DOE/ER/40561-205-INT95-00-92

UW/PT 95-05

Effective Field Theories

Lectures given at the Seventh Summer School in Nuclear Physics: "Symmetries"

Institute for Nuclear Theory, June 19 - 30, 1995

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Three lectures in which I give an introduction to effective field theories in particle physics.

6/95

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1. Introduction

Physics, chemistry and biology are all sciences describing different aspects of the same world, yet their practice differs enormously...why is that? To a large extent the cultural differences between the fields can be explained in terms of fundamental difference between the physical systems being analyzed: A biological system typically has many unrelated energy scales of comparable magnitude, such as the low lying excitation spectra of various different enzymes collaborating in a biological process. While it is possible to crudely explain why the body temperature of a human is $\mathcal{O}(10^2)$ K in terms the fine structure constant α and the masses of the proton and the electron, you will not be able to explain why a human is healthy with a temperature of 310.5 K but not with a temperature of 320.5 K without a thorough understanding of human physiology.

In contrast, a physics problem tends to involve energy scales that are widely separated, which allows one — with care — to determine many of the properties of a system using the tool of dimensional analysis. To see how this works, we first choose physical units with the speed of light and Planck's constant set to unity:

$$\hbar = c = 1 \ .$$

Then all physical parameters can be said to have the dimension of mass to some power. In particular, if some quantity X has dimensions $[mass]^n$, we will just say "X has dimension n" or $[X] = n$. You should be able to convince yourself that

$$[volume] \sim [\int d^3x] = -3$$

$$[G_N] \sim [G_F] = -2$$

$$[length] \sim [time] \sim [\mu_N] = -1$$

$$[velocity] \sim [\alpha] = 0$$

$$[energy] \sim [momentum] \sim [\Lambda_{QCD}] \sim [d/dx] \sim [d/dt] = 1$$

$$[\vec{E}] \sim [\vec{B}] = 2$$

and so forth [1]. Of particular interest to us will be the dimension of a Lagrange density; since $\int d^4x\mathcal{L}$ is an action — which comes in units of $\hbar$ and is dimensionless — it follows that

$$[\mathcal{L}] = 4 \ .$$

Let us use dimensional analysis to discuss properties of the hydrogen atom. To a first approximation, the system is described in terms of one dimensionful parameter, the electron mass m_e, and one dimensionless number, the fine structure constant $\alpha = 1/137$. If we want to estimate the size a_0 of a hydrogen atom, since length has dimension $[mass]^{-1}$ it follows that $a_0 \propto m_e^{-1}$, where the proportionality constant is dimensionless. What is it? We would guess it is some number of order unity, times some power of α. Alas, dimensional analysis doesn't tell us the power...we have to look at the dynamics to realize that the appropriate power is α^{-1}. We arrive at $a_0 \simeq 1/(\alpha m_e)$, which in fact is the exact expression for the Bohr radius. What about the ground state binding energy E_0 of the hydrogen atom? E_0 has the dimensions of $[mass]$, and dynamics gives us a proportionality factor of α^2, so we estimate $E_0 \simeq \alpha^2 m_e$, which is in fact only a factor of 2 off from the correct value.

We can go on and ask what wavelength of photon allows one to examine crystal structure by means of diffraction. Since atomic sizes are given by $a_0 = 1/(\alpha m_e)$, the atomic spacing in a crystal is expected to be similar. It follows that to see crystal structure, we need photons with wavelength $\lambda \lesssim a_0$, or equivalently energy $E_\gamma \gtrsim \alpha m_e = \mathcal{O}(10\,KeV)$ — visible light won't do, we need X-rays. On the other hand, if we wish to estimate the energy of light emitted from an atomic transition in hydrogen, get $E_\gamma \lesssim E_0 \simeq \alpha^2 m_e = \mathcal{O}(10\,eV)$, corresponding to a wavelength $\lambda_\gamma \simeq a_0/\alpha$, several orders of magnitude larger than the atom itself.

The above analysis may seem familiar and unimpressive — after all, while it is nice that one can easily determine how m_e enters quantities of physical interest, one also has to keep track of powers of α which involves going back and examining the Schrödinger

[1] For practical purposes it is conventional to express everything in units of energy (eV) rather than mass (gm). Thus $m_p \simeq 940$ GeV, $m_e \simeq .511$ MeV, $1\ F = 10^{-13}\,cm \simeq (200MeV)^{-1}$, $1\,K \simeq 10^{-4}\,eV$, etc.

equation for hydrogen (which we know how to solve anyway). Yet there *is* something remarkable about the analysis, and it lies in the sentence:

> *"To a first approximation, the system is described in terms of one dimensionful parameter, the electron mass m_e, and one dimensionless number, the fine structure constant $\alpha = 1/137$."*

Why should this be true, and what is the approximation we are making? Why is the system insensitive to the proton mass? Or the W and Z boson masses? Or Newton's constant G_N? Why don't we need to take into account the bottom quark mass, $m_b \simeq 5$ GeV? The ratio $r \equiv m_b^2/m_e^2 = 10^8$ is a dimensionless number; couldn't the ground state energy of the hydrogen atom be some function of r, namely $E_0 = f(r)\alpha^2 m_e$, where $f(r)$ could as easily equal 10^8 as 10^{-8}?

The technique of constructing effective theories allows one to answer such questions[2]. The basic idea is not to attempt to construct "a theory of everything", but to construct an effective theory that is appropriate to the energy scale of the experiments one is interested in. A theory of everything is beyond our abilities to construct since we cannot probe everything experimentally, and even if we could, it would contain lots of information extraneous to any particular experiment.

Effective field theory techniques get interesting when we wish to look at effects over a wide range of energies: then we must understand how effective theories at different scales are related to each other. This is useful if one wishes to relate experiments over a large range of energy scales, or if one has a theory of high energy physics and wishes to predict the results of low energy experiments. In the example of the hydrogen atom and the b quark, one can show that E_0 depends on the b quark mass in the following way:

$$E_0 = \tfrac{1}{2}\alpha^2 m_e \left(1 + \mathcal{O}(m_e^2/m_b^2)\right) \ .$$

There is a small power law correction to the naive value $\propto (m_e^2/m_b^2) \sim 10^{-8}$, as well as a hidden dependence of α on the b quark mass. When one is only concerned with

[2] The concept of effective field theory is mainly associated with Ken Wilson, although it is an edifice with many architects. For two quite dissimilar modern treatments see the reviews by J. Polchinski [1] and H. Georgi [2].

atomic physics, one can ignore the m_b dependence of α, since it is already incorporated in the measured physical value $\alpha = 1/137$. Effective field theories however allow you to simply compute electromagnetic scattering of electrons at a center-of-mass energy ~ 100 GeV, where one finds that the appropriate value of for the fine structure "constant" is $\alpha \simeq 1/128$ — a change due in part to the effects of the b quark.

The outline of these lectures is as follows:

1. First I discuss how to construct effective theories as an expansion in operators consistent with low energy symmetries, and how to use dimensional analysis to extract the interesting physics;
2. Next I explain how the dimension of the operator determines whether it is irrelevant, relevant or "marginal" to low energy physics;
3. I then consider "matching": how one relates the parameters of a low energy theory to those of a higher energy theory;
4. I then show how quantum corrections can sometimes change the dimension of an operator and therefore radically change low energy physics.
5. Finally I mention the application of some of these ideas to the strong interactions.

Each section is followed by some exercises (of widely varying difficulty); I encourage you to work through them.

Exercise 1. *Estimate the energy scale of rotational excitations of water in terms of m_p, m_e and α. Does your answer explain why microwaves are used to heat food?*

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2. Dimensional analysis, symmetries, and the separation of scales

The basic idea behind effective field theories is that a physical process typified by some energy E can be described in terms of an expansion in E/Λ_i, where the Λ_i are various physical scales involved in the problem which have dimension 1 and which are bigger than E. In this section we show how this simple idea can be incorporated into a predictive framework.

2.1. Example 1: Why the sky is blue.

Consider the question of why the sky is blue. More precisely, consider the problem of low energy light scattering from neutral atoms in their ground state, where by "low energy" I mean that the photon energy E_γ is much smaller than the excitation energy ΔE of the atom, which is of course much smaller than its inverse size or mass:

$$E_\gamma \ll \Delta E \ll a_0^{-1} \ll M_{atom} \ .$$

Thus the process is necessarily elastic scattering, and to a good approximation we can ignore that the atom recoils, treating it as infinitely heavy. Let's construct an "effective Lagrangian" to describe this process. This means that we are going to write down a Lagrangian with all interactions describing elastic photon-atom scattering that are allowed by the symmetries of the world — namely Lorentz invariance and gauge invariance. Photons are described by a field A_μ which creates and destroys photons; a gauge invariant object constructed from A_μ is the field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The atomic field is defined as ϕ_v, where ϕ_v destroys an atom with four-velocity v_μ (satisfying $v_\mu v^\mu = 1$, with $v_\mu = (1,0,0,0)$ in the rest-frame of the atom), while $\phi_v^\dagger$ creates an atom with four-velocity v_μ. So what is the most general form for $\mathcal{L}_{eff}$? Since the atom is electrically neutral, gauge invariance implies that ϕ can only be coupled to $F_{\mu\nu}$ and not directly to A_μ. So $\mathcal{L}_{eff}$ is comprised of all local, Hermitian monomials in $\phi_v^\dagger\phi_v$, $F_{\mu\nu}$,v_μ, and ∂_μ. Certain combinations we needn't consider for the problem at hand — for example $\partial_\mu F^{\mu\nu} = 0$ for radiation (by Maxwell's equations); also, if we define the energy of the atom at rest in it's ground state to be zero, then $v^\mu\partial_\mu\phi = 0$, since $v_\mu = (1,0,0,0)$ in the rest frame, where $\partial_t\phi = 0$. Similarly, $\partial_\mu\partial^\mu\phi = 0$. Thus we are led to consider the Lagrangian

$$\begin{aligned}\mathcal{L}_{eff} &= c_1\phi_v^\dagger\phi_v F_{\mu\nu}F^{\mu\nu} + c_2\phi_v^\dagger\phi_v v^\alpha F_{\alpha\mu}v_\beta F^{\beta\mu} \\ &+ c_3\phi_v^\dagger\phi_v(v^\alpha\partial_\alpha)F_{\mu\nu}F^{\mu\nu} + \ldots\end{aligned} \tag{2.1}$$

The above expression involves an infinite number of operators and an infinite number of unknown coefficients! Nevertheless, dimensional analysis allows us to identify the leading contribution to low energy scattering of light by neutral atoms. It is straightforward to figure out that

$$[\partial_\mu] = 1 \ , \qquad [F_{\mu\nu}] = 2 \ , \qquad [\phi] = \tfrac{3}{2} \ .$$

The first follows from the fact that ∂_μ has the dimension of 1/length. The second is easily determined by noting that the Maxwell Lagrangian is $\mathcal{L}_M = -\frac{1}{4}F^2$, and that $[\mathcal{L}] = 4$. Finally ϕ is determined by writing a state with no atom as $|0\rangle$, and one atom as $|A\rangle$, where $\phi^\dagger(x)\,|0\rangle = \Psi_A(x)\,|A\rangle$, with $\Psi_A(x)$ being the normalized atomic wavefunction and $\langle 0|\,0\rangle = \langle A|\,A\rangle = 1$. Since $\int d^3x\,|\Psi_A|^2 = 1$, it follows that $[\phi] = 3/2$.

