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On the Monodromies of $N=2$ Supersymmetric Yang-Mills Theory with Gauge Group $SO(2n)$

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Abstract

We present families of algebraic curves describing the moduli space of $N = 2$ supersymmetric Yang-Mills theory with gauge group $SO(2n)$. We test our curves by computing the weak coupling monodromies and the number of $N=1$ vacua.

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Introduction

A year ago Seiberg and Witten [1] showed how to obtain an exact solution for the low energy effective action of $N=2$ supersymmetric Yang-Mills theory with gauge group $SU(2)$. Shortly afterwards the generalization to gauge groups $SU(n)$ with arbitrary n has been obtained in [2, 3]. For these cases a lot of work has been done by now. The large n-limit and the connection to $N=1$ theories has been studied in [4]. The non-perturbative effective action has been elaborated in [5]. Phases with mutually non-local massless dyons were identified in [6]. Also $N=2$ supersymmetric QCD with matter in the fundamental representation of the gauge group has been considered in [7]. Fairly recently a solution for gauge groups $SO(2n+1)$ has been presented in [8]. The purpose of this letter is to extend these previous results to gauge groups $SO(2n)$. It should be noted here that in the Cartan classification the groups $SO(2n)$ correspond to simply laced Lie algebras of type D_n whereas $SO(2n+1)$ correspond to non-simply laced ones of type B_n.

Semiclassical Monodromies

Following [1] we write the Lagrangian for $N=2$ SYM with arbitrary gauge group G in $N=1$ superspace language as

$$\mathcal{L} = \frac{1}{4\pi} Im \left(\int d^4\theta \frac{\partial \mathcal{F}}{\partial \mathcal{A}} \bar{\mathcal{A}} + \int d^2\theta \frac{1}{2} \frac{\partial^2 \mathcal{F}}{\partial \mathcal{A}^2} W_\alpha W^\alpha \right) \tag{1}$$

where the $N=2$ vectormultiplet has been decomposed into an $N=1$ chiral superfield $\mathcal{A}$ and an $N=1$ vectorfield W_α. Two facts are of utmost importance here. Firstly the whole theory is governed by a single holomorphic function, the prepotential $\mathcal{F}(\mathcal{A})$. Secondly there is a nontrivial classical potential for the scalar field, $V(\phi) = \left[\phi, \phi^\dagger\right]$. From this it follows that for generic vacuum expectation value of ϕ the gauge group is broken down to the maximal torus. Thus the low energy degrees of freedom correspond to a $U(1)^r$-gauge theory with r being the rank of G. Since $< \phi >$ can always be chosen to lie in the Cartan sub-algebra, whose generators we denote by H_i, we have $< \phi >= \sum_{i=1}^r \phi_i H_i$. The W-bosons corresponding to the roots $\vec{\alpha}$ of the gauge group acquire a mass which is proportional to $(\vec{\phi}.\vec{\alpha})^2$. Whenever the vacuum expectation value of the scalar field is orthogonal to a root, a W-boson becomes massless and therefore the low energy description is no longer valid there. This information can be compactly encoded in the zeroes of the Weyl group invariant "classical" discriminant:

$$\Delta_{cl} = \prod_{\vec{\alpha} \in \Psi_+} (\vec{\alpha}.\vec{\phi})^2. \tag{2}$$

Here Ψ_+ is the set of positive roots. Two vectors $\vec{\phi}$ should be identified if they differ by a Weyl transformation of the gauge group. In the space of the ϕ_i the zeroes of the discriminant coincide with the fixed points under the Weyl group action.

Quantum mechanically the theory is characterized by a dynamically generated scale Λ. It is weakly coupled if $<\phi>$ is large in comparison to Λ, and therefore a perturbative description is valid in this regime. The prepotential takes the form [5, 8]

$$\mathcal{F}_{pert} = \frac{i}{4\pi} \sum_{\vec{\alpha}\in\Psi_+} (\vec{\alpha}.\vec{\phi})^2 \log \frac{(\vec{\alpha}.\vec{\phi})^2}{\Lambda^2}. \tag{3}$$

It is clear that the logarithm gives rise to singularities if $(\vec{\alpha}.\vec{\phi}) = 0$. $\mathcal{F}_{pert}$ is not a single-valued function of $\vec{\phi}$, and thus gives rise to non-trivial monodromies when one encircles a singularity in the moduli space of vacuum configurations. Actually what one is interested in are monodromies acting on the vector $(\vec{\phi}^D, \vec{\phi})$ where the dual variables are defined as $\phi_i^D = \partial\mathcal{F}/\partial\phi_i$. Thus for a gauge group of rank r the monodromy group will be a subgroup of the group of duality transformations $Sp(2r, Z)$.

