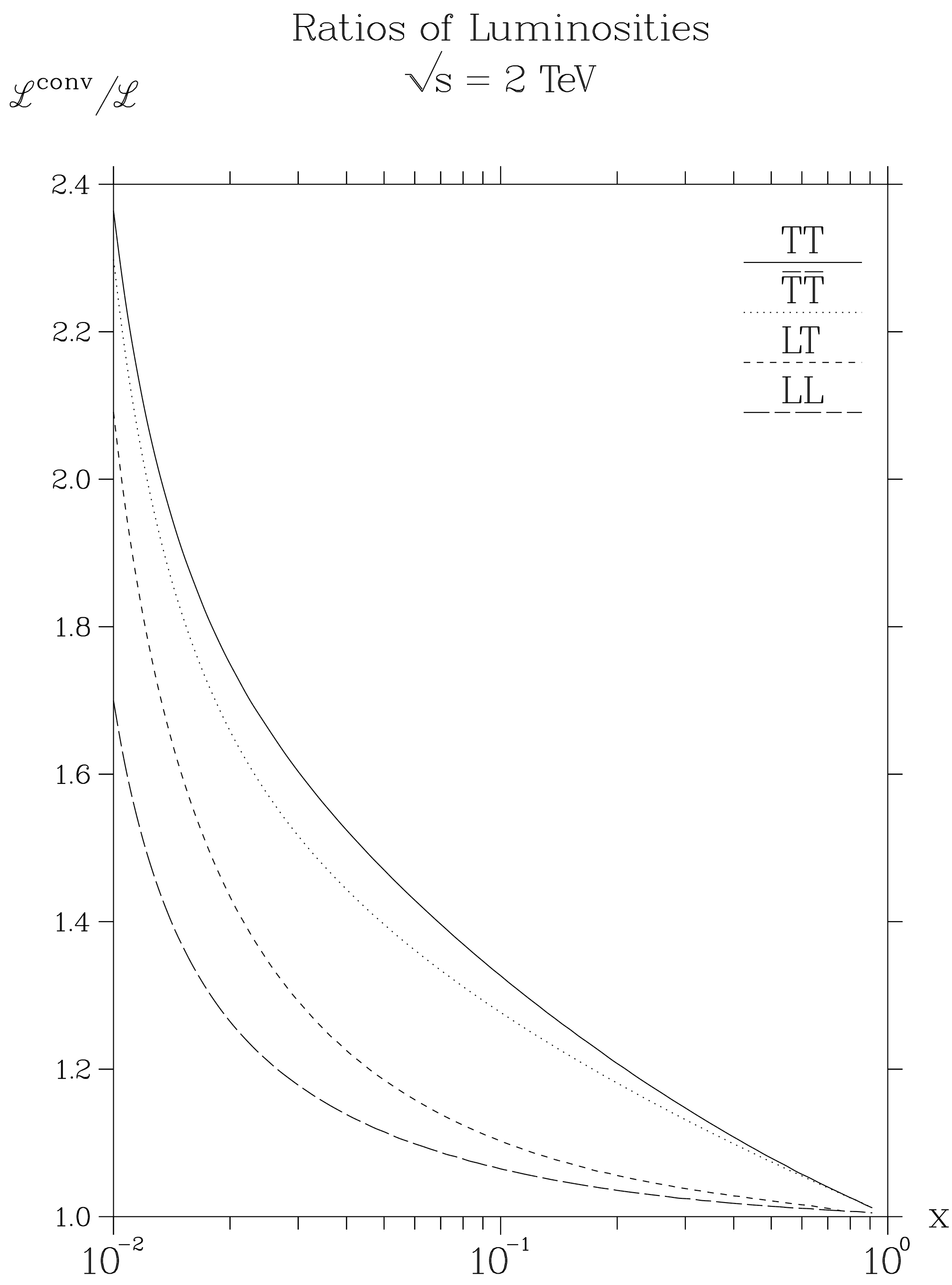

Figure 4: Ratios of the convolutions of single-vector-boson distributions Eq. (42) and the exact luminosities for $pol = TT$, $\overline{TT}$, LT and LL for a W^+W^- pair in e^+e^- collisions at $\sqrt{s} = 2$ TeV.