Since the effective Lagrangian has dimension 4, the coefficients c_1, c_2 etc. also have dimensions. It is easy to see that they all have negative mass dimensions:

$$[c_1] = [c_2] = -3\ , \qquad [c_3] = -4$$

and that operators involving higher powers of $\partial \cdot v$ would have coefficients of even more negative dimension. It is crucial to note that these dimensions must be made from dimensionful parameters describing the atomic system — namely its size r_0 and the energy gap δE between the ground state and the excited states. The other dimensionful quantity, E_γ, is explicitly represented by the derivatives ∂_μ acting on the photon field. Thus for $E_\gamma \ll \Delta E, r_0^{-1}$ the dominant effect is going to be from the operator in $\mathcal{L}_{eff}$ which has the *lowest* dimension. There are in fact two leading operators, the first two in eq. (2.1), both of dimension 7. Thus low energy scattering is dominated by these two operators, and we need only compute c_1 and c_2.

What are the sizes of the coefficients? To do a careful analysis one needs to go back to the full Hamiltonian for the atom in question interacting with light, and "match" the full theory to the effective theory. We will discuss this process of matching later, but for now we will just estimate the sizes of the c_i coefficients. We first note that extremely low energy photons cannot probe the internal structure of the atom, and so the cross-section ought to be classical, only depending on the size of the scatterer. Since such low energy scattering can be described entirely in terms of the coefficients c_1 and c_2, we conclude that

$$c_1 \simeq c_2 \simeq r_0^3\ .$$

The effective Lagrangian for low energy scattering of light is therefore

$$\mathcal{L}_{eff} = r_0^3\left(a_1 \phi_v^\dagger \phi_v F_{\mu\nu}F^{\mu\nu} + a_2 \phi_v^\dagger \phi_v v^\alpha F_{\alpha\mu} v_\beta F^{\beta\mu}\right) \tag{2.2}$$

where a_1 and a_2 are dimensionless, and expected to be $\mathcal{O}(1)$. The cross-section (which goes as the amplitude squared) must therefore be proportional to r_0^6. But a cross section σ has dimensions of area, or $[\sigma] = -2$, while $[r_0^6] = -6$. Therefore the cross section must be proportional to

$$\sigma \propto E_\gamma^4 r_0^6 \ , \tag{2.3}$$

growing like the fourth power of the photon energy. Thus blue light is scattered more strongly than red, and the sky looks blue.

Is the expression (2.3) valid for arbitrarily high energy? No, because we left out terms in the effective Lagrangian we used. To understand the size of corrections to (2.3) we need to know the size of the c_3 operator (and the rest we ignored). Since $[c_3] = -4$, we expect the effect of the c_3 operator on the scattering amplitude to be smaller than the leading effects by a factor of E_γ/Λ, where Λ is some energy scale. But does Λ equal M_{atom}, $r_0^{-1} \sim \alpha m_e$ or $\Delta E \sim \alpha^2 m_e$? The latter is the smallest scale and hence the most important. We expect our approximations to break down as $E_\gamma \to \Delta E$ since for such energies the photon can excite the atom. Hence we predict

$$\sigma \propto E_\gamma^4 r_0^6 \left(1 + \mathcal{O}(E_\gamma/\Delta E)\right) . \tag{2.4}$$

The Rayleigh scattering formula ought to work pretty well for blue light, but not very far into the ultraviolet. Note that eq. (2.4) contains a lot of physics even though we did very little work. More work is needed to compute the constant of proportionality.

2.2. Example 2: The binding energy of charmonium to nuclei.

Closely related to the above example is the calculation of the binding energy of charmonium (a $\bar{c}c$ bound state, where c is the charm quark) to nuclei. In the limit that the charm quark mass m_c is very heavy, the charmonium meson can be thought of as a Coulomb bound state, with size $\sim \alpha_s(m_c) m_c$, where $\alpha_s(m_c)$ is a small number (more on this later). When inserted in a nucleus, it will interact with the nucleons by exchanging gluons with nearby quarks. Typical momenta for gluons in a nucleus is set by the QCD scale $\Lambda_{QCD} \simeq 200$ MeV. For large m_c then, the wavelength of gluons will be much larger than the size of the charmonium meson, and so the relevant interaction is the gluon-charmonium analogue of

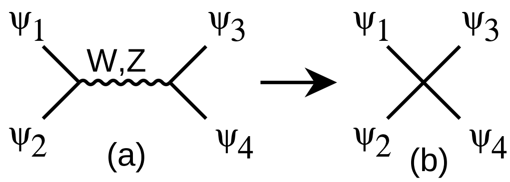

Fig. 1. (a) Tree level W and Z exchange between four fermions. (b) The effective vertex in the low energy effective theory (Fermi interaction).

photon-atom scattering considered above. The effective Lagrangian is just given by (2.2), where ϕ now destroys charmonium mesons, and $F_{\mu\nu}$ is replaced by $G^a_{\mu\nu}$, the field strength for gluons of type $a = 1, \ldots, 8$. The coefficients $a_{1,2}$ may be computed from QCD. To compute the binding energy of charmonium we need to compute the matrix element

$$\langle N, \overline{c}c| \int d^3x\, \phi^\dagger \phi G^a_{\mu\nu} G^{a\,\mu\nu}\, |N, \overline{c}c\rangle$$

(as well as the matrix element of the other operator in (2.2)), which we do not know how to do precisely since the system is strongly interacting. We can estimate its size by dimensional analysis though, getting

$$E_B \sim r_0^3 \Lambda^4_{QCD} \simeq \frac{\Lambda^4_{QCD}}{(\alpha_s m_c)^3} \ .$$

This problem is discussed with greater sophistication in ref. [3].

2.3. Example 3: The cross section for low energy neutrino interactions.

The term "weak interactions" refers in general to any interaction mediated by the W or Z bosons, whose masses are 80 GeV and 91 GeV respectively. Since their couplings are rather weak, it is usually a decent approximation to only consider first order perturbation theory, namely Feynman graphs fig. 1(a).

These interactions describe $2 \to 2$ scattering of fermions, or $1 \to 3$ decays. The W and Z propagators (in a particular choice of gauge) are given by $-ig_{\mu\nu}/(q^2 - M^2)$, where q is the four-momentum transferred. For low energy processes, $q^2 \ll M^2$ and one never has enough energy to make a physical W or Z, so there is no reason to include them in the theory. Thus the low energy effective theory just has the contact interactions shown in figure 1b:

$$\mathcal{L}_{weak} \sim G\psi_1\psi_2\psi_3\psi_4, \tag{2.5}$$

where ψ_i represent fermion fields (for either quarks or leptons). Since the Lagrangian for a noninteracting fermion is $\mathcal{L}_f = \overline{\psi}(i\partial\!\!\!/ - m)\psi$, it follows that

$$[\psi] = \tfrac{3}{2}\ ,$$

and so the coupling G in eq.(2.5) has dimension $[G] = -2$. You can estimate its size by equating the processes fig 1a and fig. 1b and it is roughly given by g^2/M^2, where g and M are the dimensionless coupling constant and mass of the W or Z. (This is "matching", and you will do this more precisely in a later exercise).

Since neutrinos only interact through the weak force, it follows that low energy neutrinos ($E_\nu \ll M_W$) interact with matter through an operator of the form (2.5), where two of the ψ's are neutrino fields, and the other two are either quark or lepton fields. Thus the neutrino cross-section σ, which has dimension -2, must be proportional to G^2 which has dimension -4. Therefore the cross-section must scale with energy as

$$\sigma_\nu \simeq G^2 s \tag{2.6}$$

for low energy neutrinos, where s equals the square of the total energy in the center of momentum frame.

Exercise 2. *Use the effective Lagrangian to explain why the force between to static neutral atoms at a separation $R \gg a_0$ scales like $1/R^7$. You should be able to get this from dimensional analysis of the two photon exchange process. Can you explain why there isn't any contribution from one photon exchange due to the operator $\phi_v^\dagger i\overleftrightarrow{\partial}_\mu\phi_v v_\nu F^{\mu\nu}$? Can you explain why the approximations made in the effective field theory are expected to be invalid for $R \lesssim a_0/\alpha$? For a detailed discussion of why one finds $1/R^7$ instead of the nonrelativistic result $1/R^6$, see ref. [4].*

Exercise 3. *The μ and the τ have the same weak interactions, and so the amplitudes for decay via W exchange $\mu \to e\overline{\nu}_e\nu_\mu$ and $\tau \to e\overline{\nu}_e\nu_\tau$ are equal. Since the τ is heavier, it has more ways to decay than the μ. The mass and lifetimes of the two particles are*

$$m_\mu = 106\ MeV\ , \qquad T_\mu = 2.2 \times 10^{-6}\ sec.,$$

$$m_\tau = 1777\ MeV\ , \qquad T_\tau = 3.0 \times 10^{-13}\ sec.$$

Given that the μ decays 100% of the time via $\mu \to e\overline{\nu}_e\nu_\mu$, calculate the fraction of τ decays which are of the form $\tau \to e\overline{\nu}_e\nu_\mu$. All you need to know is that $[G] = -2$ in eq. (2.5). How does your answer compare with the observed branching ratio $BR_{\tau\to e\overline{\nu}_e\nu_\tau} = 18.01 \pm 0.18\%$?

