YITP Uji Research Center

YITP/U-95-22
June 1995
hep-th/9507001

Instability of Solitons in imaginary coupling affine Toda Field Theory

S. Pratik KHASTGIR [1] and Ryu SASAKI [2]

Uji Research Center, Yukawa Institute for Theoretical Physics,
Kyoto University,
Uji 611, Japan.

Abstract

Affine Toda field theory with a pure imaginary coupling constant is a non-hermitian theory. Therefore the solutions of the equation of motion are complex. However, in $1+1$ dimensions it has many soliton solutions with remarkable properties, such as real total energy/momentum and mass. Several authors calculated quantum mass corrections of the solitons by claiming these solitons are stable. We show that there exists a large class of classical solutions which develops singularity after a finite lapse of time. Stability claims, in earlier literature, were made ignoring these solutions. Therefore we believe that a formulation of quantum theory on a firmer basis is necessary in general and for the quantum mass corrections of solitons, in particular.

[1]Monbusho Fellow.

[2]Supported partially by the grant-in-aid for Scientific Research, Priority Area 231 "Infinite Analysis" and General Research (C) in Physics, Japan Ministry of Education.

1 Introduction

In this paper we will address the problem of the stability of solitons in the imaginary coupling affine Toda field theory, which is obtained from the affine Toda field theory by replacing the real coupling constant β by a purely imaginary one $i\beta$, $i = \sqrt{-1}$. The affine Toda field theory is one of the best understood field theories at the classical [1] and at the quantum levels [2, 3, 4, 5, 6][7, 8], thanks to its integrability. It is the close connection between the affine Toda field theory and the conformal field theory in 2 dimensions, another group of best understood quantum field theories, (integrable deformation of conformal field theory [9, 10]) that led to the interesting but controversial "imaginary coupling" affine Toda field theory.

This apparently tiny change $\beta \to i\beta$ brings huge differences between the affine Toda field theory and its imaginary coupling counterpart. Among them the following two aspects are most prominent: The first is the emergence of soliton solutions and other interesting exact solutions in the imaginary coupling theory, just as in the well known sine-Gordon theory, which is the simplest example of the imaginary coupling affine Toda field theories. In contrast, the affine Toda field theory is known to have no solitons. The second is the lack of reality/hermiticity of the Lagrangian and action in all the imaginary coupling theories except for the sine-Gordon theory.

Many interesting and beautiful results on solitons have been obtained by various authors. By applying Hirota's method, Hollowood [11] obtained various simple soliton solutions in the $a_n^{(1)}$ theory. The total energies of these one soliton solutions are real, although the solutions themselves and the energy densities are complex. The masses of the solitons are found to be proportional to the masses of the fundamental particles of the corresponding affine Toda field theory. This result was further developed by many authors [12, 13, 14, 15, 16, 17]. A complete set of soliton solutions was obtained by invoking representation theory of affine Lie algebras by Olive and collaborators [14, 16]. The mass spectrum of the one soliton solutions is now known and it is related to the mass spectrum of the fundamental particles in the real coupling theories in a very interesting way.

Hollowood [18] then set a new trend by calculating "quantum mass corrections" to the solitons in $a_n^{(1)}$ theory. It was reported that the soliton mass ratios in the $a_n^{(1)}$ theories were unchanged after one-loop corrections. Then "quantum mass corrections" to the solitons in all the affine Toda theories were also investigated by Watts [19], Delius and Grisaru [20] and MacKay and Watts [21]. They obtained similar but slightly differing results and the relationship between the classical soliton masses and the quantum mass corrections seemed to be more involved.

On the other hand, the lack of reality/hermiticity of the Lagrangian does not seem to have attracted much attention. This is a rather strange situation, since the hermiticity or reality is sacrosanct in any physical theory and especially in quantum physics, in which non-hermitian Lagrangian or Hamiltonian implies non-unitarity and non-conservation of probability. The classical theory is well defined mathematically, even if the Lagrangian is non-hermitian. Although the physical interpretation of the solutions of the equation of motion is dubious, the concept of solutions is solid. In contrast, to the best of our knowledge, the quantum field theory or even quantum mechanics of non-hermitian systems simply does not exist. Thus, at present, quantum affine Toda field theory with imaginary coupling should be considered to be of heuristic nature. Therefore the clarification of the hermiticity issue is essential for any serious treatments of the quantum solitons, especially for those of quantum mass corrections. The lack of unitarity in quantum field theory is usually related with the lack of stability of the solutions of the corresponding classical field theory, which are the solitons in the present case.

In this context, certain stability arguments of soliton solutions connected with the "twisted reality" or "twisted hermiticity" relations were produced by Hollowood [18], Evans [22] and Delius and Grisaru [20]. We will show that none of these arguments is satisfactory. We also give various explicit solutions which develop singularity after a finite lapse of time. Moreover, these singular solutions are far more abundant than the non-singular ones. To be more precise, the moduli space of the singular 1-soliton solutions is two dimensional whereas the non-singular 1-soliton solutions have one dimensional moduli space. These results cast a big question mark on the works of quantum mass corrections of solitons, in particular and on those of the soliton physics/mathematics in the imaginary coupling affine Toda field theory in general. The problem of the hermiticity and/or unitarity in the imaginary coupling affine Toda field theory deserves a far greater attention than has been given, since the stake is very high.

This paper is organised as follows: in section 2 we briefly review the essentials of affine Toda field theory in order to set the stage and to introduce notation. In section 3 derivation of the simple solitons are given and some of their salient properties are recapitulated. In Section 4 various existing stability arguments of solitons are examined and shown to have flaws. In Section 5 and 6 instability of "vacuum" and single soliton solutions are shown by examining the time evolution of the explicit solutions. The moduli space of generalised single-soliton solutions are also introduced in order to show that the generic solutions are unstable. Section 7 is devoted to a brief summary. Appendix A gives a simple example of 2×2 matrix satisfying the "twisted reality condition" but failing to have real eigenvalues. Appendix B gives another simple example of 2×2 symmetric matrices which are not hermitian. It is shown that its eigenvectors are not guaranteed to span the entire vector space.