In the perturbative regime there will be r simple monodromies corresponding to the simple roots of the gauge group. They generate all the other semiclassical monodromies by conjugation. The monodromy matrix M_i induced by the Weyl reflection on the root α_i from the prepotential $\mathcal{F}_{pert}$ is

$$M_i = \begin{pmatrix} 1 - 2\frac{\vec{\alpha}_i\otimes\vec{\alpha}_i}{\vec{\alpha}_i^2} & -\vec{\alpha}_i \otimes \vec{\alpha}_i \\ 0 & 1 - 2\frac{\vec{\alpha}_i\otimes\vec{\alpha}_i}{\vec{\alpha}_i^2} \end{pmatrix}. \tag{4}$$

To be specific we write down the simple monodromies for $SO(8)$. We chose an orthogonal basis such that the four simple roots are given by [9]

$$\begin{aligned} \vec{\alpha}_1 &= (1, -1, 0, 0), \quad \vec{\alpha}_2 = (0, 1, -1, 0), \\ \vec{\alpha}_3 &= (0, 0, 1, -1), \quad \vec{\alpha}_4 = (0, 0, 1, 1). \end{aligned} \tag{5}$$

The four simple monodromies are

$$M_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad M_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{6}$$

$$M_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad M_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix} \tag{7}$$

According to the Dynkin diagram of D_4, they fulfill the braid group relations

$$M_i.M_2.M_i = M_2.M_i.M_2 \tag{8}$$

with $i = 1, 3, 4$.

The Curves

Our aim is to reproduce these monodromies from an algebraic curve. There are several ways to arrive at the desired expression. First let us calculate $P = det(x.\mathbb{1} - \phi)$ where the matrices are in the fundamental representation of $SO(2n)$.

$$P(x) = \prod_{i=1}^{n} (x^2 - e_i^2) \tag{9}$$

We distinguished here the roots e_i of the polynomial P from the vacuum expectation values ϕ_i. This is because in the quantum case both can be identified only in the semiclassical regime. The Weyl group of $SO(2n)$ is the semi-direct product of $\mathcal{S}_n \ltimes Z_2^{n-1}$. The first factor are permutations of n elements and the second denotes simultaneous sign changes of two elements e_i, e_j. This distinguishes the D_n series from the non-simply laced B_n case. In the latter one can flip the signs of each e_i individually. The important consequence is that the D_n series has an exceptional Casimir of order n (cfg. (12) below). Clearly $P(x)$ is invariant under the Weyl group. Therefore we can equally well write it in terms of gauge invariant variables

$$P(x) = x^{2n} - x^{2(n-1)}u_2 - x^{2(n-2)}u_4 - ... - x^2 u_{2(n-1)} - t^2 \,, \tag{10}$$

with

$$u_{2m} = (-1)^{m-1} \sum_{i_1 < i_1 < .. < i_m} e_{i_1}^2 e_{i_2}^2 .. e_{i_m}^2 \tag{11}$$

$$t = (i)^{n-1} \prod_{i=1}^{n} e_i \,. \tag{12}$$

Note that due to the structure of the Weyl group the polynomial P depends quadratically on t!

It has been suggested in [5] that the algebraic curves describing the moduli space of $N=2$ SYM with gauge groups based on the simply laced Lie algebras of ADE-type should be intimately related to the simple singularities of the same type. Indeed the $SU(n)$ curves are determined by the "LG-potentials" of type A_n. Also the curves given in [8] are determined in exactly the same manner by the well known potentials for the boundary singularities B_n [10]. How can we establish such a relationship for the case at hand?

