

# Friedmann Universes and Exact Solutions in String Cosmology

S.N. Roshchupkin

*Simferopol State University*  
333036, Simferopol, Ukraina

and A.A. Zheltukhin\*

*Kharkov Institute of Physics and Technology*  
310108, Kharkov, Ukraina

We show that the classical null strings generate the Hilbert-Einstein gravity corresponding to  $D$ -dimensional Friedmann universes.

During last years much attention is devoted to the study of different physical mechanisms of the inflation expansion in cosmological backgrounds [1-5]. Recently the new string source of the inflationary scenario has been considered [6-8] for the  $D$ -dimensional Friedmann-Robertson-Walker (FRW) spaces describing by the metric

$$ds^2 = G_{MN} dx^M dx^N = (dx^0)^2 - R^2(x^0) dx^i \delta_{ik} dx^k, \quad (1)$$

where  $M, N = 0, 1, \dots, D-1$ . Such a possibility is provided by the existence of asymptotic string configuration describing an approximate solution of the Hilbert-Einstein (HE) and string equations in the large  $R(t)$  limit. This solution corresponds to the phase of unstable non-oscillating strings with proper sizes growing like the scale factor  $R(t)$  (1). In addition to the inflationary solution a new approximate solution valid in the small  $R(t)$  limit for a negatively accelerated contraction ( $d^2R/dt^2 < 0$ ,  $dR/dt < 0$ ) of the universes was found in [8]. The last solution describes the phase of perfect gas of shrunk strings ( $R(t) \rightarrow 0$ ,  $\tau \rightarrow 0$ ) with the approximate equation of state

$$\rho \approx p(D-1). \quad (2)$$

Here we want to show that the classical equations for null strings together with gravity equations are exactly solvable for the case of the FRW metric (1). These solutions correspond to the phase of perfect gas of shrunk or stretched ( $dR/dt > 0$ ,  $d^2R/dt^2 < 0$ ) null strings with the exact equation of state

$$\rho = p(D-1), \quad (3)$$

Due to this fact we may think that the perfect gas of null strings is an alternative source of the FRW gravity with  $k = 0$ . To prove this result note that the action for null string in a cosmological background  $G_{MN}(x)$  may be written as [9,10]

$$S = \int d\tau d\sigma \frac{\det(\partial_\mu x^M G_{MN}(x) \partial_\nu x^N)}{E(\tau, \sigma)}, \quad (4)$$

---

\*E-mail: kfti@kfti.kharkov.ua

In the world-sheet gauge fixed by the conditions [9,10]

$$\dot{x}^M G_{MN} \dot{x}'^N = 0, \quad \frac{x'^M G_{MN}(x) x'^N}{E(\tau\sigma)} = \frac{1}{\gamma^* \pi}, \quad (5)$$

where  $\gamma^*$  – constant with the dimension  $L^2$  (if  $\hbar = c = 1$ ), the motion equations and constraints produced by  $S$  (4) have the following form

$$\begin{aligned} \ddot{x}^M + \Gamma_{PQ}^M \dot{x}^P \dot{x}^Q &= 0 \\ \dot{x}^M G_{MN} \dot{x}^N &= 0, \quad \dot{x}^M G_{MN} \dot{x}'^N = 0, \end{aligned} \quad (6)$$

where  $\dot{x} = \partial x / \partial \tau$ ,  $\dot{x}' = \partial x / \partial \sigma$ . To solve Eqs. (6) it is convenient to turn to a conformal time  $\tilde{x}^0(\tau, \sigma)$ , defined by

$$dx^0 = C(\tilde{x}^0) d\tilde{x}^0, \quad C(\tilde{x}^0) = R(x^0), \quad \tilde{x}^i = x^i \quad (7)$$

In the gauge of the conformal time the metric (1) is presented in the conformal-flat form

$$ds^2 = C(\tilde{x}^0) \eta_{MN} d\tilde{x}^M d\tilde{x}^N, \quad \eta_{MN} = \text{diag}(1, -\delta_{ij}) \quad (8)$$

with the Christoffel symbols  $\tilde{\Gamma}_{PQ}^M(\tilde{x})$  [11]

$$\tilde{\Gamma}_{PQ}^M(\tilde{x}) = C^{-1}(\tilde{x}) [\delta_P^M \tilde{\partial}_Q C + \delta_Q^M \tilde{\partial}_P C - \eta_{PQ} \tilde{\partial}^M C] \quad (9)$$

