A Normal Coordinate Expansion for the Gauge Potential *

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Abstract

In this pedagogical note, I present a method for constructing a fully covariant normal coordinate expansion of the gauge potential on a curved space-time manifold. Although the content of this paper is elementary, the results may prove useful in some applications and have not, to the best of my knowledge, been discussed in the literature.

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1 Introduction

Riemannian normal coordinates are first introduced in the geometric interpretation of gravitation as a realization of the equivalence principal which requires the existence of an inertial reference frame at every point in space-time in which the effects of gravity can be locally neglected.

In addition to their axiomatic significance, normal coordinates have found a very useful place in perturbative quantum field theory on curved manifolds, and perhaps, in quantum gravity. Generally covariant Taylor-type expansions in normal coordinates have proved useful in the path-integral environment, for example, to perform loop calculations for the non-linear sigma model [1, 2] and to analyze trace anomalies [3]. Generally, one can use these normal coordinate expansions to perform the so-called proper time heat kernel expansion (sometimes known as the DeWitt expansion [4]) both with [5] and without [6] the benefit of worldline path-integral methods. Within the framework of the background field method, the latter expansions can be very useful for straightforward determinations of the ultraviolet behaviour of radiative corrections [7].

A good review of the generally covariant normal coordinate expansion for a tensor field can be found in [1] where the expansion is given explicitly to fourth order. However, as far as the author is aware, very little has been published concerning the fully covariant expansion of a gauge field. The aim of this paper is to provide such a covariant expansion.

The following section reviews the construction of normal coordinates, and introduces the notation which will be used thereafter. In section 3 the gauge field is introduced, along with the so-called *radial gauge* condition which turns out to fit very well in the normal coordinate system and can be used to construct a generally gauge-covariant normal coordinate expansion.

2 Normal Coordinates

Suppose we have some curved space-time manifold with a local coordinate system q^α defined in the neighbourhood of a fixed point ϕ, and a corresponding metric tensor $g_{\alpha\beta}(q)$ with affine connection $\Gamma^\alpha_{\beta\gamma}(q)$. We would like to define what is meant by a normal coordinate system with ϕ at the origin.

For any given point q^α we construct a geodesic $\lambda^\alpha(q,t)$ which connects q with ϕ. Then λ can be taken to satisfy the equation equation

$$\ddot{\lambda}^\alpha(q,t) + \Gamma^\alpha_{\beta\gamma}\dot{\lambda}^\beta(q,t)\dot{\lambda}^\gamma(q,t) = 0\,, \tag{1}$$

for $t \in [0,1]$, with end-points

$$\begin{aligned} \lambda^\alpha(q,0) &= \phi^\alpha \\ \lambda^\alpha(q,1) &= q^\alpha\,. \end{aligned}$$

The *normal coordinates* of any point q are defined to be the components of the tangent vector $\xi(q)$, at the origin ϕ, of the geodesic ending at q, i.e.

$$\xi^\alpha(q) = \dot{\lambda}^\alpha(q,0)\,. \tag{2}$$

Despite the suggestive notation, $\xi(q)$ is not a vector field since the right hand side of equation (2) is a tangent vector at the origin ϕ, not at q. However, a corresponding vector field $\rho(q)$ can be defined by parallel transporting $\xi(q)$ to q along the geodesic. The result,

$$\rho^\alpha(q) = \dot{\lambda}^\alpha(q,1)\,, \tag{3}$$

may be referred to as the *radial vector* field.

Now, in what follows it is convenient to write non-covariant tensor component equations which are valid only when the indices refer to the normal coordinate basis; in such cases, following the notation of [1], we will put a bar over the appropriate quantities. For example, using this notation it is clear that $\rho(q)$ and $\xi(q)$ are related by

$$\xi^\alpha = \overline{\rho}^\alpha \,, \tag{4}$$

which follows from the fact that the geodesics through ϕ are straight lines ($\overline{\lambda} = \xi t$) in the normal coordinate system. From (1), we must also have

$$\overline{\Gamma}^\alpha_{\beta\gamma}\xi^\beta\xi^\gamma = 0 \,,$$

which allows us to write

$$\xi^\beta \left(\frac{\partial}{\partial \xi^\beta}\xi^\alpha + \overline{\Gamma}^\alpha_{\beta\gamma}\xi^\gamma \right) = \xi^\alpha \,.$$

Using (4) this can be written covariantly (with the bars removed) giving

$$\rho^\beta \nabla_\beta \rho^\alpha = \rho^\alpha \,, \tag{5}$$

where ∇_β indicates generally covariant differentiation with respect to the metric $g_{\alpha\beta}$.