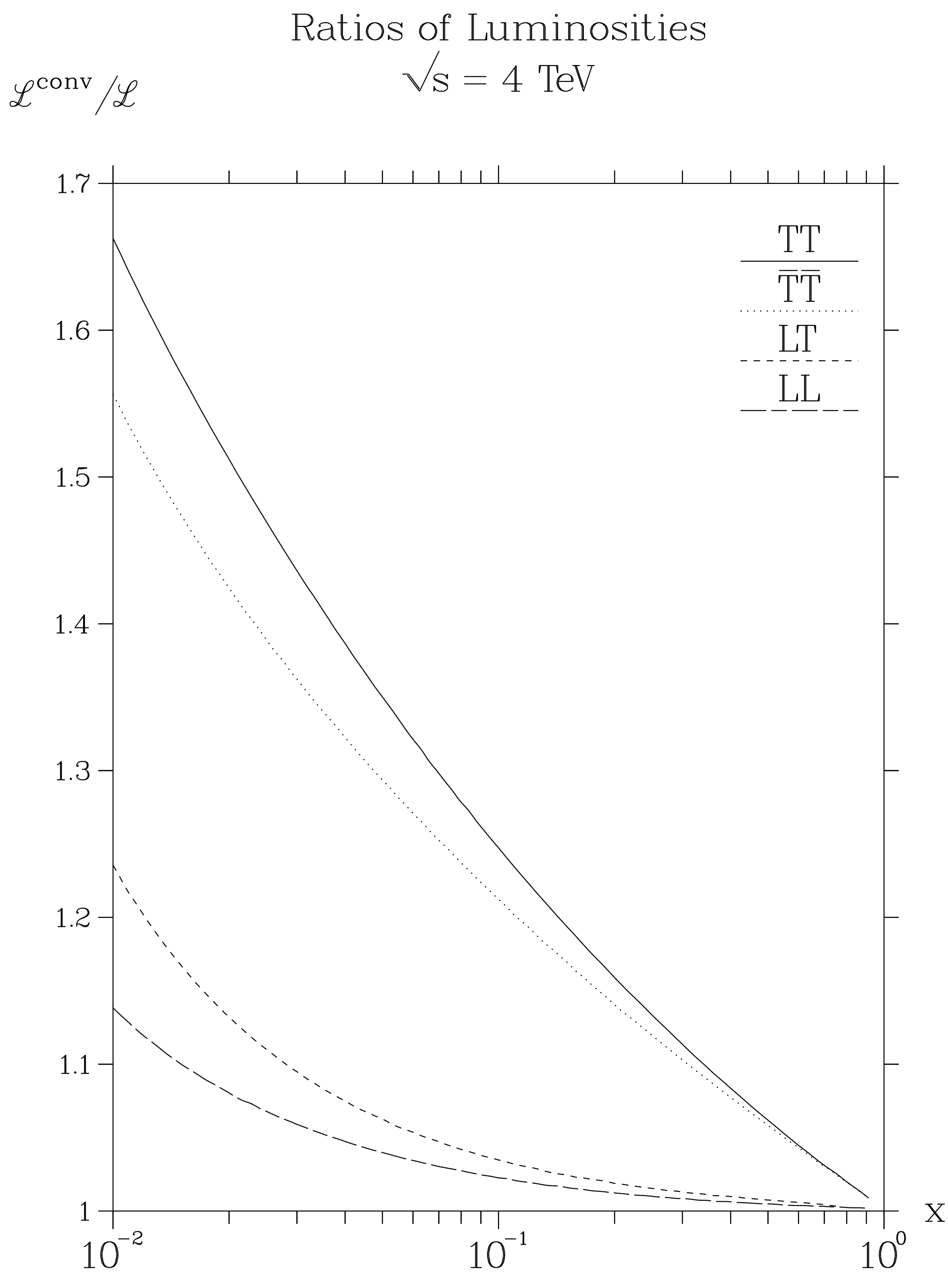

Figure 5: Ratios of the convolutions of single-vector-boson distributions Eq. (42) and the exact luminosities for $pol = TT$, $\overline{TT}$, LT and LL for a W^+W^- pair in e^+e^- collisions at $\sqrt{s} = 4$ TeV.