Taking into account the relation (8,9) we transform Eqs. (6) to the form

$$\ddot{\tilde{x}}^M + 2C^{-1} \dot{C} \dot{\tilde{x}}^M = 0 \quad (10)$$

$$\eta_{MN} \dot{\tilde{x}}^M \dot{\tilde{x}}^N = 0, \quad \eta_{MN} \dot{\tilde{x}}^M \dot{\tilde{x}}'^N = 0 \quad (11)$$

The first integration of Eqs. (10) results in the first order equations

$$H^* C^2 \dot{\tilde{x}}^0 = \psi^0(\sigma), \quad H^* C^2 \dot{\tilde{x}}^i = \psi^i(\sigma), \quad (12)$$

the solutions of which have the form

$$\tau = H^* \psi_0^{-1} \int_{t_0}^t dt R(t), \quad (13)$$

$$x^i(\tau, \sigma) = x^i(0, \sigma) + H^{*-1} \psi^i \int_0^\tau d\tau R^{-2}(t),$$

where  $H^*$  is a metric constant with the dimension  $L^{-1}$  and  $t_0 \equiv x^0(0, \sigma)$ ,  $x^i(0, \sigma)$  and  $\psi^M(\sigma)$  are the initial data. The solution (13) for the space world coordinates  $x^i(t) (i = 1, \dots, D-1)$  as a function of the cosmic time  $t = x^0$ , may be written in the equivalent form as

$$x^i(t, \sigma) = x^i(t_0, \sigma) + \nu^i(\sigma) \int_{t_0}^t dt R^{-1}(t), \quad (14)$$

where  $\nu^i(\sigma) \equiv \psi_0^{-1} \psi^i$ .

The explicit form of the solutions (13) allows to transform the constraints (11) into those for the Cauchy initial data:  $\nu^i(\sigma)$ ,  $t_0(\sigma)$  and  $x'_0 \equiv x'(t_0, \sigma)$

$$\nu^i(\sigma) \nu^k(\sigma) \delta_{ik} = 1, \quad t'_0(\sigma) = R(t_0) (x_0'^i \nu^k \delta_{ik}). \quad (15)$$

The energy-momentum tensor  $T^{MN}(x)$  of null string is defined by the variation of the action (4) with respect to  $G_{MN}(x)$

$$T^{MN}(X) = \frac{1}{\pi\gamma^*\sqrt{|G|}} \int d\tau d\sigma \dot{x}^M \dot{x}^N \delta^D(X - x). \quad (16)$$

The non-zero  $T_{MN}$  components (16) have the following form

$$T^{00}(X) = \frac{1}{\pi\gamma^*H^*} R^{(-D)}(t) \int d\sigma \psi_0(\sigma) \delta^{D-1}(X^M - x^M(\tau, \sigma)),$$

$$T^{ik}(X) = \frac{1}{\pi\gamma^*H^*} R^{(-D-2)} \int d\sigma \nu^i(\sigma) \nu^k(\sigma) \psi_0(\sigma) \delta^{D-1}(X^M - x^M(\tau, \sigma), \quad (17)$$

where the time dependence  $T^{MN}$  is factorized and accumulated in the scale factor  $R(t)$ . Since the vector  $\dot{x}^M$  is a light-like one, the trace  $T^{MN}$  equals to zero

$$Sp T = T_0^0 + G_{ij} T^{ij} = 0. \quad (18)$$

To consider the null strings (17) as a source of the FRW gravity (1) it is convenient to pass from the separate null string to the perfect gas of these strings, supposing that this gas is homogenous and isotropic. Then the energy density  $\rho(t)$  and pressure  $p(t)$  of the gas and its energy-momentum  $\langle T^{MN}(x) \rangle$  are connected by the relations

$$\langle T_0^0 \rangle = \rho(t), \quad \langle T_i^j \rangle = -p(t) \delta_i^j. \quad (19)$$

