# Distances on a one-dimensional lattice from noncommutative geometry

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## Abstract

In the following paper we continue the work of Bimonte-Lizzi-Sparano[1] on distances on a one dimensional lattice. We succeed in proving analytically the exact formulae for such distances. We find that the distance to an even point on the lattice is the geometrical average of the "predecessor" and "successor" distances to the neighbouring odd points.

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# 1 Distances on a one dimensional lattice from noncommutative geometry

The distance between two points a and b on a given space, is defined, from the geometrical point of view as the infimum of all lengths of paths connecting the two points on that space.

$$d(a, b) = \inf \ l_\alpha (a, b) \tag{1}$$

where $l_\alpha$ is the length of the path $\alpha$ which connects the points $a$ and $b$. This definition is simple and well understood, when dealing with smooth metric manifolds. However the above definition is not applicable to discrete spaces in which the term "path" is not well defined. In order to be able to define a distance on such spaces too, one has to use tools other than the usual geometrical ones. Those tools are found in noncommutative geometry (n.c.g). The n.c.g definition for distance is:[2]

$$d(a, b) = \sup_f \{|f(a) - f(b)| \ : \ f \in A, \ \| \ [ \ D, f \ ] \ \| \leq 1\} \tag{2}$$

where $a, b \in X, \ f \in A,$ $A$ is the algebra of functions on $X,$ $D$ is a Dirac operator (which is a self adjoint operator with compact resolvent) acting in the Hilbert space $H$, and the norm on the r.h.s is the norm of operators in $H$. Both definitions give the same result when the base space is a Riemannian manifold. However the n.c.g definition has the advantage of being applicable to discrete spaces too.

In the following we continue the work of Ref[1] (which we denote "BLS") by finding and proving the exact formula for distances on a one dimensional lattice. We use the BLS notation. As a first step we summarize briefly the BLS work relevant to a one dimensional lattice.

A one dimensional lattice is defined as:

$$x_k = ak, \quad k \in Z \tag{3}$$

In order to compute the distance between two point $x_0$ and $x_k$ one has first to define the Dirac operator. BLS used the Wilson definition for a Dirac operator:

$$(D\Psi)_x = \frac{1}{2ia} (\Psi_{x+1} - \Psi_{x-1}) \tag{4}$$

where $D$ acts on the Hilbert space of square integrable spinors $\Psi$ ($\Psi \in L_2$), and $x$ is a site on the one dimensional lattice. Let $f$ act on the Hilbert space as a linear operator. It is shown that in order to get the supremum it is enough to take the supremum over the set of real functions:

$$f^{(k)} \equiv \left\{ f_i^{(k)} \quad ; \quad i \in Z \right\} \tag{5}$$

such that:

$$
\begin{array}{ll}
\text{a)} & f_i^{(k)} = f_0^{(k)} \quad \forall \, i < 0 \\
\text{b)} & f_i^{(k)} = f_k^{(k)} \quad \forall \, i > k \\
\text{c)} & f_i^{(k)} \leq f_j^{(k)} \quad \forall \, i < j \\
\text{d)} & \left\| \left[ D , \, f^{(k)} \right] \right\| \leq 1
\end{array}
\tag{6}
$$

BLS then show that, following Eq.(2),(6) the distance becomes :

$$d_k = \sup \left\{ f_k^{(k)} - f_0^{(k)} \quad : \quad \left\| \left[ D , \, f^{(k)} \right] \right\| \leq 1 \right\} \tag{7}$$

where the norm on the r.h.s is equal to the maximum eigenvalue $r$ of a square, symmetric and real "three-diagonal" matrix $H^{(k)}$ :

$$
\begin{pmatrix}
0 & \Delta_1 & 0 & 0 & 0 \\
\Delta_1 & 0 & \Delta_2 & 0 & 0 \\
0 & \Delta_2 & 0 & \ddots & 0 \\
0 & 0 & \ddots & \ddots & \Delta_k \\
0 & 0 & 0 & \Delta_k & 0
\end{pmatrix}
\tag{8}
$$

where $\Delta_i = f_i^{(k)} - f_{i-1}^{(k)}$ .