2 Affine Toda field theory

Affine Toda field theory [1] is a massive scalar field theory with exponential interactions in $1+1$ dimensions described by the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi^a\partial^\mu\phi^a - V(\phi), \tag{2.1}$$

in which the potential is given by

$$V(\phi) = \frac{m^2}{\beta^2}\sum_0^r n_j e^{\beta\alpha_j\cdot\phi}. \tag{2.2}$$

The field ϕ is an r-component scalar field, r is the rank of a compact semi-simple Lie algebra g with α_j; $j = 1, \ldots, r$ being its simple roots. The roots are normalised so that long roots have length 2, $\alpha_L^2 = 2$. An additional root, $\alpha_0 = -\sum_1^r n_j\alpha_j$ is an integer linear combination of the simple roots, is called the affine root; it corresponds to the extra spot on an extended Dynkin diagram for $\hat{g}$ and $n_0 = 1$. When the term containing the extra root is removed, the theory becomes conformally invariant (conformal Toda field theory). The simplest affine Toda field theory, based on the simplest Lie algebra $a_1^{(1)}$, the algebra of $\widehat{su}(2)$, is called sinh-Gordon theory, a cousin of the well known sine-Gordon theory. m is a real parameter setting the mass scale of the theory and β is a real coupling constant, which is relevant only in quantum theory.

Toda field theory is integrable at the classical level due to the presence of an infinite number of conserved quantities. Many beautiful properties of Toda field theory, both at the classical and quantum levels, have been uncovered in recent years. In particular, it is firmly believed that the integrability survives quantisation. The exact quantum S-matrices are known [2, 3, 4, 5, 6],[7, 8] for all the Toda field theories based on non-simply laced algebras as well as those based on simply laced algebras. The singularity structure of the latter S-matrices, which in some cases contain poles up to 12-th order [4], is beautifully explained in terms of the singularities of the corresponding Feynman diagrams [23], so called Landau singularities.

The imaginary coupling affine Toda field theory is obtained simply by replacing β by $i\beta$ ($i=\sqrt{-1}$) in the Lagrangian

$$\mathcal{L}_I = \frac{1}{2}\partial_\mu\phi^a\partial^\mu\phi^a - V_I(\phi), \quad V_I(\phi) = -\frac{m^2}{\beta^2}\sum_0^r n_j\left(e^{i\beta\alpha_j\cdot\phi}-1\right), \tag{2.3}$$

and by reinterpreting the fields ϕ as *complex*. The Lagrangian is *not hermitian* except for the $a_1^{(1)}$ theory ($r=1$, $n_1=n_0=1$, $\alpha_1=-\alpha_0=\sqrt{2}$) with a real field. In the rest of this paper, $a_1^{(1)}$ theory is excluded. The equation of motion obtained from the above Lagrangian reads

$$\partial_\mu^2\phi = -\frac{m^2}{i\beta}\sum_0^r n_j\alpha_j e^{i\beta\alpha_j\cdot\phi}. \tag{2.4}$$

It is easy to see that it has no real solutions except for the trivial constant solutions corresponding to the "minima" of the imaginary potential[3]:

$$\frac{2\pi}{\beta}\sum_{j=1}^r k_j\lambda_j, \quad k_j: \text{ integer.} \tag{2.5}$$

Here $\{\lambda_j\}$, $j=1,\ldots,r$ is the dual basis to the simple roots $\{\alpha_j\}$:

$$\alpha_j\cdot\lambda_k = \delta_{jk}. \tag{2.6}$$

It should be noticed that the structure of the Lax pair, the existence of an infinite set of conserved quantities in involution, the corner stone of the integrability, and etc. etc are the same in the imaginary coupling theory as in the real coupling theory [18, 22]. However, their actual contents are markedly different. In the real coupling theory, the conserved energy is *positive definite*, namely each term is positive. Thus if one follows the time evolution of a regular initial data $\phi(x,0)$, $\partial_t\phi(x,0)$ with finite energy, the field $\phi(x,t)$ and its first derivatives are always finite everywhere, since any singularity would violate the conservation of energy. In the imaginary coupling theory, the conservation of energy still holds but the energy has negative as well as positive terms. The conservation of energy fails to prevent the singularities in the time evolution. We show this phenomenon by explicit examples in section 5 and 6. Other simple examples of singularities in the case of non-positive definite energy caused by integrable boundary interactions were given in Ref.[24].

In the rest of this paper we will discuss the $a_n^{(1)}$ theory only ($r=n$, $n_j=1$, $j=0,1,\ldots,n$) for definiteness and simplicity. This also makes it easy to concentrate on the fundamental and universal problems of (non) hermiticity and stability without being bothered by the Lie algebra technicalities.

[3]Points in (2.5) simply correspond to the stationary points of the potential V_I, $\frac{\partial V_I}{\partial\phi}=0$. The second derivative matrix $\frac{\partial^2 V_I}{\partial\phi^a\partial\phi^b} = m^2\sum n_j\alpha_j^a\alpha_j^b$, which is equal to the classical (mass)^2 matrix, is positive definite. Since the potential is an analytic function of the field ϕ, it has neither a minimum nor a maximum in any open domain. In other words the potential V_I as a function of the complex field ϕ is *not bounded from below*, another sign of instability.

3 Solitons

In this section we recapitulate some of the results on the explicit soliton solutions which are necessary for our purposes. Only the very fundamental features of the soliton solutions are relevant here, so we follow the elementary method of Hollowood [11]. Like the sine-Gordon solitons, these solitons interpolate various "vacua" or the "minima" of the complex potential (2.5). We start from the following Hirota ansatz

$$\phi(x,t) = -\frac{1}{i\beta}\sum_{j=0}^{n}\alpha_j \log\tau_j. \tag{3.1}$$

In terms of τ_j the equation of motion can be decoupled into

$$\ddot{\tau}_j\tau_j - \dot{\tau}_j^2 - \tau_j''\tau_j + {\tau_j'}^2 = m^2(\tau_{j-1}\tau_{j+1} - \tau_j^2). \tag{3.2}$$

The label on τ_j is understood modulo $n+1$, which reflects the periodicity of the $a_n^{(1)}$ Dynkin diagram.