The tensor  $\langle T_{MN} \rangle$  is derived from  $T_{MN}$  by means of its space averaging when a set of null strings is introduced instead of a single null string. As a result of this procedure we find that  $\rho$  and  $p$  are

$$\langle T_0^0 \rangle = \rho(t) = \frac{A}{R^D(t)}, \quad \langle T_i^j \rangle = -p(t) \delta_i^j = -\frac{\delta_i^j}{D-1} \frac{A}{R^D(t)}, \quad (20)$$

where  $A$  is a constant with the dimension  $L^{-D}$ . Eqs.(20) show that the equation of state of null string gas is just the equation of state for a gas of massless particles

$$\langle Sp T \rangle = \langle T_M^M \rangle = 0 \iff \rho = (D-1)p. \quad (21)$$

Now assume that the gas of null strings is a dominated source of the FRW gravity (1). For the validity of the last conjecture it is necessary that the HE equations

$$R_M^N = 8\pi G_D \langle T_M^N \rangle \quad (22)$$

with the non-zero Ricci tensor  $R_M^N$  components defined by  $G_M^N$  (1)

$$R_0^0 = -\frac{D-1}{R} \frac{d^2 R}{dt^2},$$

$$R_i^k = -\delta_i^k \left[ \frac{1}{R} \frac{d^2 R}{dt^2} + \frac{D-2}{R^2} \left( \frac{dR}{dt} \right)^2 \right] \quad (23)$$

should contain the tensor  $\langle T_M^N \rangle$  (20) as a source of the FRW gravity. Moreover, the constraints (20), i.e.

$$\rho R^D - A = 0$$

must be a motion integral for the HE system (22). It is actually realized because

$$\frac{d}{dt}(\rho R^D) = -\frac{D-2}{16\pi G_D} R^{D-1} \frac{dR}{dt} R_M^M = 0,$$

since the trace  $R_M^M \sim \langle T_M^M \rangle = 0$  (see (21)). In view of this fact it is enough to consider only one equation of the system (22)

$$\left( \frac{1}{R} \frac{dR}{dt} \right)^2 = \frac{16\pi G_D}{(D-1)(D-2)} \frac{A}{R^D}, \quad (24)$$

which defines the scale factor  $R(t)$  of the FRW metric (1). Note that in the case  $D = 4$  Eq. (24) turns into the well-known Friedmann equation for the energy density in the radiation dominated universe with  $k = 0$ . The solutions of Eq. (24) are

$$\begin{aligned} R_I(t) &= [q(t_c - t)]^{2/D}, & t < t_c, \\ R_{II}(t) &= [q(t - t_c)]^{2/D}, & t > t_c, \end{aligned}$$

where  $q = [4\pi G_D A / (D-1)(D-2)]^{1/2}$  and  $t_c$  is a constant of integration which is a singular point of the metric (1). The solution  $R_I$  describes the stage of negatively accelerated contraction of D-dimensional FRW universe. In the small  $R$  limit ( $R \rightarrow 0$ ) this solution was found in [8] as an approximate asymptotic solution for the gas of strings with non-zero tension. For the case of null strings this solution is exact. The second solution (24)  $R_{II}$  describes the stage of the negatively accelerated expansion of the FRW universe from the state with space volume equals to "zero". Thus we see that the perfect gas of noninteracting null strings may be considered as an alternative source of the gravity in the FRW universes with  $k = 0$ . From the view point of string cosmology it seems important to find other exclusive metrics selfconsistently connected with the dynamics of null strings and allowing to consider the latter as a dominant source of gravity. The most interesting are the metrics which could describe the inflation of spaces. The solution of the problem under question is divided into two steps. At the first step it is necessary to exactly solve the null string motion equation in some cosmological background. The existence of such solution produces the constraints between  $T_{MN}$  and  $G_{MN}$ . From this moment the source in the HE equations becomes fixed as function on the background metric  $G_{MN}$ . At the second step we must solve these HE equations. The number of the independent functions characterizing  $G_{MN}$  may be less than the number of the independent HE equations. Then the condition of the selfconsistency of the considered scenario demands that the constraints (or some part of them) between  $T_{MN}$  and  $G_{MN}$  should be conserved integrals of the HE equations. The last condition may turn out to be incorrect. Then we must deform our initial background metric and again repeat the first and second steps with the new metric  $\tilde{G}_{MN}$ . As an example of above discussed inconsistency let us consider the  $(d+1)$ -dimensional FRW space  $(t, x^i)$  extended by the addition of  $n$  internal compactified dimensions  $y^a$  [8]. Choose the metric  $G_{MN}(z^L)$  (where  $z^L = (t, x^i, y^a)$ ) of this extended space-time in the form [8]:

$$ds^2 = G_{MN} dx^M dx^N = (dt)^2 - R^2(t) \sum_{ik=1}^d dx^i \delta_{ik} dx^k - r^2(t) \sum_{a,b=d+1}^{d+n} dy^a \delta_{ab} dy^b. \quad (25)$$

In the metric (25) the classical equation of motion (6) (where  $z^M$  is substituted instead of  $x^M$ ) are exactly integrable, so that

$$\begin{aligned}\tau &= \int_{t_0}^t \frac{dt}{\sqrt{\lambda(\sigma) + AR^{-2} + ar^{-2}}}, \\ x^i(\tau, \sigma) &= x^i(0, \sigma) + \mu^i(\sigma) \int_0^\tau d\tau R^{-2}(t), \\ y^a(\tau, \sigma) &= y^a(0, \sigma) + \mu^a(\sigma) \int_0^\tau d\tau r^{-2}(t),\end{aligned}\tag{26}$$

where  $\lambda$ ,  $\mu^i$ ,  $\mu^a$  and  $A = \mu^i \delta_{ik} \mu^k$ ,  $a = \mu^a \delta_{ab} \mu^b$  are the "constants" of integration and their functions. The substitution of the solution (26) into the constraints (6) transforms them into the constraints for the initial data

$$t'(\sigma) = \frac{\mu^i \delta_{ik} x_0^k + \mu^a \delta_{ab} y_0^a}{\sqrt{AR^{-2}(t_0) + ar^{-2}(t_0)}}, \quad \lambda(\sigma) = 0\tag{27}$$

The non-zero components of the energy-momentum tensor  $T_M^N$  (16) corresponding to the solutions (26) have the form

$$\begin{aligned}T_0^0(Z) &= \frac{1}{\pi\gamma^*} \int d\tau d\sigma R^{-d} r^{-n} (AR^{-2} + ar^{-2}) \delta^D(Z^M - z^M(\tau, \sigma)), \\ T_i^k(Z) &= -\frac{1}{\pi\gamma^*} \int d\tau d\sigma R^{-(d+2)} r^{-n} \mu_i \mu^k \delta^D(Z^M - z^M(\tau, \sigma)), \\ T_a^b(Z) &= -\frac{1}{\pi\gamma^*} \int d\tau d\sigma R^{-d} r^{-(n+2)} \mu_a \mu^b \delta^D(Z^M - z^M(\tau, \sigma)).\end{aligned}\tag{28}$$

As in the case of the FRW metric (1), the components of  $T_M^N$  are connected by the condition  $T_M^M = 0$ . The time dependence of  $T_M^N$  (28) is factorized and becomes evident after the integration with respect to  $t$  using the relation  $dt = \sqrt{AR^{-2} + ar^{-2}} d\tau$ . As in the previous case, the transition to the perfect gas of non-interacting null strings is realized by the space averaging  $T_M^N$  (28). As a result of this averaging we obtain the following expressions for the energy density  $\rho$  and pressures  $p$  and  $q$  acting in  $x$ - and  $y$ -subspaces

$$\begin{aligned}\langle T_0^0 \rangle &= \rho = \frac{1}{\gamma R^d r^n} \sqrt{AR^{-2} + ar^{-2}}, \\ \langle T_i^k \rangle &= -p \delta_i^k = -\frac{\delta_i^k}{\gamma R^d r^n} \frac{A}{R^2 d} \frac{1}{\sqrt{AR^{-2} + ar^{-2}}}, \\ \langle T_a^b \rangle &= -p \delta_a^b = -\frac{\delta_a^b}{\gamma R^d r^n} \frac{a}{r^2 n} \frac{1}{\sqrt{AR^{-2} + ar^{-2}}}.\end{aligned}\tag{29}$$