Since all the elements of $H^{(k)}$ are not negative, one can use a theorem[3] according to which, in such case the maximum eigenvalue of $H^{(k)}$ is less than 1, if and only if, all the leading principal minors $H_n$ ($\Delta_1, \ldots, \Delta_{n-1}$) where $n = 1, \ldots, k+1$ of the matrix $I - H$ are positive :

$$H_1^{(k)} = 1 > 0, \ldots, H_i^{(k)} > 0, \ldots, H_k^{(k)} > 0, \, H_{k+1}^{(k)} = \det \left( I^{(k)} - H^{(k)} \right) = 0 \tag{9}$$

It is then shown that in order to find the distance, i.e. the supremum of $\sum_i \Delta_i$ satisfying all conditions set by (9), one can use the method of

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Lagrange multipliers, and solve the following set of equations in the unknowns $\left( \triangle_1^{(k+1)} \ldots \triangle_k^{(k+1)} \right)$ and $\alpha$ (the Lagrange multiplier) :

$$\left\{ \begin{array}{ll} \frac{\partial}{\partial \triangle_j} \left[ \sum_{i=1}^k \triangle_i + \alpha \mathrm{H}_{k+1}^{(k)} \right]_{\triangle_j = \triangle_j^{(k+1)}} = 0, & \forall \; j \le k \\ \mathrm{H}_{k+1}^{(k)} = 0 & \end{array} \right\} \tag{10}$$

to find what the distance is.

In the following we solve exactly the above equations.

## 2   Exact solution

As the first step in solving the above equations we have to use some of the properties of real, symmetric, three-diagonal matrices:

The first property is :

$$\mathrm{H}_{k+1} = \mathrm{H}_k - \triangle_k^2 \mathrm{H}_{k-1} \tag{11}$$

from which it follows that:

$$\text{a)} \quad \triangle_k = \left( \frac{\mathrm{H}_k - \mathrm{H}_{k+1}}{\mathrm{H}_{k-1}} \right)^{\frac{1}{2}} \qquad \text{if } \mathrm{H}_{k-1} \neq 0$$

$$\text{b)} \quad \frac{\partial}{\partial \triangle_k} \mathrm{H}_{k+1} = -2\triangle_k \mathrm{H}_{k-1} \qquad \forall \; k \tag{12}$$
$$\text{(because there is no } \triangle_k \text{ in } \mathrm{H}_k\text{)}$$

$$\text{c)} \quad \frac{\partial}{\partial \triangle_{k-1}} \mathrm{H}_{k+1} = \frac{\partial}{\partial \triangle_{k-1}} \mathrm{H}_k \qquad \forall \; k$$
$$\text{(because there is no } \triangle_{k-1} \text{ in } \mathrm{H}_{k-1}\text{)}$$

We now make use of the fact that :

$$\frac{\partial}{\partial x} \det A_{k+1} = \sum_{i=1}^{k+1} A_{i,k+1}^{(x)} \tag{13}$$

where $A_{k+1}$ is a $(k+1) \times (k+1)$ matrix and :

$$A_{i,k+1}^{(x)} \equiv \det \begin{pmatrix} \begin{pmatrix} \vdots \end{pmatrix}_{1 \le j \le i-1} \\ \left( \frac{\partial}{\partial x}, \ldots, \frac{\partial}{\partial x} \right)_{j=i} \\ \begin{pmatrix} \vdots \end{pmatrix}_{k+1 \ge j \ge i+1} \end{pmatrix} \tag{14}$$

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i.e. a determinant of the matrix where the derivation is imposed on the $i$ $-th$ row.

However since $I - H$ is also a three-diagonal matrix, $\triangle_j$ is found only in the $j - th$ and $(j+1) - th$ rows. One thus gets :

$$\frac{\partial}{\partial \triangle_j} H_{k+1} = H_{j,k+1}^{(\triangle_j)} + H_{j+1,k+1}^{(\triangle_j)} \tag{15}$$

One can show (using an even number of inter-changes between rows and columns) that: $H_{j,k+1}^{(\triangle_j)} = H_{j+1,k+1}^{(\triangle_j)}$, yielding,

$$\frac{\partial}{\partial \triangle_j} H_{k+1} = -2\triangle_j H_{j,k+1}^{(\triangle_j)} = -2\triangle_{j+1,k+1} H_{j+1,k+1}^{(\triangle_j)} \tag{16}$$