One soliton solution is obtained by assuming

$$\tau_j = 1 + \tau_j^{(1)}. \tag{3.3}$$

By substituting (3.3) into (3.2) we get

$$\begin{aligned}\ddot{\tau}_j^{(1)} &- {\tau''}_j^{(1)} - m^2(\tau_{j-1}^{(1)} + \tau_{j+1}^{(1)} - 2\tau_j^{(1)}) \\ &+ \ddot{\tau}_j^{(1)}\tau_j^{(1)} - (\dot{\tau}_j^{(1)})^2 - {\tau''}_j^{(1)}\tau_j^{(1)} + ({\tau'}_j^{(1)})^2 - m^2(\tau_{j-1}^{(1)}\tau_{j+1}^{(1)} - (\tau_j^{(1)})^2) = 0.\end{aligned} \tag{3.4}$$

A characteristic feature of the Hirota method is that this equation is decomposed into linear and quadratic parts:

$$\ddot{\tau}_j^{(1)} - {\tau''}_j^{(1)} - m^2(\tau_{j-1}^{(1)} + \tau_{j+1}^{(1)} - 2\tau_j^{(1)}) = 0, \tag{3.5}$$

$$\ddot{\tau}_j^{(1)}\tau_j^{(1)} - (\dot{\tau}_j^{(1)})^2 - {\tau''}_j^{(1)}\tau_j^{(1)} + ({\tau'}_j^{(1)})^2 - m^2(\tau_{j-1}^{(1)}\tau_{j+1}^{(1)} - (\tau_j^{(1)})^2) = 0, \tag{3.6}$$

and the quadratic part is always satisfied by the solution of the linear part. The linear equation is solved by

$$\tau_j^{(1)} = \exp(\sigma x - \lambda t + x_0 + j\rho), \tag{3.7}$$

for constants σ, λ, x_0 and ρ. The periodicity in the label j on τ_j then implies

$$\rho = \frac{2\pi i a}{n+1}, \qquad a: \text{ integer } \quad 1 \le a \le n. \tag{3.8}$$

The parameters σ, λ and the integer a are constrained by

$$\mathcal{F}(\sigma,\lambda,a) \equiv \sigma^2 - \lambda^2 - 4m^2\sin^2\frac{\pi a}{n+1} = 0, \tag{3.9}$$

in order to satisfy (3.5). It is very easy to verify that the quadratic equation is also satisfied.

Thus we arrive at the explicit one soliton solution

$$\phi_a(x,t) = -\frac{1}{i\beta}\sum_{j=0}^{n}\alpha_j \log[1+\omega^{aj}\exp(\sigma x - \lambda t + x_0)], \tag{3.10}$$

in which the parameters σ and λ should satisfy

$$\sigma^2 - \lambda^2 = 4m^2 \sin^2 \frac{\pi a}{n+1}, \tag{3.11}$$

and ω is a primitive root of unity $\omega = e^{\frac{2\pi i}{n+1}}$, $\omega^{n+1} = 1$. The right hand side of (3.11) is is simply the mass^2 of the fundamental particles in $a_n^{(1)}$ Toda field theory. The above 1-soliton solutions are classified into three different types as follows:

$$\begin{array}{lll} 1)\ \sigma,\lambda : \text{Real} & \text{r}-\text{soliton} \\ 2)\ \sigma,\lambda : \text{pure Imaginary} & \text{i}-\text{soliton} \\ 3)\ \sigma,\lambda : \text{Complex} & \text{c}-\text{soliton.} \end{array} \tag{3.12}$$

The c-solitons contain all the 1-soliton solutions not belonging to the r-solitons and i-solitons; for example, σ: real and λ: pure imaginary. The parameters (σ, λ) have one real degree of freedom in the cases of r-solitons and i-solitons, whereas (σ, λ) have two real degrees of freedom in the c-soliton case. The r- and i-solitons are located at the boundaries of the moduli space of the c-soliton solutions. As we will see in section 5, these three types of 1-soliton solutions have very different characters. The other parameter x_0 is in general complex and its real part x_{0R} is related to the location of the soliton at $t = 0$. The imaginary part of x_0, x_{0I} can be restricted to $0 \le x_{0I} < 2\pi$ without loss of generality.

In the rest of this section we give the two soliton solutions without derivation [11]. Let us define

$$y_j^{(p)} = \sigma_p x - \lambda_p t + x_0^{(p)} + \frac{2\pi i a_p}{n+1} j, \tag{3.13}$$

in which the parameters σ_p, λ_p and the integer a_p satisfy the constraint

$$\mathcal{F}(\sigma_p, \lambda_p, a_p) = 0.$$

Then a general two soliton solution is given by

$$\tau_j = 1 + e^{y_j^{(1)}} + e^{y_j^{(2)}} + e^{y_j^{(1)}+y_j^{(2)}+\gamma_{(12)}}, \tag{3.14}$$

in which the interaction function $e^{\gamma_{(pq)}}$ is given by [11]

$$e^{\gamma_{(pq)}} = -\frac{\mathcal{F}(\sigma_p-\sigma_q, \lambda_p-\lambda_q, a_p-a_q)}{\mathcal{F}(\sigma_p+\sigma_q, \lambda_p+\lambda_q, a_p+a_q)}. \tag{3.15}$$

4 "Stability" of 1 Soliton Solutions

As a "prerequisite" for calculating "quantum mass corrections" of solitons, Hollowood [18], Evans [22] and Delius and Grisaru [20] produced certain arguments that the soliton solutions are classically "stable". In this section we recapitulate the essence of their "stability" arguments and show that these are flawed. They picked up a stationary r-soliton solution located at the origin

$$\bar{\phi}_a(x) = -\frac{1}{i\beta}\sum_{j=0}^{n}\alpha_j \log[1+\omega^{aj}e^{m_a x}], \quad m_a = 2m\sin\frac{\pi a}{n+1}, \tag{4.1}$$

and considered a small perturbation around it:

$$\phi(x,t) = \bar{\phi}(x) + \eta(x,t). \tag{4.2}$$

From the equation of motion for ϕ, a linearised equation for the small perturbation η was derived

$$\partial_\mu^2\eta + m^2\sum_{j=0}^{n}\alpha_j(\alpha_j\cdot\eta)\exp(i\beta\alpha_j\cdot\bar{\phi}) = 0. \tag{4.3}$$

Assuming simple time dependence

$$\eta(x,t) = \eta(x)e^{i\nu t},$$

the linearised equation of motion was reduced to an eigenvalue problem

$$\mathcal{D}\eta = \nu^2\eta, \tag{4.4}$$

in which $\mathcal{D}$ was a *non-hermitian* second order differential operator of the following form

$$\mathcal{D} = -\frac{d^2}{dx^2} + m^2\sum_{j=0}^{n}\alpha_j\otimes\alpha_j\exp(i\beta\alpha_j\cdot\bar{\phi}). \tag{4.5}$$

They argued that "if the spectrum of $\mathcal{D}$ –for bounded eigenfunctions– was real and positive; hence, the frequencies ν were real, then the small perturbations to $\bar{\phi}$ would not diverge".