Exercise 4. *The partial mean lifetime of the proton in the decay $p \to e^+\pi^0$ is known to be greater than 1.3×10^{32} years. Suppose that new physics at a scale Λ does give rise to this decay (for example, through the tree level exchange of a particle with mass Λ, analogous to the interaction in fig. 1). What is an approximate lower bound on Λ ? (Hint: Find the lowest dimension operators made up of quark and lepton fields that could give rise to this decay mode).*

Exercise 5. *Suppose that there are $\Delta B = 2$ baryon violating operators due to new physics at a scale Λ, but no $\Delta B = 1$ operators, so that the proton is stable, but $n - \overline{n}$ oscillations can occur. Such oscillations have not been seen, and the lower bound on the oscillation rate is 1.2×10^8 sec. How does this translate into a bound on the scale Λ?*

Exercise 6. *Estimate the cross-section for photon-photon scattering at energies well below the electron mass, $E_\gamma \ll m_e$. Since $\alpha = 1/137$, counting powers of α matters!*

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3. The relevant, the irrelevant, and the marginal

So far I have only discussed examples where the operator has dimension greater than 4, so that the coefficient has negative dimension and the resulting cross-section or decay width therefore becomes smaller as the energy scale E of the interaction gets smaller. Even though these are often the most interesting interactions — since they are harbingers of new physics at energies well above E — these sorts of interactions are called *irrelevant*. The rationale is that at low energies, their effects are small (for example, see eqs. (2.4), (2.6).). In contrast, operators with dimension less than 4, whose coefficients have positive dimension, are called *relevant* operators because they become more relevant at lower E. Ignoring quantum corrections, the only relevant operators one can write down in a relativistic field theory in four dimensions are

- The unit operator (whose coefficient is the cosmological constant) which is dimension 0;
- Boson mass terms, which are dimension 2;
- Fermion mass terms, which are dimension 3;
- 3-scalar (ϕ^3) interactions, also dimension 3.

(Terms linear in a scalar field can be removed by shifting its value).

An example is the electron mass, arising from the dimension 3 operator $\overline{\psi}\psi$ with coefficient m_e. In high energy scattering ($E_e \gg m_e$) the effects of the electron mass are negligible. However, the effects of the electron mass are very important at energies comparable to m_e. In fact, exercise 3. is only simple if one not only assumes that the momentum scales in μ and τ decay are low compared to M_W, but also that they are high compared to m_e, so that one could ignore the electron mass. As another example, consider two real scalar fields ϕ and Φ with a Lagrangian of the form

$$\mathcal{L} = \tfrac{1}{2}(\partial\phi)^2 + \tfrac{1}{2}(\partial\Phi)^2 - \tfrac{1}{2}m^2\phi^2 - \tfrac{1}{2}M^2\Phi^2 - \tfrac{1}{2}\kappa\phi^2\Phi \ . \tag{3.1}$$

We will assume

$$m \simeq \kappa \ll M \ .$$

We can see that unlike fermion fields, scalar fields have dimension 1, which means that the coupling κ does as well:

$$[\phi] = [\Phi] = [\kappa] = 1 \ .$$

By our definition above, the three scalar interaction is *relevant*. Consider $\phi\phi \to \phi\phi$ scattering at tree level in this model. First take the case where the center of mass energy E_ϕ

is much greater than m, M, and κ. Then the scattering amplitude from a graph like fig. 1a — with the ψ_i replaced by ϕ and the W, Z replaced by Φ — is proportional to κ^2 and the cross section must go as

$$\sigma_{2\phi\to 2\phi}|_{E_\phi \gg m, M, \kappa} \propto (\frac{\kappa}{E_\phi})^4 \frac{1}{E_\phi^2}$$

which goes rapidly to zero for large E_ϕ. Now look at the scattering cross section at an energy E_ϕ satisfying $m \ll E_\phi \ll \kappa, M$, so that the ϕ particles are still relativistic, but the Φ propagator and be contracted to a point as in figure 1b. Now the cross section goes as

$$\sigma_{2\phi\to 2\phi}|_{m \ll E_\phi \ll \kappa, M} \propto (\frac{\kappa}{M})^4 \frac{1}{E_\phi^2}$$

Contrasting this low energy cross section with that for neutrinos in §2.3 explains why κ interaction is said to be relevant at low energies, while Fermi interaction is called irrelevant.

Operators with dimension 4 lie between relevancy and irrelevancy and are called *marginal*. Examples of marginal interactions are

- ϕ^4 interactions;
- Yukawa interactions ($\overline{\psi}\psi\phi$);
- Gauge interactions (interactions of a gauge boson with itself, a scalar, or a fermion).

As we will see, marginality is an insecure position to be in, and quantum corrections will almost always change such operators from marginal to either relevant or irrelevant.

In each of the examples in the previous section we focussed on irrelevant interactions. The only reason why this was interesting was that in each case, irrelevant operators gave the leading contribution to the process... and because they weren't *too* irrelevant. For example, neutrinos *only* interact with matter through irrelevant operators...so if one sees any evidence of low energy neutrino scattering, one is seeing irrelevant operators. In contrast, e^+e^- scattering has an electromagnetic contribution from photon exchange. Since the photon-electron coupling is a marginal operator, at low energies electromagnetic interactions dominate the weak interaction contribution. (No coincidence that these are called weak interactions!). Now imagine a world where the W and Z masses were 10^{16} GeV. In this world there would be practically no discernible weak interaction effects. The neutron would have a lifetime greater than 10^{30} years, and there would be no radioactivity;

no one would have guessed that the neutrino existed, because it would not interact with anything. All we would discern in particle collisions and spectra would be the strong and electromagnetic interactions.

In fact, in any situation where there is a large gap between the energy where one is doing experiments and the energy scale of new physics, the effective theory one constructs will only consist of marginal and relevant operators... such theories are called "renormalizable" and are a natural outcome when there is a large hierarchy of physical scales. This typically results in a vast simplification of the physics one needs to consider, as seen in the next example.

3.1. Example: the success of Landau liquid theory.

A condensed matter system can be a very complicated environment; there may be various types of ions arranged in some crystalline array, where each ion has a complicated electron shell structure and interactions with neighboring ions that allow electrons to wander around the lattice. Nevertheless, the low energy excitation spectrum for many diverse systems can be described pretty well as a "Landau liquid", whose excitations are fermions with some complicated dispersion relation but no interactions. Why this is the case can be simply understood in terms of effective field theories, modifying the dimension counting used above to suit a nonrelativistic system with a Fermi surface [3].

Let us assume that the low energy spectrum of the condensed matter system has fermionic excitations with arbitrary interactions above a Fermi surface characterized by the fermi energy ϵ_F ; call them "quasi-particles". Ignoring interactions, the action can be written as

$$S_{free} = \int dt \int d^3p \sum_{s=\pm\frac{1}{2}} \left[\psi_s(p)^\dagger i\partial_t\psi_s(p) - (\epsilon(p) - \epsilon_F)\psi_s^\dagger(p)\psi_s(p)\right] \qquad (3.2)$$

where an arbitrary dispersion relation $\epsilon(p)$ has been assumed. Now let us consider higher dimension operators...but how should we count "dimension"? In the relativistic case, we defined mass dimension in a simple way, since we wanted to do an expansion in E/Λ,

[3] The treatment here follows that of Polchinski in ref. [1].

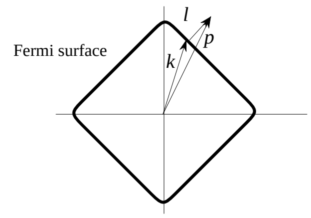

Fig. 2. The momentum $\vec{p}$ of an excitation above the Fermi surface is divided into a component $\vec{k}$ on the Fermi surface, and a component $\vec{\ell}$ perpendicular to the surface. The length of $|\vec{\ell}|$ is the quantity one wants to scale.

where E was the scale of the experiment and Λ was a physical scale associated with the system being probed. In a nonrelativistic system we identify the scaling dimension with momentum, in which case energy scales like p^2. Furthermore, it doesn't make sense to expand around $p = 0$ since an excitation cannot have a momentum vector inside the Fermi surface. So we write the momentum as

$$\vec{p} = \vec{k} + \vec{\ell}$$

where $\vec{k}$ lies on the Fermi surface and $\vec{\ell}$ is perpendicular to it (fig. 2). Then $\vec{\ell}$ is the quantity we vary in experiments and so we define the dimension of operators by how they must scale so that the theory is unchanged when we change $\vec{\ell} \to r\vec{\ell}$. If an object scales as r^n, then we say it has dimension n. Then $[k] = 0$, $[\ell] = 1$, and $[\int d^3p = \int d^2k d\ell] = 1$. And if we define the Fermi velocity as $\vec{\nabla}_p \epsilon$, then for $\ell \ll k$,

$$\epsilon(\vec{p}) - \epsilon_F = \vec{\ell} \cdot \vec{v}_F(\vec{k}) + \mathcal{O}(\ell^2) \ ,$$

and so $[\epsilon - \epsilon_f] = 1$ and $[\partial_t] = 1$. Given that the action (3.2) isn't supposed to change under this scaling,

$$[\psi] = -\tfrac{1}{2} \ .$$

Now consider an interaction of the form

$$S_{int} = \int dt \int \prod_{i=1}^{4} (d^2k_i d\ell_i) \delta^3(\vec{P}_{tot}) C(k_1, \ldots, k_4) \psi_s^\dagger(p_1) \psi_s(p_2) \psi_{s'}^\dagger(p_3) \psi_{s'}(p_4) \ .$$

This will be relevant, marginal or irrelevant depending on the dimension of C. Apparently $[\delta^3(P_{tot})C] = -1$. So how does the δ function scale? For generic $\vec{k}$ vectors, $\delta(\vec{P}_{tot})$ is a constraint on the $\vec{k}$ vectors that doesn't change much as one changes ℓ, so that $[\delta^3(\vec{P}_{tot})] = 0$. It follows that $[C] = -1$ and that the four fermion interaction is irrelevant...and that the system is adequately described in terms of free fermions (with an arbitrarily screwy dispersion relation). This effect is known in nuclear physics, where Pauli blocking allows a strongly interacting system of nucleons to have single particle excitations.