And by using the block structure of $H_{j,k+1}^{(\triangle_j)}$ one gets that :

$$\frac{\partial}{\partial \triangle_j} H_{k+1} = -2\triangle_j H_{j-1}(\triangle_1, \ldots, \triangle_{j-2}) \cdot H_{k-j}(\triangle_i \to \triangle_{k-i+1} : \forall\, i \le k-j-1) \tag{17}$$

but since it was shown[1] that the maximum is unique, and since the equations (10) have a symmetry under $\triangle_i \longleftrightarrow \triangle_{k-i+1}$ the solution must fix $\triangle_i = \triangle_{k-i+1}$. Thus what one essentially gets is simply that:

$$\frac{\partial}{\partial \triangle_j} H_{k+1} = -2\triangle_j H_{j-1} H_{k-j} \tag{18}$$

Equations (10) take the following form:

$$\left\{ \begin{array}{c} 1 - 2\alpha\triangle_j H_{j-1} H_{k-j} = 0 \\ \vdots \\ H_{k+1} = 0 \end{array} \right\}_{1 \le j \le k} \tag{19}$$

It can be verified, that if $H_{k-1}$ and $H_k$ are both $\ne 0$, it follows from the last equation that $\triangle_k = \left(\frac{H_k}{H_{k-1}}\right)^{\frac{1}{2}}$; while from the $j = k$ equation one gets, $\alpha = \frac{1}{2\sqrt{H_k H_{k-1}}}$ .

Until now what we have done is just to simplify the equations that we have to solve. From now on we assume what the solutions for the unknowns

should be. Since the solution is unique, then by setting the assumed solutions (i.e. the $\triangle_i$ 's) into the set of equations (19) and showing that they really solve the equations, we are essentially proving that the assumed solutions are correct. Knowing the solutions for the $\triangle_i$ 's one can then find what the distance is. As the first case we will solve for $k =$ even.

## 2.1 The $k = even$ case

Let us assume that the solution for the $\triangle_i$ 's is:

$$\triangle_i^{(k)} = \frac{\frac{1}{2}\left(1 - (-1)^i\right)\left(\frac{k}{2}+1\right) + (-1)^i\left[\frac{i+1}{2}\right]}{\sqrt{\frac{k}{2}\left(\frac{k}{2}+1\right)}} \qquad \forall\, i \leq k \qquad (20)$$

where the $[\ ]$ bracket stand for the integral value of the term within; and the Lagrange multiplier is :

$$\alpha^{(k)} = \frac{\left[k\left(k+2\right)\right]^{\frac{k-1}{2}}}{2^{k-1} k\left[\left(\frac{k}{2}-1\right)!\right]^2} \qquad (21)$$

By using the recursion relation (11), and from the fact that $H_0 = H_1 = 1$ , one can prove by induction that:

$$H_{2l+1}^{(k)} = \frac{2^{2l} l!\left(\frac{k}{2}-1\right)!}{\left[k\left(k+2\right)\right]^l\left(\frac{k}{2}-l-1\right)!} \qquad :\ 1 \leq l \leq \frac{k}{2} \qquad (\Rightarrow\ H_{k+1} = 0) \qquad (22)$$

and

$$H_{2l}^{(k)} = \frac{2^{2l} l!\left(\frac{k}{2}\right)!}{\left[k\left(k+2\right)\right]^l\left(\frac{k}{2}-l\right)!} \qquad :\ 1 \leq l \leq \frac{k}{2} \qquad (23)$$

We now have all the ingredients we need for the equations. All we have to do is to check that essentially all the equations are fulfilled; and indeed they are, as can be verified by a few arithmetical steps.

After proving that the $\triangle_i$ 's are the solution for the set of equations (19) we can now find exactly, that the distance in the $k = even$ case is :

$$d_k = 2a \cdot \sum_{i=1}^{k} \triangle_i^{(k)} = 2a \cdot \sqrt{\frac{k}{2}\left(\frac{k}{2}+1\right)} = a\sqrt{k\left(k+2\right)} \qquad (24)$$

Q.E.D. for the $k = even$ case.