If $\mathcal{D}$ is hermitian, then obviously its eigenvalues ν^2 are real, the eigenfunctions belonging to different eigenvalues are orthogonal to each other and they constitute a complete basis of the function space. However, $\mathcal{D}$ *is non-hermitian.* Its eigenvalues are in general complex and the eigenfunctions are not guaranteed to form a complete orthogonal basis of the entire function space. Therefore, the "stability argument" based on the eigenfunctions of $\mathcal{D}$ is in general incomplete.

Hollowood and Evans' [18, 22] argument that "ν^2 are real" goes as follows: First $\bar{\phi}$ satisfies the following "twisted reality" condition

$$\bar{\phi}^*(x) = -M\bar{\phi}(x), \quad * \text{ denotes complex conjugation}, \tag{4.6}$$

in which M acts as a Z_2 symmetry of the roots

$$M\alpha_j = \alpha_{n+1-j}, \quad \alpha_{n+1}\equiv\alpha_0, \tag{4.7}$$

and it also satisfies the conditions

$$M^2 = 1, \quad M^t = M. \tag{4.8}$$

From this it follows that

$$\mathcal{D}^\dagger = M\mathcal{D}M, \qquad \dagger \text{ denotes hermitian conjugation.} \tag{4.9}$$

Hence, they argue that "$\mathcal{D}$ is hermitian" with respect to the following "inner product":

$$\langle f, g\rangle = \int_{-\infty}^{\infty} dx\, f^\dagger(x)\cdot Mg(x). \tag{4.10}$$

In fact, it is easy to see

$$\langle f, \mathcal{D}g\rangle = \int_{-\infty}^{\infty} dx f^\dagger(x)\cdot M\mathcal{D}g(x) = \int_{-\infty}^{\infty} dx f^\dagger(x)\cdot \mathcal{D}^\dagger Mg(x) = \langle \mathcal{D}f, g\rangle. \tag{4.11}$$

Based on this they assert that the spectrum of $\mathcal{D}$ is real.

However, *the flaw lies in the point that $\langle f, g\rangle$ does not define an inner product* since it is not *positive definite*. Namely, $\langle f, f\rangle$ can be positive, zero or negative and $\langle f, f\rangle = 0$ does not imply $f = 0$. Supposing that f is an eigenfunction of $\mathcal{D}$ with eigenvalue ν^2, then we can calculate $\langle f, \mathcal{D}f\rangle$ in two ways to obtain

$$\left(\nu^2 - (\nu^2)^*\right)\langle f, f\rangle = 0. \tag{4.12}$$

But we cannot conclude from this that ν^2 is real:

$$\nu^2 \neq (\nu^2)^*, \quad \text{in general,}$$

because of the possibility of $\langle f, f\rangle = 0$, see (A.4). The fact that the "inner product" $\langle f, g\rangle$ is not positive definite can be easily seen when one takes a basis of the n-dimensional vector space (the Cartan subalgebra of a_n) such that M is diagonal. Since $M^2 = 1$, M has eigenvalue ± 1 subspaces and the -1 subspace violates the positive definiteness. In appendix A we give a simple example of M and $\mathcal{D}$ satisfying the conditions (4.8) and (4.9), but $\mathcal{D}$ failing to produce real eigenvalues, or failing to give a complete orthogonal basis consisting of its eigenvectors.

Delius and Grisaru's argument [20] is slightly different. They assumed a *complete set of orthonormal eigenfunctions* $\eta_k(x)$ of $\mathcal{D}$

$$\mathcal{D}\eta_k(x) = \nu_k^2\eta_k(x), \quad \langle\langle \eta_k, \eta_{k'}\rangle\rangle = \delta_{kk'}, \tag{4.13}$$

with respect to an "inner product" without complex conjugation

$$\langle\langle f, g\rangle\rangle = \int_{-\infty}^{\infty} dx f^t(x)\cdot g(x), \qquad t \text{ denotes transpose.} \tag{4.14}$$

They argued that $\mathcal{D}$ was not hermitian but symmetric

$$\langle\langle f, \mathcal{D}g\rangle\rangle = \langle\langle \mathcal{D}f, g\rangle\rangle.$$

However, $\langle\langle f, g\rangle\rangle$ cannot define an inner product, since $\langle\langle f, f\rangle\rangle$ is neither real nor positive. Therefore, the "stability analysis" and "quantisation" based on a "complete set" of eigenfunctions of $\mathcal{D}$ cannot be justified. In appendix B we give a simple example of 2×2 matrix which is symmetric but non-hermitian and show that its eigenvectors need not form a complete orthogonal basis.

Further they went on calculating the explicit eigenfunctions of $\mathcal{D}$ with real eigenvalues for the evaluation of the "quantum corrections" to the masses of solitons. Such "quantisation procedure" is not well founded because the eigenfunctions do not span the complete function space. For, if we assume that $\mathcal{D}$ has real eigenvalues only and that the corresponding eigenfunctions form a complete orthogonal basis, then we can easily prove that $\mathcal{D}$ is hermitian, which is a contradiction.

5 Blowing up Solutions 1

In the previous section we have confirmed that the linear operator $\mathcal{D}$ (4.5) describing small perturbations around a stationary r-soliton solution is not hermitian. Thus the "small perturbations" always contain certain components which grow exponentially in time and the linear approximation eventually breaks down. In this and subsequent sections we will show that most solutions of the imaginary coupling affine Toda field theory really develop singularities after certain time and that the theory is *unstable*.