It is amusing that when a pair of $\vec{k}_i$ vectors are within $\mathcal{O}(\ell)$ of cancelling each other, then the scaling dimension of the delta function changes from 0 to -1. To see this, fix set the ℓ's to zero, and fix the incoming momenta $\vec{k}_1$ and $\vec{k}_2$. The δ-function then generically constrains three out of the four degrees of freedom in the outgoing momenta $\vec{k}_3$ and $\vec{k}_4$ in terms of $\vec{k}_1 + \vec{k}_2$. However, if $\vec{k}_1 + \vec{k}_2 = 0$, then $\vec{k}_3 + \vec{k}_4$ must equal zero, but that only constrains two of the four degrees of freedom (assuming a parity symmetric Fermi surface). Therefore the delta function $\delta^3(p)$ must scale like $\delta^2(k)\delta(\ell)$, and so for these head-on collisions between particles at opposite sides of the Fermi sea, $[C] = 0$, and the interaction is marginal. Quantum corrections either make it either irrelevant of relevant; it turns out that for C attractive, the interaction becomes relevant, and if it is repulsive it becomes irrelevant. In the former case, the interaction between such quasiparticles becomes strong near the Fermi surface, and can lead to pairing and superconductivity. See ref. [1] for more about this.

Exercise 7. *How would you couple phonons to the fermions in a Landau liquid? Would the phonon - fermion coupling be relevant, irrelevant, or marginal?*

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4. Quantum corrections and renormalization

It is fine to call a higher dimension operator irrelevant when one is computing amplitudes at tree level, and the momenta flowing through the vertices is small. But what happens when one calculates quantum corrections (loop graphs) involving these irrelevant interactions and integrates over intermediate states of all energies? Do the irrelevant operators become important? A field theory with irrelevant operators used to fill field theorists

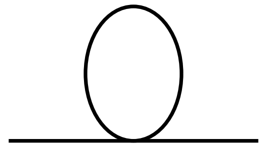

Fig. 3. A divergent one-loop radiative correction to the fermion mass and kinetic term in a theory with a $(\overline{\psi}\psi)^2$ interaction.

with horror, since they were "nonrenormalizable". This meant that rather than having a finite number of counterterms that had to be fixed by some experimental measurement, one needed an infinite number. Such theories were thought to be unpredictive. QED is a good example of a renormalizable theory: Only two measurements are needed to fix the counterterms, namely α and m_e. Once these quantities are measured in one set of experiments, all other QED processes can be predicted. In a theory with irrelevant operators, however, extra insertions of the operator in a graph makes it more divergent. In a theory with a Fermi interaction, for example — $(\overline{\psi}\psi)^2$ — one finds one needs counterterms for all $(\overline{\psi}\psi)^{2n}$ operators. Furthermore, these operators can in general renormalize relevant operators, such as the fermion mass, so it seems that all of these infinite number of interactions must be fit to experiment and nothing can be predicted.

This quandary is avoided if one uses a mass independent renormalization scheme (dimensional regularization), and thinks of the effective theory not as an expansion in operators, but as an expansion in inverse powers of some large physical scale Λ. Let us assume that we wish to do experiments at some momentum scale p and that the relevant operators have coefficients set by a scale $m \lesssim p$. In contrast, the irrelevant operators have coefficients which are inverse powers of $\Lambda \gg m, p$. For example, a theory of a fermion with mass m and higher order interactions:

$$\mathcal{L} = \overline{\psi} i \partial\!\!\!/ \psi - m\overline{\psi}\psi - \frac{a}{\Lambda^2}(\overline{\psi}\psi)^2 - \frac{b}{\Lambda^4}(\overline{\psi}\psi)^3 - \dots$$

Now consider the divergent graph in fig. 3.

This graph gives a divergent contribution to the mass operator $\overline{\psi}\psi$ proportional to

$$\frac{i}{\Lambda^2}\int \frac{d^4q}{(2\pi)^4}\frac{m}{q^2 - m^2} \ .$$

When Wick rotated into Euclidian space and defined by dimensional regularization, the above integral equals (see eq. (A.1) in the Appendix)

$$\frac{m^3}{16\pi^2\Lambda^2}\left(-\frac{1}{\epsilon}+\gamma-1+\ln\left[\frac{m^2}{4\pi\mu^2}\right]\right)$$

where we are in $4-2\epsilon$ dimensions, and μ is the renormalization scale that creeps into the problem[4]. In a mass independent subtraction scheme we put in a one-loop counterterm that cancels the infinite part of this graph, as well as a mass independent finite part. For example, in the $\overline{MS}$ scheme, we subtract the part proportional to

$$\frac{am^3}{16\pi^2\Lambda^2}\left(-\frac{1}{\epsilon}+\gamma-1-\ln 4\pi\right) \ .$$

We are left with a finite contribution to the fermion mass equal to (up to an $\mathcal{O}(1)$ numerical factor which I have dropped)

$$\delta m \sim \frac{am^3}{16\pi^2\Lambda^2}\ln\left[\frac{m^2}{\mu^2}\right] \ . \tag{4.1}$$

We choose a convenient scale μ and fit $(m+\delta m)$ to experiment (once one has also calculated the one-loop wave function renormalization).

The important point I wish to make is that

$$\frac{\delta m}{m} \propto \frac{m^2}{16\pi^2\Lambda^2} \ ;$$

it is *small* — it needs to be taken into account when probing effects proportional to $\frac{1}{\Lambda^2}$, but not otherwise. Note that this would not have been the case if we had simply taken Λ to be a physical momentum cutoff and not renormalized...then, since the fermion loop graph is quadratically divergent, we would have found $\delta m \sim (m/\Lambda^2)\times\Lambda^2 \sim m$. This would be a ludicrous state of affairs — we would have to understand quantum gravity, for example, to compute radiative corrections to e^+e^- scattering.

The above example has several important features which I wish to draw your attention to: (i) The correction to the electron mass δm is suppressed by m^2/Λ^2; (ii) δm has a logarithmic dependence on the fermion mass and the renormalization scale μ; (iii) The corrections to the fermion mass are proportional to the fermion mass. Each of these three points is worth commenting on:

[4] For a discussion of dimensional regularization, see for example refs. [5], [6]. For those familiar with the concepts, some useful formulas are included as an appendix to these lecture notes.

4.1. The size of radiative corrections

Concerning the first point: it is an obvious and general result that in a mass independent subtraction scheme, corrections to low dimension operators due to high dimension operators are always suppressed by powers of p/Λ and m/Λ. This is *not* what one would finds simply putting Λ in as a momentum cutoff for one's integrals. It is an obvious result because the only new mass scale induced by dimensional regularization is μ, and that can be seen to only enter logarithms. Thus an integral with dimension n will be proportional to the n^{th} power of the physical scales in the problem p and/or m. The scale Λ only enters the problem raised to negative powers at the vertices. Thus the graph is always proportional to $(p/\Lambda)^n$ where n is the combined powers from the vertices. No positive power of Λ is generated by the loop integral.

The fermion mass in our theory *does* receive an infinite number of corrections from the infinite number of higher dimension operators, and they are only computable if I measure all of the coefficients of these operators. However the theory remains predictive, since at any finite order in $m/\Lambda \ll 1$ there are a finite number of contributions to δm.

4.2. Radiative logarithms and the scale μ

The renormalization scale μ enters through logarithms of μ/m or μ/p. If we could sum up all orders in perturbation theory, all our answers would be μ independent. However, we stop at finite order, and our choice of μ can affect how quickly the perturbative expansion converges, since higher loop graphs yield higher powers of $\ln \mu^2/p^2$. Thus we should optimize perturbation theory by choosing μ to minimize the logarithm. When comparing experiments at widely different physical scales, we may run across large logarithms then of the form $\ln(p_1^2/p_2^2)$ since the same μ cannot make the logs in the two processes simultaneously small. These large logs can be resummed using the renormalization group, discussed in a later section.

4.3. Symmetry and naturalness

I noted that $\delta m \propto m$ in eq. (4.1). This is because $m \to 0$ increases the symmetry of the theory: in the above example the symmetry $\psi \to \gamma_5 \psi$, $\overline{\psi} \to -\overline{\psi}\gamma_5$ is a symmetry of the Fermi interaction (and kinetic term), but not the mass term. If $m = 0$ it follows that δm must also vanish. Therefore it is *natural* that the fermion mass might be small compared to other physical scales in the problem. In contrast, a scalar mass term $\phi\phi^*$ does not usually break a symmetry — the only exceptions are if the theory is supersymmetric, or if the scalar is a Goldstone boson. The latter is important for pion physics and chiral perturbation theory. Even if the tree level scalar mass is zero, it will get radiatively corrected by other fields it couples to. Thus it is *unnatural* for there to be a light scalar coupled to high energy fields. Since scalars presumably couple to gravity, typified by the Planck scale $m_P = 10^{19}$ GeV, one has to wonder why the Higgs boson in the standard model has a mass in the 10^2 to 10^3 GeV mass range. (It has been suggested that in fact either there is no scalar Higgs boson, or that it is a Goldstone boson, or that it is a member of a supersymmetric multiplet of particles).

It is ironic that it used to be that people were worried about theories with irrelevant operators being sick. In fact what we see is that irrelevant operators cause no problems; it is the relevant operators that we must worry about. If relevant operators appear in the effective field theory, then they must be set by a scale much less than Λ (else they wouldn't be in the effective theory below Λ). But if their coefficients are much smaller than Λ without a symmetry reason, then we are baffled. The prime example is the cosmological constant, namely the dimension 4 coefficient of the operator $\mathbf{1}$, otherwise known as the vacuum energy density. There is no known symmetry that appears relevant to our world that is increased by setting the vacuum energy density to zero, yet from cosmological observations, the vacuum energy is known to be $\lesssim 10^{-46}\ GeV^4$ [7]. The smallness of the cosmological constant should be taken as a warning: it appears contrary to effective field theory dogma, so the dogma may be flawed.

Exercise 8. *Compute both the wavefunction and mass corrections from the graph in fig. 3, using the $\overline{MS}$ scheme. See the Appendix for dimensional regularization formulas.*

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5. Matching

Consider doing experiments with photons and electrons entirely within the context of QED. There are three different regimes for scattering experiments which one might consider:

1. Either photons or electrons or both in the incoming state, with momentum transfer large compared to m_e;
2. electrons in the incoming state, but at momentum transfer much smaller than m_e;
3. photons in the incoming state, but momentum transfer much less than m_e.

In the first case one needs to compute the relevant amplitude in the full QED theory, although at high energy one might make the approximation that the electron is massless. Furthermore, the fine structure constant has to be adjusted from its low energy value $\alpha = 1/137$, and effect due to quantum corrections which we discuss in a later section. The second case is a little funny — we can ignore much of the complexity of QED since we do not have enough energy to produce positron-electron pairs, yet we still need to include both electrons and photons in the theory; I briefly mention the techniques one uses in this case in §7. For the third case one need only consider an effective theory with photons...why include electrons if one never sees any?