The perturbed simple singularities for the D_n series is given by

$$W_{D_n} = x_2^2 x_1 + x^{n-1} - x_1^{n-2} u_2 - x_1^{n-3} u_4 - \dots - u_{2(n-1)} - x_2 t\,. \tag{13}$$

Formally integrating out the variable x_2 by its equation of motion and then substituting $x_1 \to \hat{x}^2$ one arrives at

$$\bar{W}_{D_n} = \hat{x}^{2(n-1)} - \hat{x}^{2(n-2)} u_2 - \dots - u_{2(n-1)} - \frac{t^2}{\hat{x}^2}\,. \tag{14}$$

This is in fact well-known in the context of conformal field theory [11]. Then our polynomial is simply $\hat{x}^2 \bar{W}_{D_n}$. At this stage it is useful to have an explicit look at the discriminant of P. To be specific, we chose the simplest generic case D_4 and change notation according to $u_2 \to u$, $u_4 \to v$, $u_6 \to w$

$$\begin{aligned} \Delta_{cl} &= t^2 \Delta_W \\ \Delta_W &= (256t^6 + 27t^4u^4 + 144t^4u^2v + 128t^4v^2 + 4t^2u^2v^3 + 16t^2v^4 + \\ &\quad +192t^4uw - 18t^2u^3vw - 80t^2v^2w - 6t^2u^2w^2 - 144t^2vw^2 - u^2v^2w^2 - \\ &\quad -4v^3w^2 + 4u^3w^3 + 18uvw^3 + 27w^4)\,. \end{aligned} \tag{15}$$

Here Δ_W is the discriminant that one computes from the Landau-Ginzburg potential (13) or from the one-variable expression (14). Using the basis of roots (5) one finds that it precisely reproduces (2). Thus our polynomial P has an additional and a priori unexpected singularity at $t=0$. We will see in the following how this puzzle resolves.

Following the standard arguments given in [2, 3], we suggest for the quantum theory the following family of algebraic curves

$$y^2 = P(x)^2 - \Lambda^{(4n-4)} x^4\,. \tag{16}$$

The classical theory has a $U(1)_{\mathcal{R}}$ symmetry under which the field ϕ has charge 2. In the quantum theory it is anomalous and broken down to a discrete Z_{4C_ν}-symmetry [1]. Here

C_ν is the dual Coxeter number of the gauge group. In the case of $SO(2n)$ it is given by $C_\nu = 2n-2$. If we assign charge 2 to x and $4n$ to y, we see that our curve has precisely the required $Z_{4(2n-2)}$-symmetry. The right hand side of (16) factorizes into

$$\mathcal{C}_+.\mathcal{C}_- = (P(x) + \Lambda^{2n-2}x^2).(P(x) - \Lambda^{2n-2}x^2)\,. \tag{17}$$

From this we infer that similarly as for $SU(n)$, the quantum discriminant factorizes as well. We see that $\mathcal{C}_\pm$ has the form of the classical polynomial if we substitute $u_{2n} \to u_{2n} \pm \Lambda^{2n-2}$. The factors $\mathcal{C}_\pm$ have common roots only at a point in the moduli space where $x=0$ is a root. However this case is already contained in $\Delta_\pm$ itself and therefore no new singularity arises. Thus the quantum discriminant is given by

$$\Delta_{qu} = t^4.\Delta_+.\Delta_-\,. \tag{18}$$

Strong and weak coupling monodromies

An important feature of our curve is that there appear only even powers of x. Therefore it is invariant under a Z_2-symmetry acting on the complex x-plane $\Pi : x \to -x$. This implies that the homology can be uniquely decomposed into two subspaces $H^+ \oplus H^-$ where H^+ is the space of cycles which are invariant under Π and H^- is the anti-invariant subspace [10]. Our choice of cycles is depicted in Fig.1.

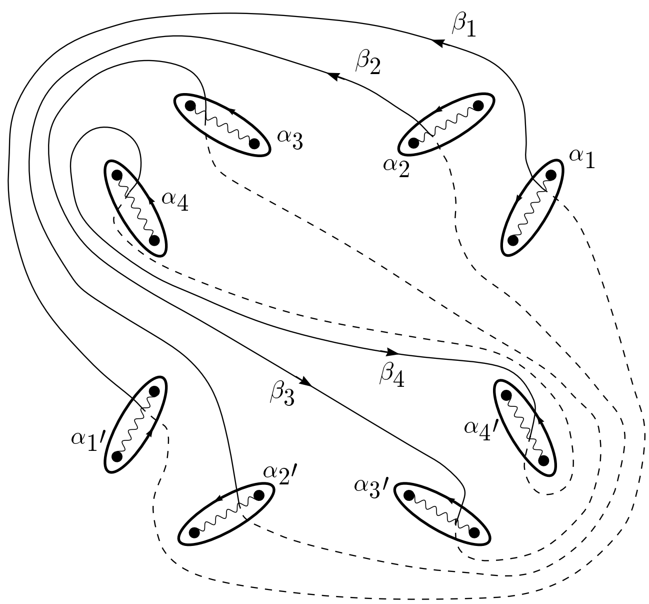

Figure 1: The basic cycles for the $SO(8)$ curve

One sees that the cycles β_i belong to H^- whereas $\Pi\alpha_i = \alpha_i\prime$. From the latter we define anti-invariant cycles by taking the differences