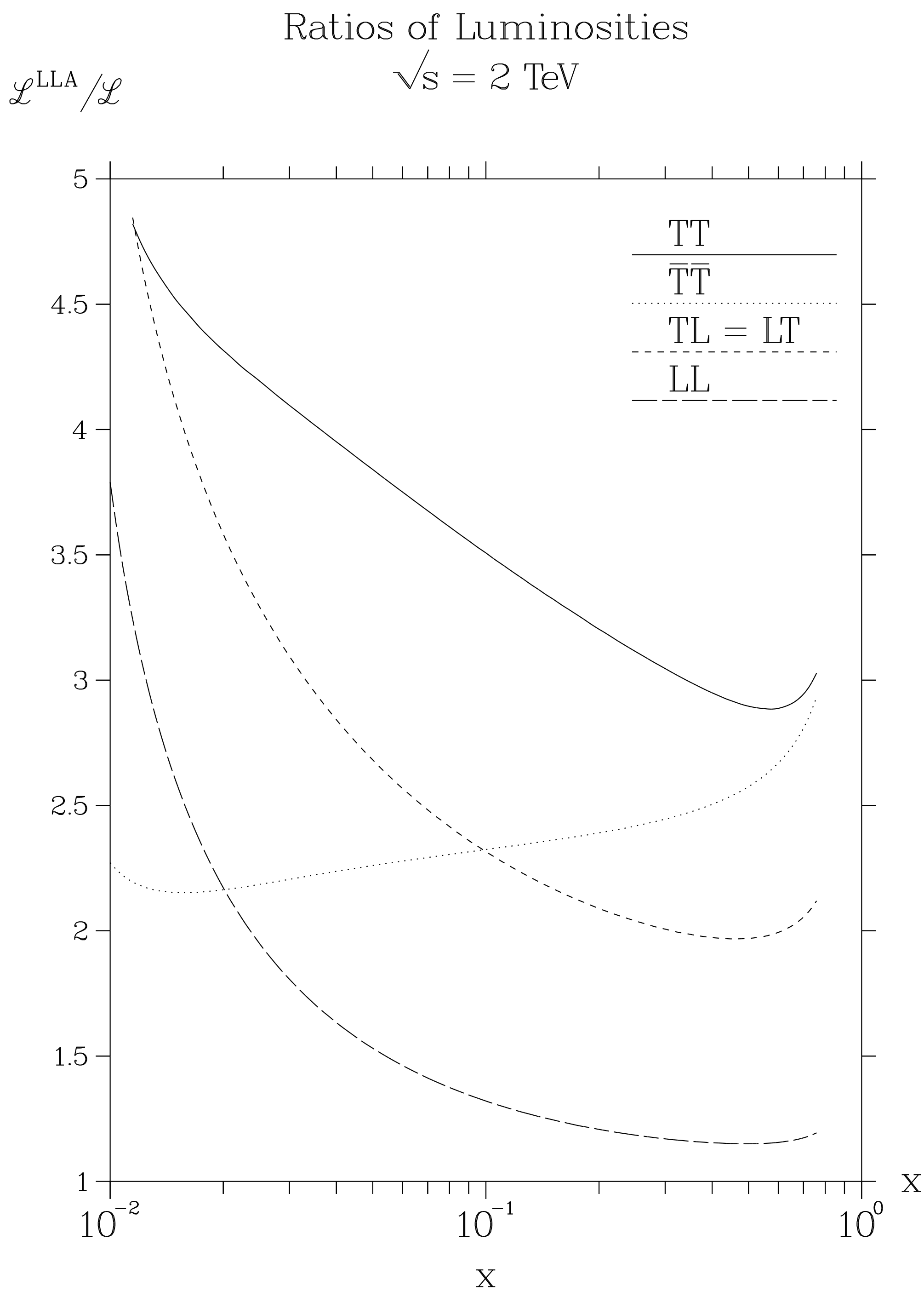

Figure 6: Ratios of the leading logarithmic approximation for vector-boson pair luminosities Eq. (46) and the exact luminosities for $pol = TT$, $\overline{TT}$, LT and LL for a W^+W^- pair in e^+e^- collisions at $\sqrt{s} = 2$ TeV.