The low energy theory of photons alone looks like

$$\mathcal{L}_{eff} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{m_e^4}\left(a(F_{\mu\nu}F^{\mu\nu})^2 + b(F_{\mu\nu}\tilde{F}^{\mu\nu})^2\right) + \mathcal{O}(1/m_e^8) ,$$

the most general local, hermitian theory invariant under Lorentz, gauge, charge conjugation and parity transformations. (Can you show why there are no irrelevant operators of dimension 6?). This is not QED because it distorts high energy physics...but we do care that it correctly reproduces low energy phenomenology If we did not know about QED, we could treat this as a phenomenological theory and try to fit a and b to measured scattering cross sections. However, we do know QED, and so we can compute a and b. To do this we simply require that $\mathcal{L}_{QED}$ and $\mathcal{L}_{eff}$ give us the same physical predictions at low energy. In general, ensuring that the effective theory agrees in its predictions with the full theory to any desired order of accuracy is called "matching". What we are matching is the value of Green's functions in the two theories. Effective field theories are designed to reproduce all of the infrared (light particle) physics of the full theory, while distorting the high energy behavior to make calculations simpler. All of the interesting infrared effects in the full theory due to light particles are explicitly included; only the effects of the heavy particles or high energy modes must be mocked up. So the correct thing to do is match all the "one light particle irreducible" (1LPI) diagrams (diagrams that do not fall apart when one light particle line is cut), since these are the graphs that contain either a heavy particle, or high energy modes of a light particle. We cannot do this exactly of course, but we can do it systematically in a "loop" expansion, which is an expansion in powers of the numbers of loops in a diagram, or equivalently, powers of $\hbar$ [5].

5.1. Example: the $\phi^2\Phi$ interaction.

Rather than discussing QED, I will consider a toy model that exhibits nicely the matching procedure. It is the theory in eq. (3.1) with a light scalar ϕ coupled to a heavy scalar Φ via the interaction $\frac{1}{2}\kappa\phi^2\Phi$. (Never mind that the vacuum energy is unbounded below; one won't see this in perturbation theory). Suppose we are interested in $2\phi \to 2\phi$ scattering at energies much below the Φ mass M. The graphs we have to match to order $\hbar$ are those in fig. 4.

At tree level, Φ exchange generates a ϕ^4 interaction in the effective theory, so we find that

$$\mathcal{L}^0_{eff} = \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{2}m^2\phi^2 - c_0\left(\frac{\kappa}{M}\right)^2\frac{\phi^4}{4!} + \ldots$$

where the number c_0 is dimensionless and $\mathcal{O}(1)$, and computable from the graphs. The $\ldots$ refers to operators such as $(\kappa^2/M^4)\phi^2\partial^2\phi^2$ that one finds expanding the one-Φ exchange diagram to order p^2. If I had included a Φ^3 interaction in the full theory, there would have been more complicated tree diagrams leading to operators with higher powers of ϕ in $\mathcal{L}_{eff}$. The tree level matching condition is shown at the top of fig. 4. The graphs on the left are Φ exchange graphs in the full theory, while the contact interaction on the right is a local operator in the effective theory. For nonrelativistic ϕ particles, the procedure is equivalent to replacing the short range Yukawa potential due to Φ exchange with a $\delta^3(\vec{r})$ potential with a suitably matched coefficient.

[5] It may seem funny expanding in a dimensionful quantity we set to unity! However the loop expansion can be seen to be consistent with a perturbative expansion in coupling constants — see Coleman's lecture "Secret Symmetries" in ref. [8].

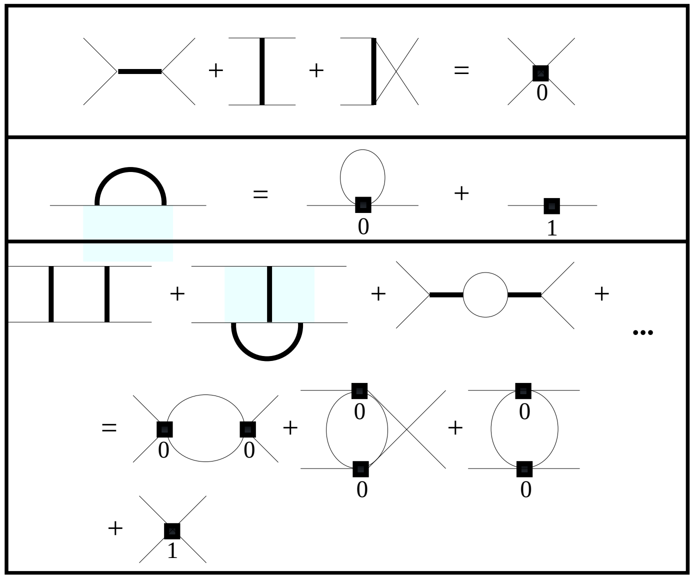

Fig. 4. Matching conditions for the theory of eq. (3.1). Diagrams on the left are in the full theory, while those on the right are in the effective theory. Heavy lines correspond to the heavy scalar propagator; numbers beneath the vertices count the loop order of the matching condition. The first row is the complete tree level matching condition; second and third rows are the one-loop matching conditions for the two- and four-point vertices respectively. Note that matching conditions are not simply the contraction of heavy propagators to contact interactions.

Now consider matching at $\mathcal{O}(\hbar)$. We must consider graphs with both 2 and 4 external ϕ fields. First consider the ones with two external ϕ fields. The mass renormalization graphs are divergent in both theories and are computed in $\overline{MS}$ To avoid large logarithms in the matching conditions of the form $M^2 \ln M^2/\mu^2$ we choose the renormalization scale $\mu = M$. Then the loop graphs in the second line of fig. 4 are well defined, finite objects, and the equation defines the $\mathcal{O}(\hbar)$ ϕ^2 interactions of the effective theory, labeled by a "1" on the right side of the equation. Including these terms, the kinetic term of $\mathcal{L}_{eff}$ becomes

$$\tfrac{1}{2}\left(1 + a_1 \frac{\kappa^2}{16\pi^2 M^2}\right)(\partial\phi)^2 - \tfrac{1}{2}\left(m^2 + b_1 \frac{\kappa^2}{16\pi^2}\right)\phi^2$$

where a_1 and b_1 are again dimensionless, $\mathcal{O}(1)$, and computable from the graphs. I have explicitly pulled out of the graphs the dimensionful quantities and the factors of $1/16\pi^2$ that arise from the loop integration.

Some of the graphs with four external ϕ's are shown on the third line in fig. 4.

With zero external momentum, the graphs are approximately equal to $\kappa^4/(16\pi^2 M^4)$ times logarithms. The logarithms blow up in the limit that the ϕ mass m goes to zero (an "infrared divergence"). However, the loop graphs in the effective theory have the *exact* same infrared divergence. Therefore the $\mathcal{O}(\hbar)$ contribution to a ϕ^4 interaction in the effective theory (labelled by a "1" on the last line of fig. 4) does not blow up as $m \to 0$. After 1-loop matching has been performed, the effective theory looks like:

$$\begin{aligned} \mathcal{L}^1_{eff} = \tfrac{1}{2}\left(1 + a_1 \frac{\kappa^2}{16\pi^2 M^2}\right)(\partial\phi)^2 - \tfrac{1}{2}\left(m^2 + b_1 \frac{\kappa^2}{16\pi^2}\right)\phi^2 \\ - \left[c_0\left(\frac{\kappa^2}{M^2}\right) + c_1\left(\frac{\kappa^4}{16\pi^2 M^4}\right)\right]\frac{\phi^4}{4!} \end{aligned} \tag{5.1}$$

where the coefficients a, b and c are $\mathcal{O}(1)$. In addition there are higher dimension operaotrs, such as ϕ^6, $(\phi\partial^2\phi)^2$, etc. This Lagrangian can be used to compute $2\phi \to 2\phi$ scattering up to 1 loop. One can perform an a_1-dependent rescaling of the ϕ field to return to a conventionally normalized kinetic term.

Let me close this section with several comments about the above example:

- Notice that the loop expansion is equivalent to an expansion in $(\kappa^2/16\pi^2 M^2)$. To the extent that this is a small number, perturbation theory and the loop expansion makes sense.
- We only computed relevant operators. There are in addition effects that are suppressed by powers of E^2/M^2 in an experiment with energy E (irrelevant operators). These may be as important as a subleading correction to a relevant operator's coefficient.
- We see an example of naturalness: the matching correction to the scalar mass is not proportional to m^2, so that it is "unnatural" for the physical mass to be $\ll \frac{\kappa^2}{16\pi^2}$ — that would require a finely tuned conspiracy between m^2 and κ^2. For κ and m both very small there is a symmetry regained in the full theory, namely the shift symmetry $\phi \to \phi +$ constant, which explains why ϕ can be naturally light in this limit.
- The coefficients of operators in the effective field theory are regularization scheme dependent. Their values differ for different schemes, but physical predictions do not (e.g, the relative cross sections for $2\phi \to 2\phi$ at two different energies).
- The coefficients of operators in the effective field theory are μ dependent, where μ is the renormalization scale. (More on this below).

- In the matching conditions the graphs in both theories have pieces depending nonanalytically on light particle masses and momenta (eg, $\ln m^2/M^2$ or $\ln p^2/M^2$)...these terms cancel on both sides of the matching condition so that the interactions in $\mathcal{L}_{eff}$ have a local expansion in inverse powers of $1/M$. This is an important and generic property of effective field theories.

Exercise 9. *Compute the graphs in fig. 4, using the $\overline{MS}$ scheme, and determine the coefficients a, b, and c in eq. (5.1).*

Exercise 10. *Draw a graph in the full theory that is not 1LPI ("one light particle irreducible") and convince yourself that that it is included in the effective theory, provided one matches all 1LPI graphs.*

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6. Quantum corrections: the myth of marginality

We have seen that relevant interactions — those with dimension < 4 (or $< d$ in d dimensions) — dominate physics at low energies. Marginal interactions (dimension 4) would appear to be equally important at all scales. In fact, quantum corrections change the scaling dimension of operators from their classical value. This doesn't usually have a dramatic effect on relevant or irrelevant operators, but for marginal operators it means that they become either relevant or irrelevant.