$$\Delta_i = \alpha_i - \alpha_i' \, . \tag{19}$$

Cycles of this form are called long, cycles of the form of the β_i are called short. Their intersection form is given by

$$< \Delta_i, \beta_j >= 2\delta_{ij} \, . \tag{20}$$

In order to define the variables e_i and their duals, e_i^D, as period integrals, we also need a suitable meromorphic one-form λ. For curves of the form (16) it has already been derived in [8]. In our case it is given by

$$\lambda = (2P(x) - xP'(x)) \frac{dx}{y} \, . \tag{21}$$

Since it also changes sign under the action of Π, we obtain invariant periods only by integrating over anti-invariant cycles. Thus the physically relevant subspace of the homology is H^-. Now the fields can be defined as

$$e_i = \oint_{\Delta_i} \lambda \, , \qquad e_i^D = \oint_{\beta_i} \lambda \, . \tag{22}$$

An additional subtlety arises in considering the Picard-Lefschetz formula. Since the intersection of any long anti-invariant cycle with another anti-invariant cycle is always even we have to correct this by a factor $\frac{1}{2}$. We obtain the modified Picard-Lefschetz formula [10]

$$\delta\nu \;=\; \nu + \frac{1}{2} < \nu, \mu_l > \mu_l \tag{23}$$

$$\delta\nu \;=\; \nu + < \nu, \mu_s > \mu_s \, , \tag{24}$$

where μ_l is a long anti-invariant vanishing cycle and μ_s a short one. It should be noted here that in the case of $SO(2n)$ we will never encounter short anti-invariant vanishing cycles. This is no surprise since these cycles correspond in a one-to-one manner to short roots of Lie-algebras. Also we could have chosen a basis of cycles consisting only of long anti-invariant ones. We did not do so because the choice in Fig.1 is best suited to the orthogonal basis of the Lie algebra. This will become clear from the discussion of the strong coupling monodromies. In addition, it allows most easily to compare and to work out the differences to the case when the gauge group is $SO(2n+1)$.

In order to derive the semi-classical monodromies (6,7), we use the fact that they can be written as a product of two strong coupling monodromies. More precisely, we have chosen a slice through the moduli space by fixing v, u, t and varying only u. Away from intersections or cusps of the singular lines one sees four pairs of singular lines. Fixing a base point in this plane we traced the effect on the branch points when looping around the singularities. Depending on the particular form of the loops, one finds different vanishing cycles. It suffices however to choose eight loops, each encircling different singular lines.

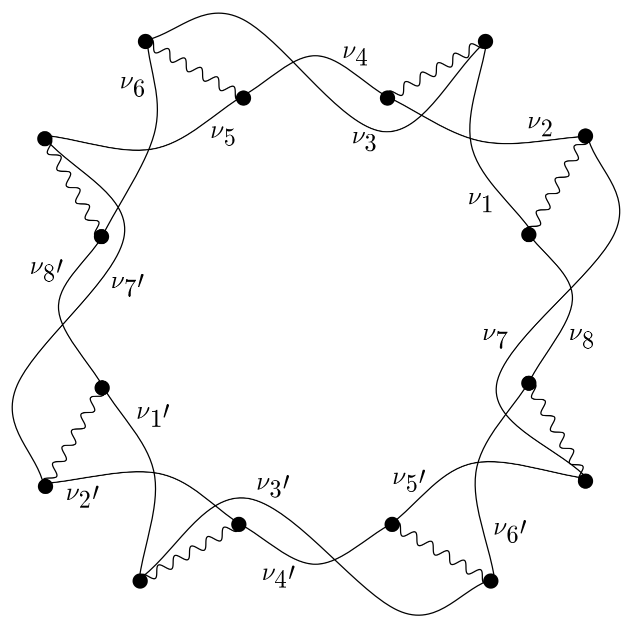

Figure 2: The strong coupling vanishing cycles

All the other monodromies can then be obtained by conjugation from the eight basic ones. The eight vanishing cycles can be seen in Fig. 2. Of physical relevance are again the differences $\delta_i = \nu_i - \nu_i'$. They can be expanded in terms of β_i and Δ_i. We find