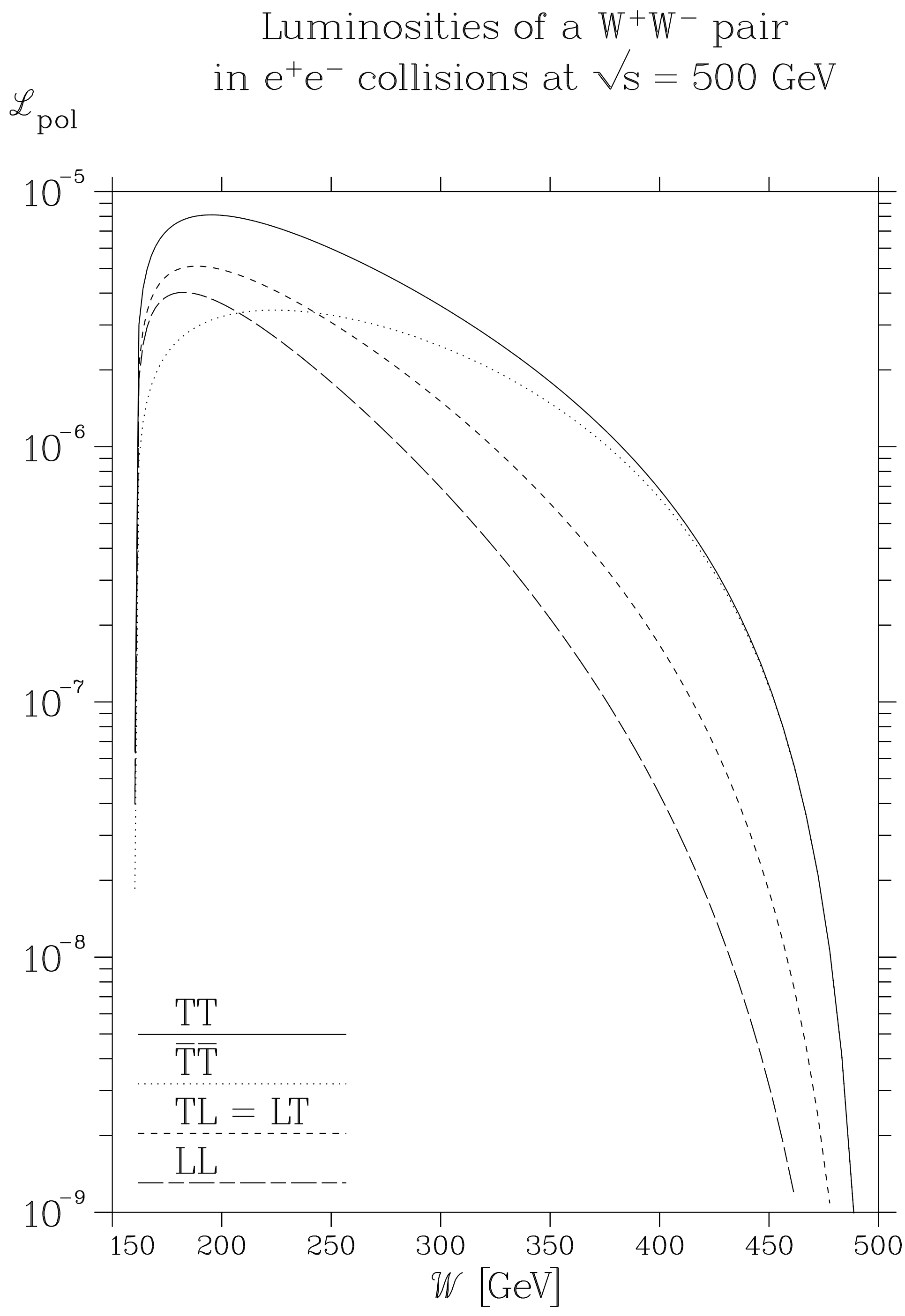

Figure 7: Luminosities $\mathcal{L}_{TT}$, $\mathcal{L}_{\overline{TT}}$, $\mathcal{L}_{LT}$, and $\mathcal{L}_{LL}$ as a function of the boson pair invariant mass $\mathcal{W}$, $\mathcal{W}^2 = xs$, for a W^+W^- pair in e^+e^- collisions at $\sqrt{s} = 500$ GeV.