6.1. Renormalization group and ϕ^4 theory

To be concrete, consider ϕ^4 theory with the Lagrangian

$$\mathcal{L} = \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \ . \tag{6.1}$$

Consider the calculation of a the 1PI Green's functions Γ_n, which are one particle irreducible graphs that have had the external propagators amputated. They can be directly related to scattering amplitudes. Ignoring the issues of renormalization, one would expect to express these Green's functions in terms of the external momenta, the particle mass m, and the coupling constant λ:

$$\Gamma_n(p_1, ..., p_n; m, \lambda) \ .$$

The dimension of this object[6] is $(4-n)$. Therefore if one scales all of the external momenta by a factor s, one expects

$$\Gamma_n(sp; m, \lambda) = s^{4-n}\Gamma_n(p; m/s, \lambda) \ . \tag{6.2}$$

This expresses precisely what I was saying earlier about how the scalar mass is a relevant operator — note that its effects become large for small momentum scales, corresponding to $s \ll 1$. On the other hand, the ϕ^4 interaction's marginality is the observation that the importance of the λ coupling is independent of scale.

This analysis is incorrect when quantum corrections are taken into account, due to the introduction of a new scale μ. When we compute in perturbation theory, we must include counterterms and define the renormalized Lagrangian [7]

$$\mathcal{L}_{ren.} = \mathcal{L} + \mathcal{L}_{ct}$$

where

$$\begin{aligned}
\mathcal{L}_{ren} &= \tfrac{1}{2}(\partial\phi_0)^2 - \tfrac{1}{2}m_0^2\phi_0^2 - \frac{\lambda_0}{4!}\phi_0^4 \\
\mathcal{L} &= \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{2}m^2\phi^2 - \mu^{2\epsilon}\frac{\lambda}{4!}\phi^4 \ , \\
\mathcal{L}_{ct} &= \tfrac{1}{2}A(\partial\phi)^2 - \tfrac{1}{2}m^2 B\phi^2 - \mu^{2\epsilon}\frac{\lambda}{4!}C\phi^4 \ .
\end{aligned}$$

Both $\mathcal{L}$ and $\mathcal{L}_{ct}$ must be regulated; here I have chosen dimensional regularization, and a factor of $\mu^{2\epsilon}$ is inserted to keep λ dimensionless, where μ is the arbitrary renormalization scale. The Lagrangian $\mathcal{L}$ is written in terms of finite parameters, but gives infinite results; $\mathcal{L}_{ct}$ gives the counterterms A, B, C which all have $1/\epsilon$ poles in dimensional regularization and blow up in the $\epsilon \to 0$ limit. Computing graphs with the sum $\mathcal{L}_{ren} = \mathcal{L} + \mathcal{L}_{ct}$, which

[6] Γ_n is the time ordered product of n scalar fields ($d = n$) Fourier transformed to momentum space ($d = -4n$) with n external propagators removed ($d = 2n$) and a factor of $\delta^4(p_{tot})$ factored out ($d = 4$)...this gives $d = 4 - n$.

[7] See Ramond's book [6] for details; also see David Gross' 1975 Les Houches lecture [9].

is written in terms of "bare" couplings and fields, yields finite answers. The obvious correspondence between bare and renormalized parameters is:

$$\phi_0 = \sqrt{1+A}\,\phi \equiv \sqrt{Z_\phi}\,\phi \ , \qquad m_0^2 = m^2(1+B)/Z_\phi \ , \qquad \lambda_0 = \lambda(1+C)/Z_\phi^2 \ .$$

We can treat λ_0, m_0, μ and ϵ as independent parameters, and express λ and m in terms of them.

We can now define either bare or renormalized Green's functions, Γ^0 and Γ respectively. The relation between the two is

$$\Gamma_n^0(p_1, ..., p_n; \lambda_0, m_0, \epsilon) = Z_\phi^{-n/2}\Gamma_n(p_1, ..., p_n; \lambda, m, \mu, \epsilon)$$

where Γ_n is finite as $\epsilon \to 0$. Using the fact that Γ_n^0 is independent of μ, so that $\mathrm{d}\Gamma_n^0/\mathrm{d}\mu = 0$, one can derive the renormalization group (RG) equation

$$\left[\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial\lambda} + \gamma_m m\frac{\partial}{\partial m} - n\gamma\right]\Gamma_n = 0 \ , \tag{6.3}$$

where $\beta = \mu\partial\lambda/\partial\mu$, $\gamma_m = \mu\partial m/\partial\mu$, $\gamma = \frac{1}{2}\mu\partial\ln Z_\phi/\partial\mu$. One can compute these functions in perturbation theory by relating m, Z_ϕ and λ to m_0, λ_0 and μ and ϵ. For ϕ^4 theory one finds to leading nonzero order in perturbation theory

$$\beta(\lambda) = \frac{3\lambda^2}{16\pi^2} \ , \qquad \gamma_m = \frac{\lambda}{16\pi^2} \ , \qquad \gamma = \frac{1}{12}\left(\frac{\lambda}{16\pi^2}\right)^2 \ . \tag{6.4}$$

The reason why the RG equation is useful is because it tells one what happens if one scales the external momenta, given that there is a new scale in the problem, μ. On rescaling momenta by s, eq. (6.2) must be modified to read

$$\Gamma_n(sp; m, \lambda, \mu) = s^{4-n}\Gamma_n(p; m/s, \lambda, \mu/s) \tag{6.5}$$

or equivalently

$$\left[s\frac{\partial}{\partial s} + m\frac{\partial}{\partial m} + \mu\frac{\partial}{\partial\mu} - (4-n)\right]\Gamma(sp; m, \lambda, \mu) = 0. \tag{6.6}$$

This can be combined with the renormalization group equation (6.3) to yield an equation which relates the scaling of s to changes in m and λ alone, and not μ:

$$\left[-s\frac{\partial}{\partial s} + \beta\frac{\partial}{\partial\lambda} + (\gamma_m - 1)m\frac{\partial}{\partial m} - n\gamma + 4 - n\right]\Gamma_n(sp; m, \lambda, \mu) = 0. \tag{6.7}$$

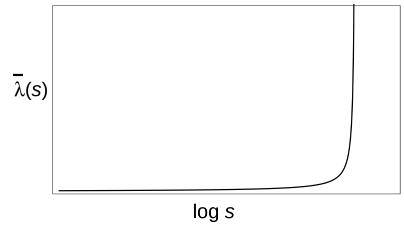

Fig. 5. The solution for the running coupling $\overline{\lambda}(s)$ *as a function of* $\ln(s)$. *The one-loop expression becomes infinite at finite* $\ln(s) = 16\pi^2/3\lambda^2$, *but the result is not to be trusted since the perturbative expansion breaks down.*

If one uses a mass independent subtraction scheme such as $\overline{MS}$, then the coefficients β, γ_m and γ depend only on λ and not on the other dimensionless quantity, m/μ. In this case, one can solve eq. (6.7), and one finds

$$\Gamma_n(sp; m, \lambda, \mu) = s^{4-n}\Gamma_n(p; \overline{m}(s), \overline{\lambda}(s), \mu)e^{-n\int_1^s \mathrm{d}s'\gamma(\overline{\lambda}(s'))/s'} \tag{6.8}$$

where $\overline{\lambda}$ and $\overline{m}$ satisfy the differential equations

$$s\frac{\partial\overline{\lambda}(s)}{\partial s} = \beta(\overline{\lambda}(s)) \ , \qquad \overline{\lambda}(1) = \lambda$$

$$s\frac{\partial\overline{m}(s)}{\partial s} = (\gamma_m - 1)\overline{m}(s) \ , \qquad \overline{m}(1) = m \ .$$

First look at this solution at tree level, where $\beta = \gamma_m = \gamma = 0$ and Γ is independent of μ. Then the solution (6.8) is equivalent to the simple scaling property (6.5). If only γ is nonzero, and it is constant, then the exponential in eq. (6.8) gives an overall factor of $s^{-n\gamma}$ to the scaling of Γ...the engineering dimension $(4 - n)$ is modified by an additional factor of $-\gamma$ for each of the n fields, hence the name "anomalous dimension" for γ. Finally, if β and γ_m are nonzero, then changing the momentum scale means one lets the mass and coupling "run". Using the β function in eq. (6.4), one finds

$$\beta(\lambda) = \frac{3\lambda^2}{16\pi^2} \quad \longrightarrow \quad \overline{\lambda}(s) = \frac{\lambda}{1 - (3\lambda/16\pi^2)\ln s} \ .$$

See fig. 5.

We see that the ϕ^4 interaction is an example of a marginal interaction that becomes irrelevant due to quantum corrections: the lower the energy scale probed in a scattering experiment, the weaker the effect of the interaction. QED is another example — the gauge interaction becomes irrelevant due to quantum corrections. There is a simple physical explanation for this: the vacuum acts as a dielectric, with virtual particle-antiparticle pairs which screen charges. The greater the impact parameter in a scattering experiment, the more screened the charge is and the weaker the interaction. This can be parametrized by a scale dependent fine structure constant, $\alpha(\mu)$. As $\mu \to 0$, $\alpha(\mu) \to 0$. In QED, the screening ceases over distances longer than the Compton wavelength of the electron, and so $\alpha(\mu) \to 1/137$ for $\mu \lesssim m_e$. Theories such as QED and ϕ^4 all by themselves are called "asymptotically unfree". they are thought to be meaningless as theories because of what happens in the ultraviolet: In ϕ^4 theory one finds nonperturbatively (ie, on the lattice) that $\lambda(\mu) \to \infty$ for $\mu \to \mu_0$ for a finite μ_0. QED probably behaves similarly, although people debate whether α may approach a constant for sufficiently large μ (a "nontrivial fixed-point").