$$\begin{aligned} \delta_1 &= (1,-1,0,0;0,0,0,0)\,, & \delta_2 &= (1,-1,0,0;1,-1,0,0)\,, \\ \delta_3 &= (0,1,-1,0;0,0,0,0)\,, & \delta_4 &= (0,1,-1,0;0,1,-1,0)\,, \\ \delta_5 &= (0,0,1,-1;0,0,-1,1)\,, & \delta_6 &= (0,0,1,-1;0,0,0,0)\,, \\ \delta_7 &= (-1,0,0,-1;1,1,1,3)\,, & \delta_8 &= (-1,0,0,-1;0,1,1,2)\,. \end{aligned} \tag{25}$$

The first entries refer to magnetic quantum numbers and the others to electric ones. A general formula for the monodromy matrix for a massless dyon of magnetic charge $\vec{g}$ and electric charge $\vec{q}$ has been given in [2]:

$$\mathcal{M}_{(\vec{g},\vec{q})} = \begin{pmatrix} 1-\vec{g}\otimes\vec{q} & -\vec{q}\otimes\vec{q} \\ \vec{g}\otimes\vec{g} & 1+\vec{g}\otimes\vec{q} \end{pmatrix} \tag{26}$$

Note that the magnetic quantum numbers of $\delta_{1,2}$ are given by the root $\vec{\alpha}_1$, that of $\delta_{3,4}$ and that of $\delta_{5,6}$ by $\vec{\alpha}_2$ and $\vec{\alpha}_3$ respectively and finally that of $\delta_{7,8}$ by the root $-\vec{\alpha}_1 - \vec{\alpha}_2 - \vec{\alpha}_4$. According to this we reproduce the weak coupling monodromies in the following manner:

$$\begin{aligned} M_1 &= \mathcal{M}_{\delta_1}.\mathcal{M}_{\delta_2} \\ M_2 &= \mathcal{M}_{\delta_3}.\mathcal{M}_{\delta_4} \\ M_3 &= \mathcal{M}_{\delta_5}.\mathcal{M}_{\delta_6} \\ M &= \mathcal{M}_{\delta_7}.\mathcal{M}_{\delta_8} \\ M_4 &= M_1.M_2.M.M_2^{-1}.M_1^{-1} \end{aligned} \tag{27}$$

Another nontrivial check of the curve (16) concerns the quantum shift matrix T. It is obtained in the following way. Performing the rotation $\Lambda^2 \to e^{2\pi it}\Lambda^2$, $t \in (0,1)$ one computes from the prepotential (3) the monodromy

$$T = \begin{pmatrix} 1 & -\sum_{\vec{\alpha}\in\Psi_+} \vec{\alpha}\otimes\vec{\alpha} \\ 0 & 1 \end{pmatrix}. \tag{28}$$

In the orthogonal basis we have $\sum_{\vec{\alpha}\in\Psi_+} \vec{\alpha}\otimes\vec{\alpha} = C_\nu.\mathbb{1}$. On the complex curve this rotation induces pure braidings of the branch cuts. A single braid of the cut i induces $\beta_i \to \beta_i - \Delta_i$. From the fact that Λ appears to the power of two times the dual Coxeter number we see that the number of braiding for each branch cut is C_ν. Thus we find the quantum shift matrix in the orthogonal basis.

The Monodromy around $t = 0$?

Now we come to a subtle and crucial point in our discussion. From the expression of the quantum discriminant (18) one expects that some BPS state becomes massless as we approach $t = 0$ for generic values of u , v and w. A look at the algebraic curve (16) tells us that four branch points i.e. two branch cuts will collide at $x = 0$ in this case, which seems to be a nasty, unstable situation. But what does really happen as we loop around

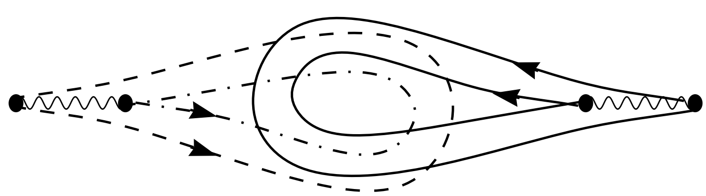

Figure 3: Monodromy around $t = 0$?