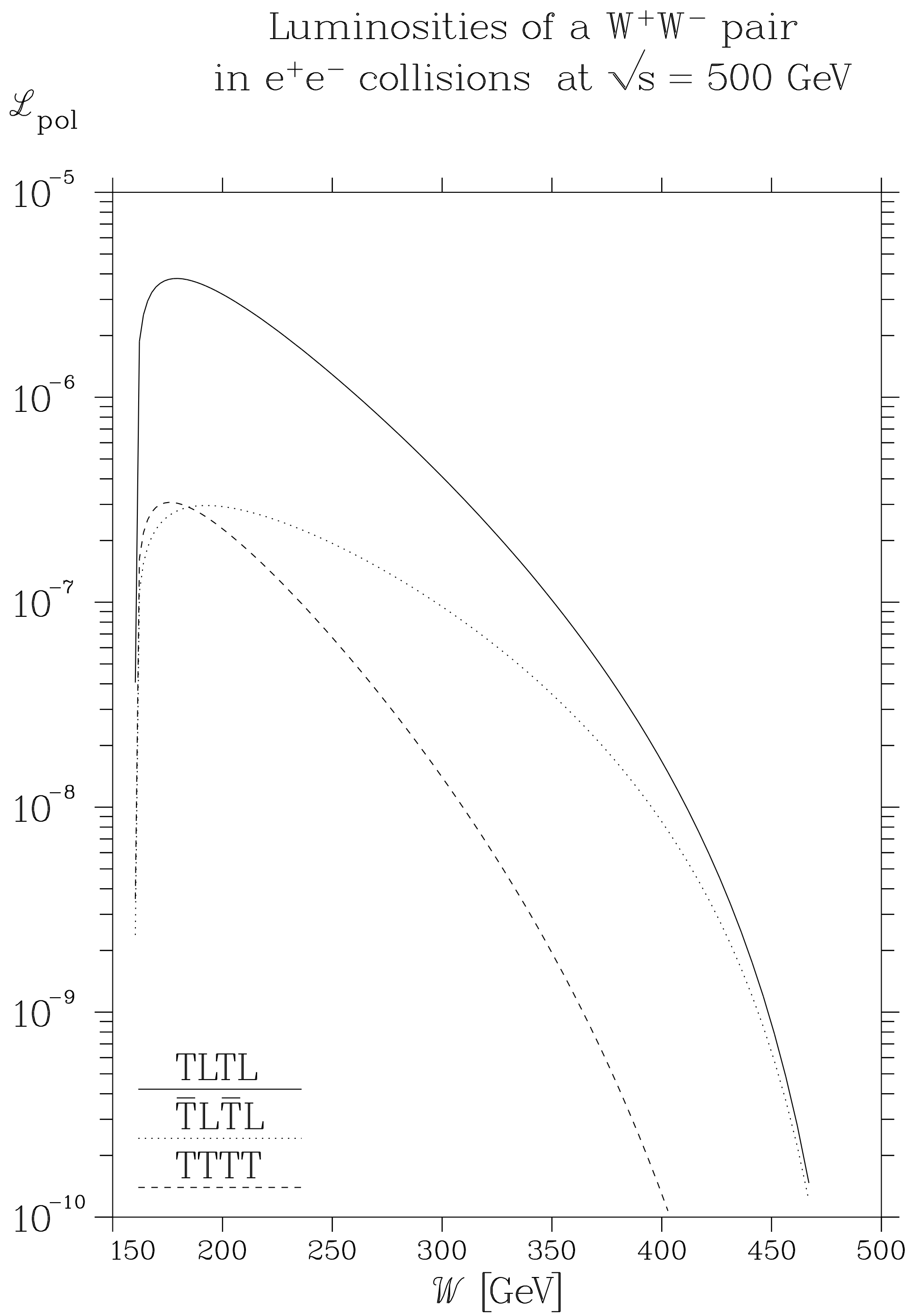

Figure 8: Luminosities $\mathcal{L}_{TLTL}$, $\mathcal{L}_{\overline{T}L\overline{T}L}$, and $\mathcal{L}_{TTTT}$ as a function of the boson pair invariant mass $\mathcal{W}$, $\mathcal{W}^2 = xs$, for a W^+W^- pair in e^+e^- collisions at $\sqrt{s} = 500$ GeV.