6.2. Renormalization group and QCD

In contrast, Yang-Mills theories such as QCD have a negative β-function and are asymptotically free: the gauge interactions, which are marginal at tree level, become relevant. The important physical difference between QED and Yang-Mills theories that accounts for the different sign of the β-function is that Yang-Mills gauge bosons carry charge, while photons do not. For QCD, the β function at one loop order with N_f flavors of (Dirac) quarks is

$$\beta(g) = \mu\frac{\partial g}{\partial \mu} = \frac{g^3}{16\pi^2}\left[-11 + \frac{2N_f}{3}\right] \equiv -\frac{b_0 g^3}{2} \ . \tag{6.9}$$

For $N_f \leq 16$ this is negative, and so it is negative in the standard model where $N_f = 6$ (u,d,s,c,b,t). Defining $\alpha_s = g^2/4\pi$, eq. (6.9) can be integrated to give

$$\alpha_s(\mu) = \frac{1}{1/\alpha_s(\mu_0) + 4\pi b_0 \ln(\mu/\mu_0)} \equiv \frac{1}{4\pi b_0 \ln(\mu/\Lambda_{QCD})} \ .$$

Notice that a new scale has crept into the theory — Λ_{QCD}. It has been defined as

$$\Lambda_{QCD} = \mu_0 E^{1/4\pi b_0 \alpha(\mu_0)} \ , \tag{6.10}$$

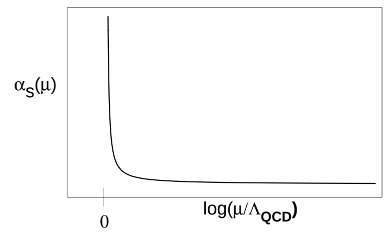

Fig. 6. The solution for the running coupling $\alpha_s(\mu)$ in QCD using the one-loop β function. The behavior near $\mu = \Lambda_{QCD}$ is unreliable where α_s is large.

and is independent of μ_0 (to the order we are working in perturbation theory). This is the scale that determines where the strong interactions get strong, what the proton and ρ masses are, etc. It's value is scheme dependent, and depending on which sensible scheme one uses, it can range from $100 - 250$ MeV [8]. See fig. 6 for a plot of $\alpha_s(\mu)$.

The reason why we call the strong interactions "strong" is because the beta function is negative, and there are some light quarks. Even though the the gauge coupling α_s has positive dimension, it is small when α_s is small and the interaction is only barely relevant. In contrast, quark masses start off with a classical dimension 1. Assuming for the moment that QCD with explicit quark masses was the true theory (ie, ignoring the weak interactions and the Higgs), let us look at the theory from the vantage point of the Planck scale. We would see a small α_s with sluggish logarithmic scaling racing against tiny quark masses with linear scaling properties. Which one wins in the infrared? For the top quark, the mass term wins — $\alpha_s(m_t) \simeq 0.1$ and the toponium ($\bar{t}t$) mass is determined by $2m_t$ with only small ($\alpha_s^2 m_t$) Coulomb corrections. In contrast, the gauge interaction wins in the race against the u and d quarks, which have masses of order 10 MeV. The proton — a uud bound state — has a mass equal to 940 MeV, which is scarcely affected by the u and d quark masses. Its mass is attributable to the effects of the strong gauge interaction, which is associated with a nonperturbative (scheme dependent) mass scale $\Lambda_{QCD} \sim 200$ MeV ($\overline{MS}$). The strong interactions are strong because gluon interaction is relevant and

[8] One does not determine it by measuring where α_s blows up! Instead one determines α_s at some large scale, such as the Z mass, where QCD is weakly coupled and a perturbative calculation of β makes sense. Then Λ is defined (at one-loop order) by eq. (6.10).

the u, d and s quarks are light [9]

6.3. The RG and perturbation theory

The β function for marginal interactions can be computed without much problem in perturbation theory; similarly one can see how relevant and irrelevant operator dimensions are modified (anomalous dimensions). My favorite treatment of the subject — called renormalization group analysis — is in the book by Ramond [6], §4.5. Unfortunately, perturbation theory, with its attendant divergences and counterterms, may be a practical computational tool, but it tends to obscure the beautiful physics behind the renormalization group. Wilson thought of the effective action S_Λ as a theory with all modes of frequency $\omega > \Lambda$ removed. $S_{\Lambda-\delta\Lambda}$ could be defined as

$$e^{-S_{\Lambda-\delta\Lambda}} = \int_{\delta\Lambda} \mathrm{d}\phi_\Lambda e^{-S_\Lambda}$$

where the integral is a path integral over all modes with frequency $\Lambda - \delta\Lambda < \omega < \Lambda$. Since the integration is over a finite number of modes (assume the system is in a box), the integration is finite and one needn't ever discuss counterterms, renormalization, etc. The action can then be shown to obey a differential equation

$$\frac{\partial S_\Lambda}{\partial \Lambda} = F(S_\Lambda)$$

where F is a functional of the action. Couplings in the effective action "flow" as one changes the cutoff Λ. Those with negative eigenvalues are irrelevant, and their coupling flows to zero in the infrared (limit of small Λ); positive eigenvalues correspond to relevant operators and their effects become stronger in the infrared. Wilson's picture is in many

[9] It should seem peculiar to the reader that the quark masses are scattered within a couple orders of magnitude of Λ_{QCD} — this doesn't seem natural from the Planck scale perspective. In fact the situation is complicated by the fact that above the weak scale, quarks don't have masses, but rather Yukawa couplings to the Higgs. The mystery then becomes, why does the Higgs get an expectation value within a couple orders of magnitude of Λ_{QCD}? Why do the Yukawa couplings range from 10^{-5} (for the u quark) to 1 (for the top quark)? There are a range of explanations to these questions in the literature, with a range of plausibility, but there is no evidence for any of them presently.

ways less confusing than the perturbative renormalization group discussion, but it is not practical for analytic computations.

At any finite order in perturbation theory, one's answers *will* be μ dependent, and one has to pick a scale. Changing $\mu \to \mu'$ corresponds to changing the coupling constant $g \to g'$. If

$$\beta = bg^2$$

for example, then

$$g' = \frac{g}{1 - bgt} = g\left[1 + bgt + (bgt)^2 + \ldots\right]$$

where $t = \ln \mu/\mu'$. So a calculation to second order in $g(\mu')$ includes an infinite number of terms in a perturbative expansion in $g(\mu) \ln \mu/\mu'$. Scaling g is said to "sum the leading logs". Using the two loop β function sums terms like $(g^2 t)^n$, the "subleading logs". So using different values for μ *does* change results of a calculation to some finite order in g. In practice then one wants to choose the value for μ that makes the perturbation expansion converge most quickly. Typically, that means choosing μ to make the logs small, which means choosing μ to be a physical scale in the process of interest, *eg* the q^2 flowing through the graph. This is good news if you are doing QCD at $q^2 = (100\ GeV)^2$, since $\alpha_s(\mu) \sim 0.1$ at that scale; it is bad news if you wish to do a QCD calculation at $q^2 = (500\ MeV)^2$ since there is no perturbative expansion in α_s at that low value for μ. To figure out the right value of μ one really needs to compute next order corrections and find the μ that minimizes them. There are interesting prescriptions for choosing μ in the literature [10].

The procedure for computing photon-photon scattering at $q^2 \ll m_e^2$ should be clear now:

- Match QED to an effective theory without photons, choosing $\mu \simeq m_e$;
- Compute the β's and γ's of the effective theory;
- Change μ to the q^2 scale of the process of interest, scaling the parameters of the effective theory;
- Compute the process of interest.

Of course, the scaling isn't very interesting in low energy QED — the only interactions are irrelevant, and the lowest order 4-photon vertex does not run with μ; only higher order operators do. Where scaling effects are important is when an interaction is pretty strong (eg, scaling effects due to gluons in the perturbative regime), or when the scaling is over a huge energy range (eg, the computation of $\sin^2\theta_w$ from GUT theories. See ref. [11].). See lecture notes by A. Manohar from the 1995 Lake Louise Winter Institute for a nice discussion of the scaling of $\Delta S = 1, 2$ operators from the weak interactions due to gluons [12].

Exercise 11. *Given that to leading nonzero order the renormalization parameters C and Z_ϕ in eq. (6.4) for ϕ^4 theory ($\overline{MS}$) are $C = -\frac{3\lambda}{32\pi^2}\frac{1}{\epsilon}$ and $Z_\phi = 1 - \frac{\lambda^2}{384\pi^2}\frac{1}{\epsilon}$, show that β is as given in eq. (6.4). Hint: use the fact that λ_0 is independent of μ, work consistently to a given order in perturbation theory, and only set $\epsilon \to 0$ at the end of the calculation).*

Exercise 12. *Compute α_s at μ= 2 GeV using the one-loop β function for QCD, given that α_s at $\mu = 90$ GeV equals 0.12. For the sake of this calculation, assume that the b quark mass is $m_b = 4.5$ GeV. All the other quarks are lighter than 2 GeV, except the top, which is heavier than 90 GeV. Assume there are no other colored particles.*

Exercise 13. *Explain how you know that the four photon vertex doesn't run with μ in effective QED below m_e, in the $\overline{MS}$ scheme.*

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7. Effective field theory with heavy stable particles

Previously I mentioned that one might be interested in scattering electrons and photons at energies much below the electron mass. One immediately encounters a problem in constructing the effective field theory in terms of local operators constructed out of the electron and photon fields and their derivatives: an operator such as $(\bar{e}\not{D}^2 e)F^2/m_e^5$ is not actually a $1/m_e^5$ effect since the time derivatives acting on an electron at rest bring powers of m_e into the numerator. The solution is similar to what we have always done in nonrelativistic physics...ignore the electron rest mass and redefine the electron field to get rid of the $exp(-imt)$ phase and assume that the remaining frequency — corresponding to the kinetic energy — is much smaller than m.

A free, charged, nonrelativistic particle obeys the Schrödinger equation:

$$i(\partial_t - ieA_0)\Psi = -\frac{(\vec{\nabla} - ie\vec{A})^2}{2m}\Psi \ .$$

in the infinite mass limit, we can ignore the kinetic energy which goes as $1/m$ and so $\partial_t\Psi = 0$ is the equation of motion. We can restore relativistic covariance by defining the four velocity vector, which equals $(1, 0, 0, 0)$ in the rest frame of the particle, and so the equation of motion becomes

$$v_\mu D^\mu \Psi = 0 \ ,$$

and the kinetic term in the Lagrangian is

$$\mathcal{L}_0 = \Psi^\dagger v \cdot D\Psi \ . \tag{7.1}$$

How does one get from the relativistic field theory to one with a kinetic term like (7.1)? One defines the momentum of the heavy particle to be

$$p_\mu = mv_\mu + k_\mu$$

where $k \ll m$. Then for a scalar field, one rewrites ϕ as $\phi = e^{-imv\cdot x}\Psi_v$. Then one removes the positive frequency component of Ψ which creates antiparticles, and writes the most general Lagrangian in terms of the negative frequency component Ψ_v^- and its derivatives. The result is a theory which involves an expansion in k/m, assumed to be small. One then integrates of $v's$ to restore Lorentz covariance.