Let us first look at the asymptotic ($x \to \pm\infty$) properties of the three types of 1-soliton solutions (3.12) at $t = 0$. All of them are *bounded functions of* x. For r- and c-solitons, let us assume that $\mathrm{Re}\,\sigma > 0$. Then at $x \to +\infty$ the expression (3.10) can be simplified as

$$\begin{aligned} -i\beta\phi(x,t) &\approx \sum_{j=0}^{n} \alpha_j \log[\exp(\sigma x + x_0 + \frac{2\pi i a}{n+1}j))] \\ &= \sum_{j=0}^{n} \alpha_j \left(\sigma x + x_0 + \frac{2\pi i a}{n+1}j\right) \\ &= \sum_{j=0}^{n} \alpha_j \left(\frac{2\pi i a}{n+1}j\right). \end{aligned} \quad (5.1)$$

$$(5.2)$$

The parts proportional to $\sigma x + x_0$ cancel with each other due to the relation of the simple roots $\sum_{j=0}^{n} \alpha_j = 0$. At $x \to -\infty$ the r- and c-soliton solutions simply go to zero. Similar arguments can be made for the i-soliton solutions. It is easy to show that any combination of the above soliton solutions shares this property.

In this section we will show that all c-soliton solutions develop singularities after a finite time. The singularity is caused by vanishing of the arguments of the logarithms. Let us follow the time developments from the 'initial' time $t = 0$. At $t = 0$ the argument of the j th logarithm is

$$1 + \exp(\sigma x + x_0 + \frac{2\pi i a}{n+1}j). \quad (5.3)$$

In order this to vanish x must be a root of

$$\sigma x + x_0 + \frac{2\pi i a}{n+1}j = i(2m+1)\pi, \quad m : \text{integer} \quad (5.4)$$

However, this equation does not have a real root in general. If it has one, we could have started with a slightly different x_0 and σ. To sum up,

$$\frac{1}{\sigma}\left(i(2m+1)\pi - \frac{2\pi i a}{n+1}j - x_0\right)$$

is in general complex, unless x_0 and σ are fine tuned. So $\phi(x,0)$ is *regular everywhere*. From the continuity of (3.10) $\phi(x,t)$ is *regular everywhere* for small enough t.

As t increases from zero, the argument of the c-soliton changes. The condition for vanishing argument (5.4) now reads

$$\sigma x - \lambda t + x_0 + \frac{2\pi i a}{n+1}j = i(2m+1)\pi, \quad m : \text{integer}. \quad (5.5)$$

By introducing the real and imaginary parts of σ, λ and x_0

$$\sigma = \sigma_R + i\sigma_I, \quad \lambda = \lambda_R + i\lambda_I, \quad x_0 = x_{0R} + ix_{0I},$$

(5.5) can be rewritten as

$$\begin{pmatrix} \sigma_R & -\lambda_R \\ \sigma_I & -\lambda_I \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} = \begin{pmatrix} -x_{0R} \\ (2m+1)\pi - \frac{2\pi a}{n+1} j - x_{0I} \end{pmatrix}, \tag{5.6}$$

which has always a real root, unless

$$\det \begin{pmatrix} \sigma_R & -\lambda_R \\ \sigma_I & -\lambda_I \end{pmatrix} = -\sigma_R \lambda_I + \sigma_I \lambda_R = 0. \tag{5.7}$$

The above condition means that

$$\frac{\sigma_I}{\sigma_R} = \frac{\lambda_I}{\lambda_R}, \qquad \text{or} \quad \sigma = k\lambda, \quad k : \text{real}. \tag{5.8}$$

In this case the condition $\sigma^2 - \lambda^2 = 4m^2 \sin^2 \frac{\pi a}{n+1}$ (3.11) can never be satisfied by complex σ and λ. So we need not worry about the above situation in the case of c-solitons. The above result also shows that the r-soliton and the i-soliton solutions are essentially singularity free, since for them the determinant (5.7) vanishes.

Let us choose among the roots of (5.6) for various j and m, the one having the smallest $|t|$ and call it t_M. If $t_M < 0$ then we change $\lambda \to -\lambda$ and get $t_M > 0$. Therefore we have shown that the c-soliton solutions always develop singularity as time increases.

Next let us remark that the above $\phi(x,0)$ can be made as small as we wish within a given finite interval $[-L, L]$. Suppose $\sigma_R > 0$, then by choosing x_{0R} sufficiently large and negative, we can make

$$|\exp(\sigma x + x_0 + \frac{2\pi i a}{n+1} j)| < \epsilon \quad \text{for} \quad |x| < L. \tag{5.9}$$

In fact

$$\text{Max} |\exp(\sigma x + x_0 + \frac{2\pi i a}{n+1} j)| = e^{\sigma_R L + x_{0R}},$$

so that we have to choose x_{0R} such that

$$e^{\sigma_R L + x_{0R}} < \epsilon = e^{\log \epsilon}.$$

That is

$$x_{0R} < -\sigma_R L + \log \epsilon. \tag{5.10}$$

Due to the fact that the influence of the c-soliton solution on $\phi(x,0)$ can be made as small as we wish in any finite region, the above result (blowing up of a c-soliton solution) can also be regarded as the "instability" of the "vacuum", $\phi(x,t) \equiv 0$.

This instability can be naively "understood" if we approximate (3.10) in the region $x < L$,

$$-i\beta\phi(x,t) = \sum_{j=0}^{n} \alpha_j \, e^{\sigma x - \lambda t + x_0 + \frac{2\pi i a}{n+1} j} \tag{5.11}$$

which has $e^{-\lambda t}$, an exponentially growing or decaying factor for complex λ. It should be noted, however, that this approximation is not valid for $x > L$.

Before concluding this section let us remark on the vacuum on which quantum states should be built. In any Lorentz invariant quantum field theory, the vacuum is a classical configuration (namely a solution of the equation of motion) satisfying the following two conditions: 1) time and space translational invariance. 2) having the lowest energy (which can be chosen to be zero). In the affine Toda field theory with real coupling it is $\phi(x,t) \equiv 0$ and unique.

But the situation is very different in imaginary coupling theory. There is no classical solution satisfying these two conditions. In short there is no stable vacuum. Firstly the candidates satisfying the condition 1) are infinite in number, that is the points (2.5). Therefore we call them "vacua". The degeneracy of "vacua" usually indicates instability in ordinary quantum field theory context.