Then we find the equation of state

$$\rho = dp + nq,\tag{30}$$

characterizing the perfect gas of null strings in the metric (25). The HE equations for the metric (25) have the form

$$\begin{aligned} \frac{d}{R} \frac{d^2 R}{dt^2} + \frac{n}{r} \frac{d^2 r}{dt^2} &= -8\pi G_D \rho, \\ \frac{1}{R} \frac{d^2 R}{dt^2} + \frac{d-1}{R^2} \left( \frac{dR}{dt} \right)^2 + \frac{n}{rR} \frac{dR}{dt} \frac{dr}{dt} &= 8\pi G_D p, \\ \frac{1}{r} \frac{d^2 r}{dt^2} + \frac{n-1}{r^2} \left( \frac{dr}{dt} \right)^2 + \frac{d}{rR} \frac{dR}{dt} \frac{dr}{dt} &= 8\pi G_D q, \end{aligned} \quad (31)$$

where the expressions (29) for  $\rho$ ,  $p$  and  $q$  are implied. The system (31) consists of three equations for two functions  $R(t)$  and  $r(t)$ . Unlike the Friedmann system (22), the constraint (29) between  $\rho$  and  $R$  is non conserved by the dynamics defined by Eqs.(31). Therefore the perfect gas of null strings can not be considered as a dominant gravity source for the space–time (25).

Returning to the case of the FRW universes note that null strings may apparently be considered as an ordered soliton–like state of a large number of coherent photons. The formation of such string–like condensate of radiation can reflect the possible spontaneous breaking of the space–time symmetry in the FRW universes.

The second conclusion concerns the possibility of the generalization of null string cosmology to the null  $p$ –brane [9,10] cosmology. Such a generalization is actually possible because the classical dynamics of null  $p$ –branes is described by the similar (6) equations of motion together with trivially extended system of constraints. Therefore the results derived here for the gas of null strings in the FRW universes are conserved also and for the perfect gas of null  $p$ –branes in the FRW spaces.

A new moment connected with the null  $p$ –branes is that they can be considered as  $p$ –dimensional ordered structures, in contrast to the 1–dimensional null string structures. So the world hypervolumes of the null  $p$ –branes may be discussed as  $(p+1)$ –dimensional galaxies. When null  $p$ –brane achieves the horizon it is splitted into a large number of ”small” pieces which are detected as the separate minihalaxies with the practically identical physical characteristics. May be the consideration of the visible galaxies as such pieces of null  $p$ –branes could shed new light in solution of the horizon problem.

The authors wish to thank D.V.Volkov, Yu.P.Stepanovskij, Yu.P.Peresun’ko, A.P.Rekalo and B.I.Savchenko for the interest to this work and useful discussions.

This work was supported in part by the International Science Foundation (grant RY9200), International Association for the Promotion of Cooperation with Scientists from the independent states of the former Soviet Union (grants INTAS-93-127, 93-633, 94-2317) and by the Ukrainian State Committee in Science and Technologies, Grant N 2.3/664.

## References

1. A.D.Linde, Physics of elementary particles and inflation cosmology, Nauka, Moscow, 1990.
2. K.A.Olive, Phys.Rep.**190** 307 (1990).
3. G.Veneziano, Strings, cosmology, ... and a particle, CERN Preprint TH.7502/94.
4. N.Sanchez and G.Veneziano, Nucl.Phys.**B333** 253 (1990).
5. M.Gasperini, N.Sanchez and G.Veneziano, Highly unstable fundamental strings in inflationary cosmologies, CERN preprint TH.5893/90.
6. M.Gasperini and G.Veneziano, Mod.Phys.Lett. **A8** 3701 (1993)  
Phys.Rev. **D50** 2519 (1994).
7. M.Gasperini, N.Sanchez and G.Veneziano, Self-sustained inflation and dimensional reduction from fundamental strings, CERN preprint TH.5010/91
8. A.A.Zheltukhin, Pis'ma Zhur.Eksp.Teor.Fiz. **46** 262 (1987) (in Russian); Yad.Fiz. **48** 375 (1988) (in Russian).
9. I.A.Bandos and A.A.Zheltukhin, Fortschr. Phys.**41** 619 (1993).
10. A.Z.Petrov, New methods in general relativity, Nauka, Moscow, 1966.