$t = 0$? We have checked this situation for several one-dimensional slices through the moduli space and in Fig. 3 we have depicted what happens to the branchpoints. Six of the branch cuts stay where they are, whereas two such cuts rotate around each other and end up at their original positions. Since no additional braidings occur, we see that the basic cycles are unchanged and the associated monodromy matrix is simply the identity. There are no massless particles associated with the "singularity around $t = 0$ " !

This is in strong contrast to what is happening for the non-simply laced gauge groups $SO(2n + 1)$. In that case the monodromy around $u_{2n} = 0$ [1] has a massless particle associated with it and one finds a strong coupling monodromy. To be more specific we want to discuss $SO(5)$ in some detail. We use basic cycles analogous to Fig. 1 with β_2 being ζ (they are the same as in [8]). The algebraic curve and the quantum discriminant read as follows:

$$y^2 = (x^4 - ux^2 - v)^2 - x^2\Lambda^6 \ , \tag{29}$$

$$\Delta_{qu}(u, v) = v^2(256v^3 - 128u^2v^2 + 16u^4v + 4\Lambda^6u^3 - 144\Lambda^6uv - 27\Lambda^{12}) \ . \tag{30}$$

In Fig. 4 we have drawn two anti-invariant vanishing cycles $\xi - \xi'$ and ζ associated with a short root.

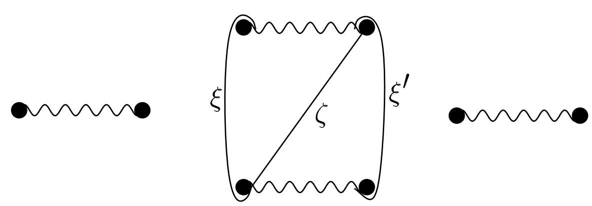

Figure 4: SO(5) vanishing cycles corresponding to a short root

For the vanishing cycle $\xi - \xi'$ we can calculate the monodromy in the usual way and the magnetic-electric quantum numbers read $\delta_{\xi-\xi'} = (0, 2; 0, 1)$. Some difficulty arises for ζ, which corresponds to a loop around [2] $v = 0$ in the moduli space. In this case the two branch points encircled by the vanishing cycle are not simply exchanged, but they rotate around $x = 0$ and move back to their starting points, picking up a non-trivial braid. The monodromy $\mathcal{M}_\zeta$ obtained by using the Picard-Lefschetz formula (24) is not the correct one, because it corresponds to the exchange of the two branch points. In fact one has to take its square $\mathcal{M}_\zeta^2$ with quantum numbers $(0, 2; 0, 0)$ i.e. the effective vanishing cycle is therefore $2\beta_2$. As expected, one obtains a weak coupling monodromy by multiplying two strong coupling monodromies. $\mathcal{M}_\zeta^2\mathcal{M}_\xi$ is precisely the monodromy matrix M_2 of [8] and

[1] By u_{2n} we mean the highest order gauge invariant operator of $SO(2n + 1)$, which is of order $2n$

[2] v is the highest order Casimir for SO(5)

corresponds to a Weyl reflection on a short root ! In the case of $SO(2n+1)$ the singularity around $u_{2n} = 0$ is essential to account for the presence of short roots. Of course, short roots are not present for $SO(2n)$.

The $N=1$ Vacua

Now we want to take a closer look at the moduli space of $SO(2n)$ gauge theories and the singularities living in it. The vanishing of the quantum discriminant (18)[3] defines an $n-1$ dimensional singular submanifold in our n dimensional moduli space. Since the quantum discriminant factorizes into two pieces, one expects to find at least two branches of the singular submanifold that intersect in some $n-2$-dimensional manifold. Furthermore, the branches split themselves into several branches and have self intersections and singularities, like cusps. For arbitrary n one will find a discrete set of points where n branches intersect simultaneously. In the following we will concentrate on these singular points, but it should be mentioned that there are also higher dimensional singular submanifolds which would be well worth to study.

At these singular points we can choose local coordinates and the quantum discriminant will factorize into n factors (to lowest order). If some of the factors are linear dependent, we are at a cusp singularity where several of the n massless dyons are mutually non-local. Such points have been studied for the case of $SU(3)$ in [4]. At some special points all branches intersect transversely i.e. the factors are all linearly independent. All n massless particles are mutually local, and one can break $N=2$ supersymmetry down to $N=1$ by adding a mass term.