Secondly, there are infinitely many solutions of equation of motion which have lower energies than the $\phi(x,t) \equiv 0$ (or $\frac{2\pi}{\beta}\sum k_j\lambda_j$), configuration. Therefore these "vacua" are unstable since they will decay into lower energy states by quantum tunneling. Suppose ϕ_0 is a constant such that $V_I(\phi_0) = -v < 0$. Then consider the solution of the initial value problem

$$\phi(x,0) = \phi_0, \quad \partial_t\phi(x,0) = 0, \tag{5.12}$$

which has energy lower than 0. In this construction the total energy is in fact minus infinity, $E = -v \times$ spacevolume.

There are also solutions having finite negative energy. Consider the solution of the following initial value problem:

$$\phi(x,0) = 0, \quad \text{everywhere} \qquad \partial_t\phi(x,0) = if(x), \tag{5.13}$$

in which $f(x)$ is a real function and finite everywhere and square integrable

$$\int_{-\infty}^{\infty} f(x)^t \cdot f(x) dx = F > 0.$$

which has a negative total energy

$$E = -\frac{1}{2}F < 0.$$

None of these negative energy solutions are time and space translational invariant. We do not know if the solutions of the initial value problems (5.12),(5.13) reman finite or not. The existence of these negative energy states is another evidence of the instability of "vacua" and it gives another difficult hurdle for constructing the quantum field theory, if any.

6 Blowing up Solutions 2

Next let us show that any r-soliton solution is unstable in the same manner as the "vacuum" is unstable by an addition of a small c-soliton solution. In other words we show that 2-soliton solutions consisting of a 1 r-soliton and a 1 c-soliton solutions develop singularity after a finite time.

For simplicity, let us assume that the r-soliton is at rest near the origin,

$$\Psi_j^{(a)} = 1 + e^{m_a x + x_0^{(a)} + \frac{2\pi i a}{n+1} j} \equiv 1 + e^{y_j^a}, \tag{6.1}$$

$$x_{0R}^{(a)} = 0. \tag{6.2}$$

Let us add to it a c-soliton from the right (meaning $\sigma_R > 0$),

$$\Psi_j^{(C,b)} = 1 + e^{\sigma x - \lambda t + x_0^{(b)} + \frac{2\pi i b}{n+1} j} \equiv 1 + e^{y_j^b}, \tag{6.3}$$

$$\sigma^2 - \lambda^2 = 4m^2 \sin^2 \frac{\pi b}{n+1}, \quad 1 \leq b \leq n. \tag{6.4}$$

The total solution is

$$-i\beta\phi(x,t) = \sum_{j=0}^{n} \alpha_j \log[1 + e^{y_j^a} + e^{y_j^b} + e^{\gamma_{ab} + y_j^a + y_j^b}], \tag{6.5}$$

in which γ_{ab} is the interaction function (cf. (3.15))

$$e^{\gamma_{ab}} = -\frac{m_a^2 + m_b^2 - m_{a-b}^2 - 2m_a\sigma}{m_a^2 + m_b^2 - m_{a+b}^2 + 2m_a\sigma},$$

a complex function of σ.

First let us consider the initial form of the solution at $t = 0$. Suppose $m_a > \sigma_R > 0$, then as in the previous case, we can make the influence of the c-soliton as small as we wish

$$|\exp(\sigma x + x_0^{(b)} + \frac{2\pi i b}{n+1} j)| < \epsilon \quad \text{for} \quad |x| < L, \tag{6.6}$$

by choosing $x_{0R}^{(b)}$ sufficiently large and negative. Let us also require that

$$L \gg \frac{1}{m_a}. \tag{6.7}$$

Then at $x > L$ we have

$$|e^{y_j^a}| \gg 1$$

and the argument of the logarithm in (6.5) can be well approximated by

$$e^{y_j^a}(1 + e^{\gamma_{ab} + y_j^b}). \tag{6.8}$$

So at $t = 0$ and $x > L$

$$-i\beta\phi(x,0) \approx \sum_{j=0}^{n} \alpha_j \log[e^{y_j^a}(1 + e^{\gamma_{ab} + \sigma x + x_0^{(b)} + \frac{2\pi i b}{n+1} j})]. \tag{6.9}$$

Thus by the same argument as in the previous section, $\phi(x,0)$ is regular for $x > L$. And for $x < L$, the effect of the c-soliton is negligible and $\phi(x,0)$, $x < L$ is given by the r-soliton solution, which is regular. Therefore $\phi(x,0)$ is regular everywhere. By continuity in t, $\phi(x,t)$ is regular for sufficiently small t.

As t increases, it becomes possible that at $x > L$

$$1 + e^{\gamma_{ab}+\sigma x-\lambda t+x_0^{(b)}+\frac{2\pi ib}{n+1}j}$$

vanishes. The solution is obtained by solving

$$\begin{pmatrix} \sigma_R & -\lambda_R \\ \sigma_I & -\lambda_I \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} = \begin{pmatrix} -x_{0R}^{(b)} - (\gamma_{ab})_R \\ (2m+1)\pi - \frac{2\pi a}{n+1}j - x_{0I}^{(b)} - (\gamma_{ab})_I \end{pmatrix}. \tag{6.10}$$

The singularity occurs at t_M and the system is unstable.

As before this instability can be naively "understood" if we approximate (6.5) in the region $x < L$,

$$-i\beta\phi(x,t) = \sum_{j=0}^{n} \alpha_j \, e^{\sigma x-\lambda t+x_0^{(b)}+\frac{2\pi ia}{n+1}j+\gamma_{ab}} \tag{6.11}$$

which has $e^{-\lambda t}$, an exponentially growing or decaying factor for complex λ. It should be noted, however, that this approximation is not valid for $x > L$.

As in the previous section we show the existence of solutions having lower energies than the single r-solitons. In other words these r-solitons do not have a "mass gap". Let $\phi_r(x,t)$ be an explicit 1 r-soliton solution. Consider the solution of the following initial value problem:

$$\phi(x,0) = \phi_r(x,0), \quad \partial_t\phi(x,0) = \partial_t\phi_r(x,0) + if(x), \tag{6.12}$$

in which $f(x)$ is a real function and finite everywhere. It is chosen to be orthogonal to the real and imaginary parts of $\partial_t\phi_r(x,0)$;

$$\int_{-\infty}^{\infty} \partial_t\phi_{rR}^t(x,0)\cdot f(x)dx = 0, \quad \int_{-\infty}^{\infty} \partial_t\phi_{rI}^t(x,0)\cdot f(x)dx = 0,$$

and square integrable

$$\int_{-\infty}^{\infty} f(x)^t \cdot f(x)dx = F > 0.$$

The solution has a real total energy

$$E = E_r - \frac{1}{2}F,$$

in which E_r is the total energy of the 1 r-soliton solution.