When we come to construct a normal coordinate power series expansion in the next section, it will be important to know the various covariant derivatives of $\rho(q)$ at the origin ϕ. The results (equations (6)-(8), below) are not surprising, but their derivation is short enough to be included for completeness.

First note that the trivial geodesic $\lambda(\phi, t) \equiv \phi$ leads immediately to the result

$$\xi(\phi) = \rho(\phi) = 0 \,. \tag{6}$$

Furthermore, we can write

$$\overline{\nabla}_\beta \overline{\rho}^\alpha = \frac{\partial}{\partial \xi^\beta}\xi^\alpha + \overline{\Gamma}^\alpha_{\beta\gamma}\xi^\gamma$$

which leads to the covariant result

$$\nabla_\beta \rho^\alpha(\phi) = \delta^\alpha_\beta \,. \tag{7}$$

Iterative differentiation of (5) along with equations (6) and (7) can be used to prove that all higher covariant derivatives of $\rho(q)$ vanish at the origin,

$$\nabla_{\beta_1}\nabla_{\beta_2}\cdots\nabla_{\beta_n}\rho^\alpha(\phi) = 0 \,, \quad n \geq 2 \,. \tag{8}$$

3 The Gauge Field $A_\alpha = A^a_\alpha T^a$

We turn our attention to the task at hand, that of developing a fully covariant normal coordinate expansion for the gauge potential. Firstly, we appeal to the generally covariant normal coordinate expansion of a tensor. A systematic derivation of such an expansion can be found in reference [1][1]; here we simply quote the result for a vector field:

$$\begin{aligned}\overline{A}_\alpha(q) \;=\; & A_\alpha(\phi) + \frac{1}{1!}\left\{A_{\alpha;\beta}\right\}\xi^\beta + \frac{1}{2!}\left\{A_{\alpha;\beta_1\beta_2} + \frac{1}{3}R^\gamma_{\beta_1\beta_2\alpha}A_\gamma\right\}\xi^{\beta_1}\xi^{\beta_2} \\ & + \frac{1}{3!}\left\{A_{\alpha;\beta_1\beta_2\beta_3} + R^\gamma_{\beta_1\beta_2\alpha}A_{\gamma;\beta_3} + \frac{1}{2}R^\gamma_{\beta_1\beta_2\alpha;\beta_3}A_\gamma\right\}\xi^{\beta_1}\xi^{\beta_2}\xi^{\beta_3} \\ & + \frac{1}{4!}\left\{A_{\alpha;\beta_1\beta_2\beta_3\beta_4} + 2R^\gamma_{\beta_1\beta_2\alpha}A_{\gamma;\beta_3\beta_4} + 2R^\gamma_{\beta_1\beta_2\alpha;\beta_3}A_{\gamma;\beta_4}\right. \\ & \qquad\qquad \left. + \frac{3}{5}R^\gamma_{\beta_1\beta_2\alpha;\beta_3\beta_4}A_\gamma + \frac{1}{5}R^\delta_{\beta_1\beta_2\alpha}R^\gamma_{\beta_3\beta_4\delta}A_\gamma\right\}\xi^{\beta_1}\xi^{\beta_2}\xi^{\beta_3}\xi^{\beta_4} \\ & + \mathcal{O}\xi^5\,,\end{aligned} \tag{9}$$

where $R^\gamma_{\beta_1\beta_2\alpha}$ is the curvature tensor and semicolons (;) indicate generally covariant differentiation at the origin $\nabla|_\phi$. Although (9) is generally covariant, it clearly does not exhibit manifest gauge covariance. The purpose of this note is to attempt to rewrite the coefficients of this expansion in a gauge-covariant manner (i.e. in terms of the field strength and its covariant derivatives).