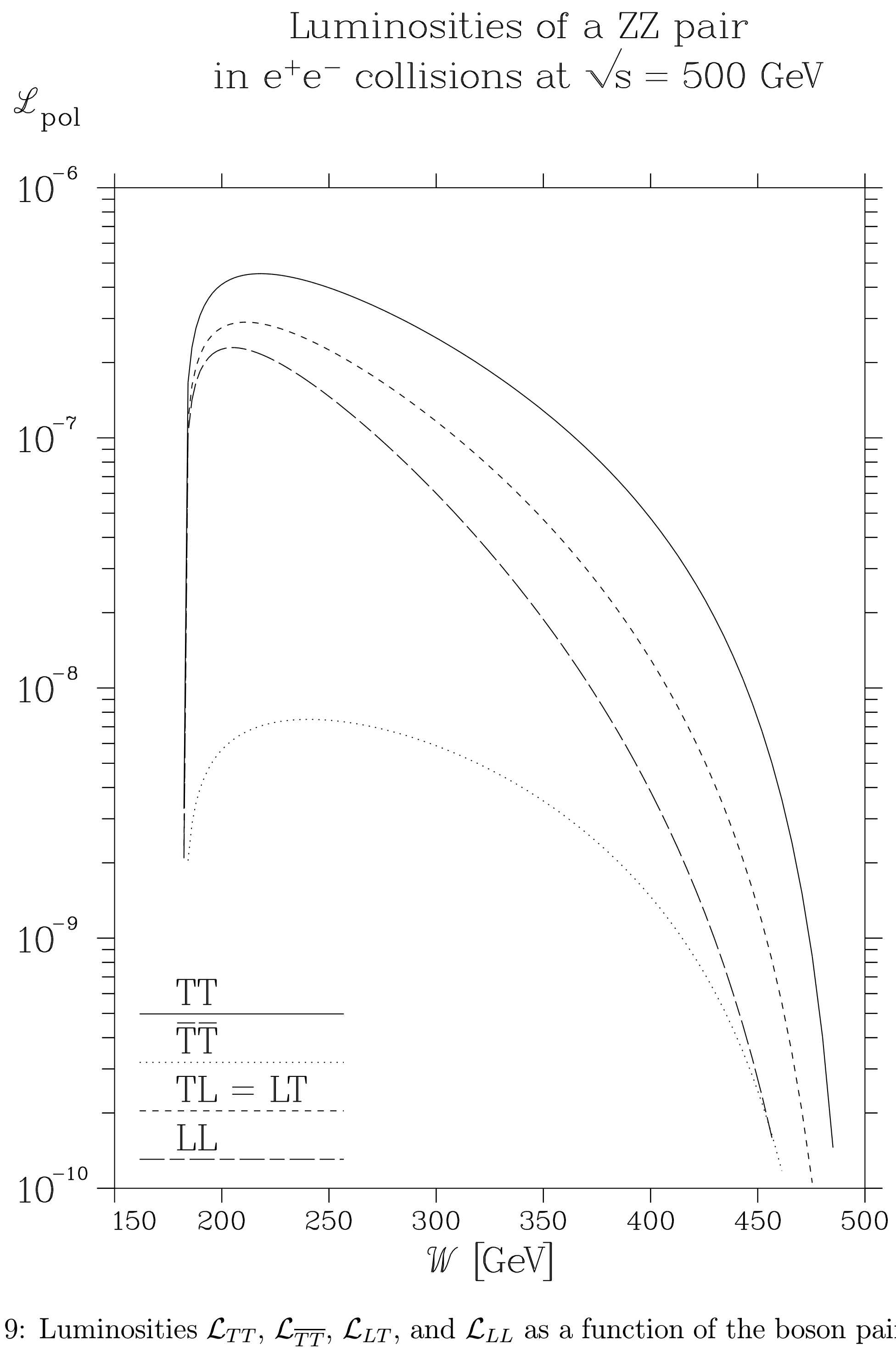

Figure 9: Luminosities $\mathcal{L}_{TT}$, $\mathcal{L}_{\overline{TT}}$, $\mathcal{L}_{LT}$, and $\mathcal{L}_{LL}$ as a function of the boson pair invariant mass $\mathcal{W}$, $\mathcal{W}^2 = xs$, for a ZZ pair in e^+e^- collisions at $\sqrt{s} = 500$ GeV.