The procedure for fermions is to define [13]

$$h_v = \frac{1 + \not{v}}{2} e^{im\not{v} v\cdot x}\psi \ ,$$

where ψ is the heavy fermion field. The $(1 + \not{v})/2$ projection operator eliminates the "small components" of the spinor, which are suppressed by $1/m$. In the large mass limit processes do not change v, and so one then constructs the effective lagrangian $\mathcal{L}_v$ out of h_v and expands in powers of ∂_μ/m; then one integrates $\mathcal{L}_v$ over velocities v. All applications of interest have extensive symmetries that limit the form of the higher dimension operators that one can write down in the effective theory.

This procedure has been used extensively over the past few years to analyze hadrons containing a heavy (b or c) quark by constructing an effective theory in powers of m_b and m_c. Another application has been the interaction of pions with baryons, treating the baryons as heavy fields. See [14] for a discussion of heavy quark effective field theory, and [15] for the heavy baryon formalism.

8. Effective field theory for the strong interactions: chiral Lagrangians

Symmetry is the only reason we know about that can explain why mass hierarchies occur. The proton mass is much lighter than the Planck scale, and we can qualitatively understand that by noting that m_p arises from spontaneous breaking of chiral symmetry. Chiral symmetry breaking is a nonperturbative effect which is expected to occur at a scale $\mu e^{-a/\alpha_s}$ (see eq. (6.10)), where a is some number and α_s is the strong coupling at the scale μ, which might be the Planck scale. Then the reasonable number $a/\alpha \simeq 40$ explains the observed (enormous) hierarchy. Attempts have been made to similarly explain why M_W is so much lighter than the Planck scale, but no convincing theory exists.

Chiral symmetry is a symmetry that keeps fermions light. Symmetries can also keep bosons light, but only if they are spontaneously broken. Then Goldstone's theorem guarantees that there will be massless Goldstone bosons. If what is broken is only an approximate symmetry, then one finds "pseudo Goldstone bosons" which are light but not massless. As you have heard in Professor Holstein's lecture, this is the explanation for why the pion is much lighter than the rho meson, and that one can construct an effective field theory of pseudoscalars and baryons to describe low energy strong interactions. Although the hierarchy here is not too large, the chiral effective theory pioneered by Weinberg has had many successes. And even though it is easy to probe physics far above its range of validity, our theoretical failings mean that we cannot analytically compute the coupling constants of the chiral Lagrangian by matching with QCD, but must determine them phenomenologically.

I don't have time to say much about chiral Lagrangian calculations, but I do want to make a comment about power counting. The pions have two mass scales associated with them: their mass $m_\pi = 140$ MeV, and their decay constant f_π. This is variously defined;

I take

$$\langle 0|\, j_A^{\mu\, a} \left|\pi^b\right\rangle = i f_\pi p^\mu \delta^{ab}$$

$$j_A^{\mu\, a} = \overline{q}\gamma^\mu\gamma_5 T^a q \qquad T^a = \tfrac{\sigma^a}{2}$$

$$f_\pi \simeq 93\ MeV$$

The leading operator in the chiral Lagrangian is

$$\frac{f^2}{4}\,\mathrm{Tr}\,\partial\Sigma\partial\Sigma^\dagger$$

where

$$\Sigma = e^{2i\pi^a T^a/f}\ .$$

This term is *universal*; it depends on the scale f and on the symmetry breaking pattern $SU(2)\times SU(2)\to SU(2)$. It does not depend on any other detail of QCD. In fact, the system is so highly constrained by symmetry that one gets the same effective theory from QCD, a linear σ model, or the NJL model! This makes the chiral Lagrangian one of the preeminent examples of how an effective field theory "loses" information about short distance physics. However, the pion mass term ($\mathrm{Tr}\, M_q\Sigma$) and higher dimension operators (eg, $\mathrm{Tr}(\partial\Sigma\partial\Sigma^\dagger)^2$) are not universal, and measuring their coefficients tells us (indirectly) about QCD dynamics. But what is the mass scale these higher dimension operators are being expanded in? if the theory is an expansion in p_π/f_π then it isn't of any use in the real world. In fact the expansion should probably be in inverse powers of m_ρ or some higher scale. A nice power counting scheme was developed by Weinberg and discussed in detail by Georgi and Manohar [16]. They argue that the "natural" scale for the derivative expansion in powers of ∂/Λ is $\Lambda \lesssim 4\pi f_\pi$. The argument is based on the requirement that the coefficient of an operator receive radiative corrections no larger than the tree level value. In the real world, it seems that $\Lambda \simeq 4\pi f_\pi$ works pretty well, and so chiral perturbation theory (ie, exploitation of the derivative expansion) works pretty well for pions, up to ~ 500 MeV in some channels. Certain features work well for kaons as well. Chiral Lagrangians have been applied to nuclear matter for both pion and kaon condensation (eg, refs. [17]) and nuclear forces [18].

9. Conclusions: Why effective field theory?

Effective field theory is a useful tool to be learned. Using effective field theory makes computations simpler — one needn't compute features of a complicated field theory that rare of no interest in low energy physics; and in conjunction with the renormalization group one can simply solve problems that involve disparate scales. To learn how to do this one must work through examples. Here are a handful of effective field theory calculations in the literature which I think are instructive:

1. Renormalization group calculation of $\sin^2\theta_w$ from a theory at 10^{14} GeV: ref. [11].
2. The matching and renormalization group scaling of $\Delta S = 1$ operators from the weak scale down to the hadronic scale: ref. [19].
3. Computation of the charmonium binding energy in nuclei: ref. [3].
4. Parity violating operators for nuclear physics in the chiral Lagrangian: ref. [20].
5. Chiral perturbation theory with heavy baryons: ref. [21].
6. Fitting properties of the $\Lambda(1405)$ to experiment at the 1-loop level in chiral perturbation theory: ref. [22].
7. Chiral perturbation theory for hadrons with a heavy quark: ref. [23].

In addition, there are three other recent reviews I recommend on effective field theories, all with quite different content, refs. [1], [2], and [12].

Effective field theory is much more than useful tool, however — it is a paradigm for considering all of physics, illuminating the reason why physics looks “simple”: To a first approximation we needn't understand quantum gravity to understand the top quark; we needn't know about the top quark to understand the hydrogen atom; the details of atomic structure are irrelevant for hydrodynamics; and we needn't understand hydrodynamics to compute the orbits of the celestial bodies. Some may hungrily await the final theory of everything, but effective field theory allows others of us to take small bites of something in the meanwhile.

Acknowledgements

I wish to thank M. Savage for useful conversations. This work was supported in part by DOE contracts DOE-ER-40561 and DE-FG06-91-ER40614, NSF Presidential Young Investigator award PHY-9057135, and a grant from the Sloan Foundation.

Appendix A. Dimensional Regularization Formulas

Consider the following integral in n dimensions with a *Euclidian* metric:

$$I_1 \equiv \int \mathrm{d}^n k \; \frac{1}{(k^2+a^2)^r} \; .$$

We may evaluate this making in terms of the Γ function:

$$\alpha^{-s}\Gamma(s) = \int_0^\infty \mathrm{d}x \; x^{s-1} \; \mathrm{e}^{-\alpha x}.$$

Then

$$\begin{aligned} I_1 &= \frac{1}{\Gamma(r)} \int \mathrm{d}^n k \; \int_0^\infty \mathrm{d}x \; x^{r-1} \; \mathrm{e}^{-x(k^2+a^2)} \\ &= \frac{\pi^{n/2}}{\Gamma(r)} \int_0^\infty \mathrm{d}x \; x^{r-1-n/2} \; \mathrm{e}^{-xa^2} \\ &= \pi^{n/2} a^{n-2r} \frac{\Gamma(r-n/2)}{\Gamma(r)} \end{aligned}$$

Another useful integral is

$$I_2 \equiv \int \mathrm{d}^n k \; \frac{k^2}{(k^2+a^2)^r} \; .$$

To get this we define

$$\begin{aligned} I_1(\alpha) &\equiv \int \mathrm{d}^n k \; \frac{1}{(\alpha k^2+a^2)^r} \\ &= \alpha^{-n/2} I_1 \end{aligned} \; ;$$

then by differentiating by α and setting $\alpha = 1$ we find

$$I_2 = \frac{n\pi^{n/2}a^{n-2r+2}}{2(r-1)} \frac{\Gamma(r-1-n/2)}{\Gamma(r-1)} \; .$$

Finally note that

$$\begin{aligned} I_3^{\mu\nu} &\equiv \int \mathrm{d}^n k \; \frac{k^\mu k^\nu}{(k^2+a^2)^r} \\ &= \frac{\delta^{\mu\nu}}{n} I_2 \end{aligned} \; .$$

Some Properties of Γ Functions

Γ functions have the property $\Gamma(z+1) = z\Gamma(z)$, with $\Gamma(1) = 1$. Thus for integers $n \geq 1$,

$$\Gamma(n+1) = n!, \qquad n \geq 1 \ .$$

Also useful is the value

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi} \ .$$

The Γ function is singular for non-positive integer arguments. Near these singularities it can be expanded as

$$\Gamma(-n+\epsilon) = \frac{(-1)^n}{n}\left[\frac{1}{\epsilon} + \psi(n+1) + \mathcal{O}(\epsilon)\right] \ ,$$

where

$$\psi(n+1) = 1 + \frac{1}{2} + \ldots + \frac{1}{n} - \gamma \ ,$$
$$\gamma = 0.5772\ldots$$

In particular,

$$\Gamma(\epsilon - 1) = -\frac{1}{\epsilon} + \gamma - 1$$
$$\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma$$

Useful consequences:

$$\mu^{2\epsilon}\int \frac{\mathrm{d}^{4-2\epsilon}q}{(2\pi)^{4-2\epsilon}}\frac{1}{q^2+m^2} = \frac{m^2}{16\pi^2}\left[-\frac{1}{\epsilon} + \gamma - 1 - \ln 4\pi + \ln(m^2/\mu^2)\right] \qquad \text{(A.1)}$$

$$\mu^{2\epsilon}\int \frac{\mathrm{d}^{4-2\epsilon}q}{(2\pi)^{4-2\epsilon}}\frac{1}{(q^2+m^2)^2} = \frac{1}{16\pi^2}\left[\frac{1}{\epsilon} - \gamma + \ln 4\pi - \ln(m^2/\mu^2)\right] \qquad \text{(A.2)}$$

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