To this end, we propose to work in the so-called *radial gauge* (the curved-space generalization of the Fock-Schwinger gauge [8]) which fits very well in the normal coordinate construction. The generally covariant gauge condition is

$$\rho^\alpha(q)A_\alpha(q) = 0\,. \tag{10}$$

This condition fixes the gauge relative to a global gauge transformation, and will now be shown to impose some very specific requirements on the various derivatives of A_α which

[1] The author suspects that the fourth order coefficients presented in [1] are not all correct; those presented here for the vector field should be reliable.

appear in (9). The gradient of (10) gives

$$\nabla_\beta \rho^\alpha A_\alpha + \rho^\alpha \nabla_\beta A_\alpha = 0\,. \tag{11}$$

At the origin, equations (6) and (7) reduce this to

$$A_\alpha(\phi) = 0\,. \tag{12}$$

Already it is evident that this choice of gauge has the potential to simplify (9) tremendously. A second derivative of the gauge condition yields

$$\nabla_{\beta_2}\nabla_{\beta_1}\rho^\alpha A_\alpha + \nabla_{\beta_1}\rho^\alpha \nabla_{\beta_2} A_\alpha + \nabla_{\beta_2}\rho^\alpha \nabla_{\beta_1} A_\alpha + \rho^\alpha \nabla_{\beta_2}\nabla_{\beta_1} A_\alpha = 0\,. \tag{13}$$

Using (6)-(8) along with (12) one finds that (13) yields a very simple relation at the origin,

$$A_{\beta_1;\beta_2} + A_{\beta_2;\beta_1} = 0\,. \tag{14}$$

This result can be used to write the first derivative of the vector field explicitly in terms of the field strength, $F_{\alpha\beta} = \nabla_\alpha A_\beta - \nabla_\beta A_\alpha + [A_\alpha, A_\beta]$. It is straightforward to show that,

$$A_{\alpha;\beta} = -\frac{1}{2} F_{\alpha\beta}(\phi)\,, \tag{15}$$

which provides a covariant expression for some of the coefficients in (9). Continuing in this way, we will now show that all of the coefficients can be written covariantly.

Equations (12) and (14) have a very straightforward generalization to higher derivatives. After n derivatives at the origin, the gauge condition (10) along with (6)-(8) yields the following identities,

$$\sum_{i=0}^{n} A_{\beta_i;\beta_0\cdots\beta_{i-1}\beta_{i+1}\cdots\beta_n} = 0\,, \quad n = 0, 1, 2, \cdots\,. \tag{16}$$

This equation contains somewhat more information than we actually need since, only the symmetric components of field derivatives will contribute to expansion (9). Useful corollaries arise by symmetrizing (16) on all or all-but-one of its indices respectively, to give,

$$A_{(\beta_0;\beta_1\cdots\beta_n)} = 0\,, \text{ and} \tag{17}$$

$$A_{\alpha;(\beta_1\cdots\beta_n)} + nA_{(\beta_1;\alpha\beta_2\cdots\beta_n)} = 0\,, \tag{18}$$

where, in these equations and hereafter, symmetrization $(\cdots)$ is implied over the β_i indices only. Following the sort of reasoning which led from (14) to (15), we hope that these last two equations will be useful to relate any order of symmetrized derivative of the vector field to covariant derivatives of the field strength $F_{\alpha\beta}$ at the origin.