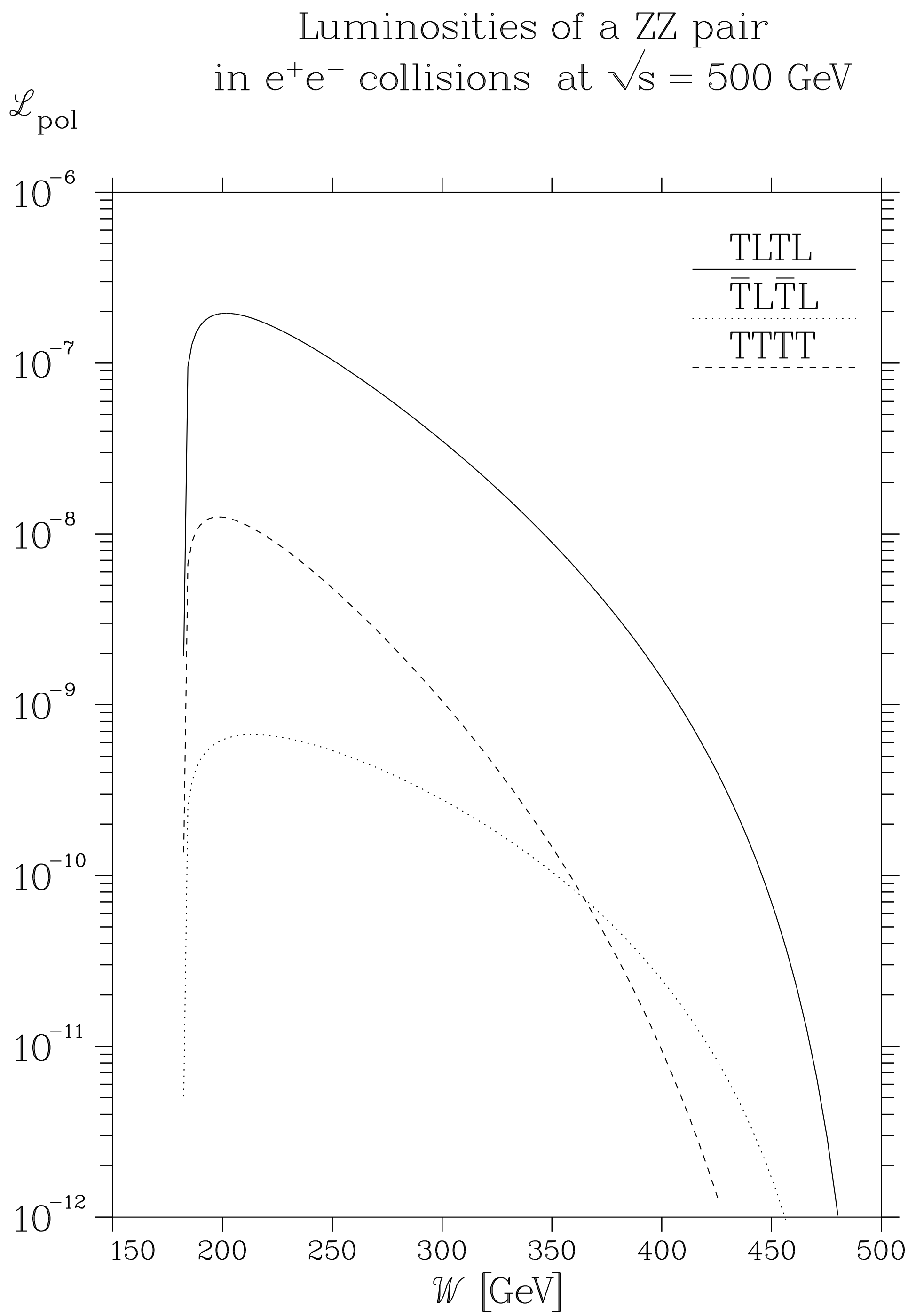

Figure 10: Luminosities $\mathcal{L}_{TLTL}$, $\mathcal{L}_{\overline{T}L\overline{T}L}$, and $\mathcal{L}_{TTTT}$ as a function of the boson pair invariant mass $\mathcal{W}$, $\mathcal{W}^2 = xs$, for a ZZ pair in e^+e^- collisions at $\sqrt{s} = 500$ GeV.