7 Summary

As expected from the non-hermiticity of the Lagrangian, the affine Toda field theory with a pure imaginary coupling constant is found to be classically unstable: It has many (almost all) solutions which develop singularity after a finite lapse of time; Its energy is not positive definite; Its potential is not bounded from below; Small perturbation around soliton solutions does not necessarily remain small as time passes.

From these it seems that it is a long way to go to construct a quantum theory of the non-hermitian affine Toda field theory with a pure imaginary coupling constant, although the theory has many beautiful classical solutions with remarkable properties. Thus the calculations of the quantum mass corrections to the solitons are ill-founded.

Although we have described many negative aspects of the affine Toda field theory with imaginary coupling, we are still fascinated by and interested in the theory, especially in the algebraic structure. We believe that the solitons have exact and factorisable S-matrices obeying non-trivial Yang-Baxter equations [25, 26, 27], reflecting the integrability of the theory. However, due to the constraints from unitarity, we expect that these S-matrices can make sense only for certain discrete values of the coupling constant β^2, at which, for example, the corresponding conformal field theories are known to be unitary [10]. Therefore, the usual method of calculation in quantum field theory, the perturbation calculation, is not applicable to the solitons and their bound states and no perturbative mass corrections to them.

Acknowledgments

We thank E. Corrigan, G. Delius, T. Hollowood, N. MacKay and G. Watts for interesting discussion. R. S thanks Department of Physics, Kyung Hee University for hospitality, where part of this work was done. He is grateful to Japan Society for Promotion of Sciences for financial support which enabled him to visit Seoul.

A Counter Example to "hermiticity argument"

In this appendix we give a simple counter example to the "hermiticity argument" produced by Hollowood and Evans. Let V_2 be a two dimensional complex vector space and $f, g \in V_2$,

$$f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \quad g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix},$$

then the ordinary inner product is given by

$$(f, g) = f^\dagger \cdot g = f_1^* g_1 + f_2^* g_2.$$

Next we choose the Z_2 symmetry matrix M as

$$M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad M^2 = 1, \quad M^t = M.$$

Then a 2×2 matrix $\mathcal{D}$

$$\mathcal{D} = \begin{pmatrix} a & b \\ -b^* & d \end{pmatrix}, \quad a, d: \text{ real}, \quad b: \text{ complex}, \tag{A.1}$$

satisfies the conjugation relation

$$\mathcal{D}^\dagger = M\mathcal{D}M. \tag{A.2}$$

In fact, one can show easily that the above $\mathcal{D}$ is the most general 2×2 matrix satisfying the relation (A.2).

According to Hollowood and Evans [18, 22], $\mathcal{D}$ is "hermitian" with respect to a new "inner product"

$$\langle f, g\rangle = f^\dagger \cdot Mg = f_1^* g_1 - f_2^* g_2.$$

Obviously $\langle f, f\rangle$ is real but not positive (semi) definite. The characteristic polynomial of $\mathcal{D}$ is

$$\det\begin{pmatrix} a-\lambda & b \\ -b^* & d-\lambda \end{pmatrix} = \lambda^2 - (a+d)\lambda + ad + |b|^2 = (\lambda-\lambda_1)(\lambda-\lambda_2), \tag{A.3}$$
$$\lambda_1 + \lambda_2 = a+d, \quad \lambda_1\lambda_2 = ad + |b|^2.$$

Thus the eigenvalues are either both real or a complex conjugate pair according to the sign of the discriminant

$$(a+d)^2 - 4(ad+|b|^2) = (a-d)^2 - 4|b|^2.$$

The eigenvalue problem reads

$$\begin{aligned} ax + by &= \lambda_1 x, \\ -b^* x + dy &= \lambda_1 y. \end{aligned}$$

Assuming $b \neq 0$, we get $y = (\lambda_1 - a)x/b$ and

$$\langle f, f\rangle = |x|^2 (1 - |\frac{\lambda_1 - a}{b}|^2) = \frac{|x|^2}{|b|^2}(|b|^2 - |\lambda_1 - a|^2).$$

If λ_1 is complex then $|\lambda_1 - a|^2 = (\lambda_1 - a)(\lambda_2 - a)$ and it is easy to see

$$|b|^2 - |\lambda_1 - a|^2 = 0.$$

Thus we find that the eigenvector belonging to a complex eigenvalue has a "zero norm" (see (4.12))

$$\langle f, f\rangle = 0. \tag{A.4}$$

B Symmetric and non-hermitian matrices

In this appendix we show that the linear analysis around a one soliton solution proposed by Delius and Grisaru [20] is incomplete with the help of a simple example.

We consider a simple symmetric and non-hermitian 2×2 matrix,

$$\mathcal{D} = \begin{pmatrix} a+ib & c+id \\ c+id & g-ib \end{pmatrix}, \quad a, b, c, d, g: \text{ real}, \quad \text{with} \quad b(g-a) = 2cd,$$

so that, the coefficients of the characteristic polynomial are real. The characteristic polynomial reads,

$$\lambda^2 - (a+g)\lambda + ag + b^2 - c^2 + d^2 = 0. \tag{B.1}$$

The roots are $\lambda_1, \lambda_2 = A \pm B$, where,

$$A = (a+g)/2, \quad \text{and}$$

$$B = \sqrt{(a-g)^2 - 4(b^2 - c^2 + d^2)}/2. \tag{B.2}$$

We show that if the eigenvalues are degenerate then the corresponding eigenspace is one dimensional. Moreover, in this case the eigenvector has "zero norm" in their symmetric "inner product" (4.14). So let us concentrate on the case of degenerate eigenvalues. In this case, $B = 0$. The eigenvalue equation is given by

$$(a+ib)x + (c+id)y = \frac{(g+a)}{2}x, \tag{B.3}$$

so that

$$y = \frac{1}{(c+id)}[(g-a)/2 - ib]x, \quad c+id \neq 0.$$

Where we have taken $f = \begin{pmatrix} x \\ y \end{pmatrix}$, as the eigenvector. Now let us examine the two cases viz $b = 0$ and $b \neq 0$ separately:

Case i) $b = 0$. In this case either c or d is zero to maintain the condition $b(g-a) = 2cd$. So we have

$$g - a = \pm 2d \quad (\pm 2ic) \qquad \text{for} \quad c = 0 \quad (d = 0)$$

and consequently $y = \pm ix$.