Since we would like to recover manifest gauge covariance in the coefficients of the expansion (9), we must introduce the notion of gauge-covariant differentiation before proceeding any further. Following the usual rules for gauge transformations, the gauge-covariant derivative of any object T (in the adjoint representation of the gauge group) is defined to be

$$\nabla_\beta^{\mathrm{gauge}} T = \nabla_\beta T + [A_\beta, T]\ .$$

Multiple covariant derivatives of T at the origin ϕ must take the form

$$T_{;\beta_1\cdots\beta_n|\mathrm{gauge}} = T_{;\beta_1\cdots\beta_n} + \sum\left[A_{\beta;\beta\cdots\beta}\left[A_{\beta;\beta\cdots\beta},\left[\cdots\left[A_{\beta;\beta\cdots\beta}, T_{;\beta\cdots\beta}\right]\cdots\right]\right]\right]\ , \tag{19}$$

where $\sum[\cdots]$ indicates some complicated sum of generic terms which involve at least one commutator of the form shown. In this gauge, such terms a guaranteed to vanish after symmetrization over the β_i indices, owing to equation (17).

Returning to the symmetrized covariant derivatives of the field strength tensor, we can now write

$$F_{\alpha(\beta_1;\beta_2\cdots\beta_n)\mathrm{gauge}} = A_{(\beta_1;\alpha\beta_2\cdots\beta_n)} - A_{\alpha;(\beta_1\cdots\beta_n)} + \sum_{l=1}^{n}\binom{n-1}{l-1}\left[A_{\alpha;(\beta_{l+1}\cdots\beta_n}, A_{\beta_1;\beta_2\cdots\beta_l)}\right]\ . \tag{20}$$

The second term in this expression is exactly of the desired form (i.e. it appears in the expansion (9)). The first term can also be written in this form by employing (18) while the commutator terms vanish by (17). The result,

$$A_{\alpha;(\beta_1\cdots\beta_n)} = -\frac{n}{n+1}F_{\alpha(\beta_1;\beta_2\cdots\beta_n)|\mathrm{gauge}}\,, \tag{21}$$

is not unexpected since it is very straightforward to verify in the flat-space limit; what is noteworthy is that it was derived in curved space without having to introduce explicit curvature-dependent terms. This absence of additional curvature terms was not obvious at the outset.

Substituting (21) and its special cases (12) and (15) into (9) yields the desired fully covariant normal coordinate expansion:

$$
\begin{aligned}
A_\alpha(q) \;=\; & \left\{\frac{1}{2}F_{\beta\alpha}\right\}\xi^\beta + \left\{\frac{1}{3}F_{\beta_1\alpha;\beta_2|\mathrm{gauge}}\right\}\xi^{\beta_1}\xi^{\beta_2} \\
& + \left\{\frac{1}{8}F_{\beta_1\alpha;\beta_2\beta_3|\mathrm{gauge}} + \frac{1}{12}R^\gamma_{\beta_1\beta_2\alpha}F_{\beta_3\gamma}\right\}\xi^{\beta_1}\xi^{\beta_2}\xi^{\beta_3} \\
& + \left\{\frac{1}{30}F_{\beta_1\alpha;\beta_2\beta_3\beta_4|\mathrm{gauge}} + \frac{1}{18}R^\gamma_{\beta_1\beta_2\alpha}F_{\beta_3\gamma;\beta_4|\mathrm{gauge}} + \frac{1}{24}R^\gamma_{\beta_1\beta_2\alpha;\beta_3}F_{\beta_4\gamma}\right\}\xi^{\beta_1}\xi^{\beta_2}\xi^{\beta_3}\xi^{\beta_4} \\
& + \mathcal{O}\xi^5\,.
\end{aligned}
\tag{22}
$$

Higher order corrections to this expansion should be very straightforward to obtain by using the methods of [1] along with equation (21).

4 Discussion

Building on the ungauged developments of ref. [1], we have been able to show how a normal coordinate expansion can be constructed for the gauge field with fully covariant coefficients in curved space.

Although this type of expansion may have additional uses, the motivation for this work was in constructing a proper time (DeWitt) expansion for the gauged heat kernel in curved space-time. In that context, these results could contribute to a covariant analysis of the renormalization group in non-Abelian gauge theory. The results of the former study are forthcoming [5].

Finally, it should be pointed out that a third-order normal coordinate expansion can be found in ref. [6] (also in the context of proper time expansions). Regrettably, the gauge field expansion falls outside of the main interests of that work, so all relevant details have been omitted. The results quoted therein do not agree with those presented here.

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