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## 2.2   The $k = odd$   case :

Let us assume that the solution in this case has the following structure:

$$\triangle_{2i+1}^{(k)} = (1 - \varepsilon)$$

$$\triangle_{2i}^{(k)} = \varepsilon \tag{25}$$

$$\alpha^{(k)} = \frac{1}{2\varepsilon^{\frac{k-1}{2}}(1-\varepsilon)^{\frac{k+1}{2}}}$$

and let us also assume temporarily that $H_1^{(k)} = 1 - \varepsilon$. It is easy to show by the recursion relation (11) that:

$$H_{2j}^{(k)} = \varepsilon^j (1 - \varepsilon)^j$$
$$\forall \, 0 \le j \le \frac{k-1}{2} \tag{26}$$
$$H_{2j+1}^{(k)} = \varepsilon^j (1 - \varepsilon)^{j+1}$$

The solutions can now be set into the equations (19). It is easy to verify that those solutions obey the set of equations. However since we know that $H_1^{(k)} = 1$  (rather then $1 - \varepsilon$)  one has to take the limit where $\varepsilon \to 0^+$. Thus, when the limit  $\varepsilon \to 0^+$  is applied, it follows that:

$$H_0^{(k)} = H_1^{(k)} = 1, \quad H_j^{(k)} = 0 \ \forall \, j \ge 2 \tag{27}$$

which means that all the conditions in Eq.(9) are fulfilled. Taking the limit $\varepsilon \to 0^+$  it follows that:

$$\triangle_{2j+1} = 1$$

$$\triangle_{2j} = 0 \quad \forall \, 0 \le j \le \frac{k-1}{2} \tag{28}$$

$$\alpha \to \infty$$

We were able to use the limiting procedure, through the fact that this is a supremum problem - so we just had to prove that the limit exists. By that we have essentially proven that the solutions are as listed in Eq.(28).

Having all the $\triangle_i^{(k)}$ 's we can find that the distance in the $k = odd$ case is:

$$d_k = 2a \sum_{i=1}^{k} \triangle_i^{(k)} = a\,(k+1) \tag{29}$$

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Q.E.D. for the $k = odd$ case.

We can summarize that the distances in the lattice are:

$$\left\{\begin{array}{l} d_{2j-1} = 2aj \\ \\ d_{2j} = 2a\sqrt{j(j+1)} \end{array}\right\} \tag{30}$$

# 3  Discussion

## 3.1  Mathematical aspects

All distances have anomalous behavior, as compared to what one expects classically. The anomaly in the odd point distances is exactly equal to one. The anomaly in the even point distances depends on the points. These are non-constant, irrational numbers, smaller than one, which asymptotically tend to 1.

The second outcome from (30) is that the distance to an even point is the geometrical average of the distances to the nearby predecessor and successor odd points (rather than an arithmetical average which one would have expected classically).

Concerning the large $k$ limit, it follows where $k = 2j \to \infty$ the distance to an even point $k = 2j$ has the following behavior:

$$2a\sqrt{j(j+1)} \longrightarrow 2aj\left(1 + \frac{1}{2j}\right) \tag{31}$$

One can thus see that the asymptotic behavior of the even point distances becomes the same as for the odd point distances (in other words the geometrical average asymptotically tends to the arithmetical average).

## 3.2  Physical aspects

One can say that the cause for the results we got to be different from what one might expect, is an outcome of using the local discrete Wilson - Dirac operator. One should perhaps try non-local Dirac operator in order to get the classical behavior that we expect.

However as far as the physical world is concerned, we have shown that asymptotically, in the large $k$ limit, the differnce in behavior of the odd

point distances and the even point distances on the lattice vanishes. Usually in nature one deals with lattices with very large $k$. Thus all these effects are not seen. One could treat the lattice as a "quantum" system, in which the quantum behavior of nature is revealed for small $k$, and for large $k$ the classical behavior is reached asymptotically (We remind the reader that the n.c.g definition for distance considers not only the base space, but also the Hilbert space). Thus, following this point of view, one can continue working with the local Wilson - Dirac operator, though revealing non-classical behavior of the lattice.

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# References

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