We have studied the case of $SO(8)$ in some detail, and found 48 points where four branches of the singular manifold intersect. Mutually non-local dyons are present at 42 of these points, whereas the remaining six points are the $N=1$ vacua

$$u = 3\,\sqrt[3]{4}\Lambda^2 e^{2\pi\imath l/6}\ ,\ v = -\frac{9}{\sqrt[3]{4}}\Lambda^4 e^{4\pi\imath l/6}\ ,\ w = \Lambda^6 e^{\pi\imath l}\ ,\ t = 0 \tag{31}$$

with $\mathrm{l} = 0 \ldots 5$. This is the correct number of $N=1$ vacua for the $SO(8)$ gauge theory. We also checked this for $SO(6)$ and found 4 $N=1$ vacua.

For larger n it becomes increasingly difficult to find the vacua directly from the quantum discriminant. We propose therefore a different approach which is similar to the discussion in [4]. If we insert the values (31) into our curve (16) we find a polynomial

[3] in the following we will suppress the factor t^4 in (18)

with two simple roots, two double roots and one sixfold root at $x = 0$. This situation can easily be generalized for arbitrary n with the help of Chebyshev polynomials. The classical polynomial (10) takes the form

$$\hat{P}_n(x) = x^2 T_{2n-2}(\frac{x}{\Lambda} 2^{\frac{1-2n}{2n-2}}) \Lambda^{2n-2} \tag{32}$$

whereas the curve (16) will be of the form

$$\hat{\mathcal{C}}_n(x) = \hat{P}(x)^2 - x^4 \Lambda^{2n-2} \ . \tag{33}$$

We obtain a total of $2n-2$ solutions by complex rotation $x \to x e^{\imath\pi l/(2n-2)}$ which is the correct number of $N=1$ ground states. As a check one can compare the expansion of (32) for $n = 4$ with the classical curve (10), and read off the values of u, v, w and t. They perfectly agree with the solutions found directly from the examination of the singular points of the quantum discriminant of $SO(8)$ given in (18). We want to point out that the vacuum points found for the D_n series, with $t = 0$, coincide with the vacua for the A_{2n-3}, series with all odd Casimirs set to zero.

In Fig. 5 we have depicted how the branch cuts arrange and what cycles vanish as we approach an $N=1$ vacuum point. The vanishing cycles have the usual anti-invariant form $\Delta\mu_i = \mu_i - \mu_i\prime$. Apparently $\Delta\mu_1$ and $\Delta\mu_2$ are non intersecting. For μ_3 and μ_4 the situation is not so obvious but the anti-invariant combinations $\Delta\mu_3$ and $\Delta\mu_4$ turn out to have vanishing intersection form. Thus we find four mutually local dyons as it should be for a $N=1$ vacuum.

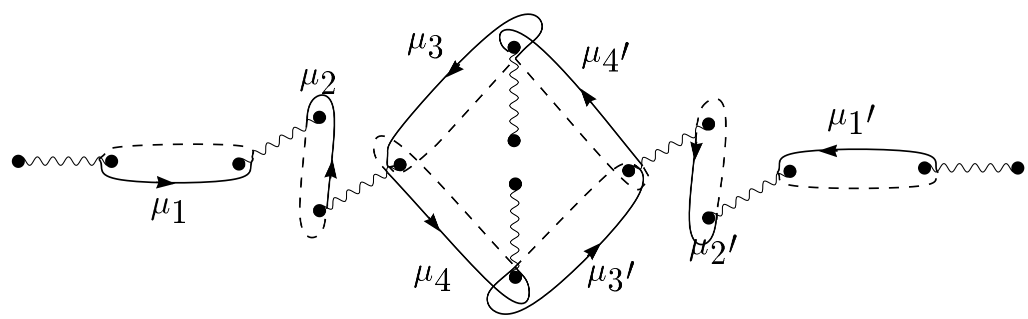

Figure 5: Near a $N=1$ vacuum point one can see two copies of the D_4 Dynkin diagram !

Finally we note that in the last figure one sees a mirror pair of the Dynkin diagram. This and the appearance of the Chebyshev polynomials is obviously related to the resolution of simple singularities [10]. It also allows for another beautiful interpretation.

Identifying the electric cycles around the branch cuts with weights and the magnetic vanishing cycles with simple roots we recognize the weight diagram of the fundamental representation of D_4!

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