Case ii) $b \neq 0$. In this case it is easy to see that vanishing discriminant implies $b = \pm c$, and in turn the condition $b(g-a) = 2cd$ gives $(g-a) = \pm 2d$. So for this case again $y = \pm ix$.

For both cases the eigenspace is spanned by only one vector. Moreover the "inner product" defined by Delius and Grisaru has vanishing norm,

$$\langle\langle f, f \rangle\rangle = f^t \cdot f = x^2 + y^2 = x^2 - x^2 = 0. \tag{B.4}$$

Thus the "complete set of orthogonal eigenfunctions" of $\mathcal{D}$ does not exist.

References

[1] A. V. Mikhailov, M. A. Olshanetsky and A. M. Perelomov, *Comm. Math. Phys.* **79** (1981) 473;
D. I. Olive and N. Turok, *Nucl. Phys.* **B265** (1986) 469.

[2] A. E. Arinshtein, V. A. Fateev and A. B. Zamolodchikov, *Phys. Lett.* **B87** (1979) 389.

[3] H. W. Braden, E. Corrigan, P. E. Dorey and R. Sasaki, *Phys. Lett.* **B227** (1989) 411.

[4] H. W. Braden, E. Corrigan, P. E. Dorey and R. Sasaki, *Nucl. Phys.* **B338** (1990) 689;
H. W. Braden and R. Sasaki, *Phys. Lett.* **B255** (1991) 343;
R. Sasaki and F. P. Zen, *Int. J. Mod. Phys.* **8** (1993) 115.

[5] P. Christe and G. Mussardo, *Nucl. Phys.* **B330**(1990) 465; *Int. J. Mod. Phys.* **A5** (1990) 4581.

[6] C. Destri and H. J. de Vega, *Phys. Lett.* **B233** (1989) 336.

[7] G. W. Delius, M. T. Grisaru and D. Zanon, *Nucl. Phys.* **B382** (1992) 365.

[8] E. Corrigan, P. E. Dorey and R. Sasaki, *Nucl. Phys.* **B408** (1993) 579.

[9] A. B. Zamolodchikov, *Advanced Studies in Pure Mathematics* **19** (1989) 641;
T. Eguchi and S-K. Yang, *Phys. Lett.* **B224** (1989) 373;
R. Sasaki and I. Yamanaka, *Advanced Studies in Pure Mathematics* **16** (1988) 271.

[10] T. J. Hollowood and P. Mansfield, *Phys. Lett.* (1989) 73;
D. Bernard and A. LeClair, *Comm. Math. Phys.* **142** (1991) 99.

[11] T. Hollowood, *Nucl. Phys.* **B384** (1992) 523.

[12] T. Hollowood, *Int. J. Mod. Phys.* **A8** (1993) 947, hep-th/9203076.

[13] N. J. MacKay and W. A. McGhee, *Int. J. Mod. Phys.* **A8** (1993) 2791, Erratum *Int. J. Mod. Phys.* **A8** (1993) 3830;
C. P. Constantinidis et al. *Phys. Lett.* **B298** (1993) 88, hep-th/9304080;
H. Aratyn, C. P. Constantinidis and L.A. Ferreira, *Nucl. Phys.* **B406** (1993) 727;
W. A. McGhee, *Int. J. Mod. Phys.* **A9** (1994) 2645;
U. Harder, A. Iskandar and W. A. McGhee, *Int. J. Mod. Phys.* **A10** (1995) 1879, hep-th/9409035;
Z. Zhu and D. G. Caldi, *Nucl. Phys.* **B436** (1995) 659, hep-th/9307175.

[14] D. I. Olive, N. Turok and J. W. R. Underwood, *Nucl. Phys.* **B401** (1993) 663; *Nucl. Phys.* **B409** (1993) 509.

[15] A. Nakamula, *Prog. Theor. Phys.* **90** (1993) 1241.

[16] M. A. C. Kneipp and D. I. Olive, *Solitons and Vertex Operators in Twisted Affine Toda Field Theories*, hep-th/9404030 (1994) ; *Nucl. Phys.* **B408** (1993) 565, hep-th/9305154;
A. Fring, P. R. Johnson, M. A. C. Kneipp and D. I. Olive, *Nucl. Phys.* **B430** (1994) 597, hep-th/9405034.

[17] J. Underwood, *On the Topological Charges of Affine Toda Solitons*, hep-th/9306031 (1993) ;
M. Freeman, *Nucl. Phys.* **B433** (1995) 657, hep-th/9408092.

[18] T. Hollowood, *Phys. Lett.* **B300** (1993) 73, hep-th/9209024.

[19] G. M. T. Watts, *Phys. Lett.* **B338** (1994) 40, hep-th/9404065.

[20] G. W. Delius and M. T. Grisaru, *Nucl. Phys.* **B441** (1995) 259, hep-th/9411176.

[21] N. J. MacKay and G. M. T. Watts, *Nucl. Phys.* **B441** (1995) 277, hep-th/9411169.

[22] J. M. Evans, *Nucl. Phys.* **B390** (1993) 225.

[23] H. W. Braden, E. Corrigan, P. E. Dorey and R. Sasaki, *Nucl. Phys.* **B356** (1991) 469.

[24] A. Fujii and R. Sasaki, *Prog. Theor. Phys.* **93** (1995) 1123, hep-th/9503083.

[25] G. W. Delius, M. Gould and Y.-Z. Zhang, Nucl.Phys. **B432** (1994), 377; hep-th/9405030

[26] T. J. Hollowood, Nucl. Phys. **B414** (1994), 379; hep-th/9305042

[27] G. M. Gandenberger and N.J. MacKay, *Exact S-matrices for $d^{(2)}_{n+1}$ affine Toda solitons and their bound states*, preprint DAMTP-95-33, hep-th/9506169, and references therein.