COSMOLOGICAL RESTRICTIONS ON CONFORMALLY INVARIANT $SU(5)$ GUT MODELS

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Abstract. Dirac's theory of constrained Hamiltonian systems is applied to the minimal conformally-invariant $SU(5)$ grand-unified model studied at 1-loop level in a de Sitter universe. For this model, which represents a simple and interesting example of GUT theory and at the same time is a step towards theories with larger gauge group like $SO(10)$, second-class constraints in the Euclidean-time regime exist. In particular, they enable one to prove that, to be consistent with the experimentally established electroweak standard model and with inflationary cosmology, the residual gauge-symmetry group of the early universe, during the whole de Sitter era, is bound to be $SU(3) \times SU(2) \times U(1)$. Moreover, the numerical solution of the field equations subject to second-class constraints is obtained. This confirms the existence of a sufficiently long de Sitter phase of the early universe, in agreement with the initial assumptions.

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1. Introduction

In recent work by the first two authors (Buccella *et al* 1992) the spontaneous-symmetry-breaking pattern of $SU(5)$ gauge theory was studied in a de Sitter universe. The main result was the proof that the technique described in Buccella *et al* 1980 to study spontaneous symmetry breaking of $SU(n)$ for renormalizable polynomial potentials in flat spacetime can be generalized, for $SU(5)$, to the curved background relevant to the inflationary cosmology (i.e. de Sitter). One thus obtained a better understanding of the result, previously found with a different numerical analysis (Allen 1985), predicting the slide of the inflationary universe into either the $SU(3) \times SU(2) \times U(1)$ or $SU(4) \times U(1)$ extremum.

The main tool used was the Wick-rotated path integral for Yang-Mills-Higgs theory at 1-loop level about curved backgrounds (leading to a de Sitter model with S^4 topology), and the corresponding 1-loop effective potential $V(r, \hat{\mathbf{\Phi}})$ first derived in Allen 1985. Assuming that the Higgs field $\hat{\mathbf{\Phi}} = \mathrm{diag}\Bigl(\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5\Bigr)$ belongs to the adjoint representation of $SU(5)$, and using the bare Lagrangian and the tree potential appearing in equations (2.1)-(2.2) of Buccella *et al* 1992, the background-field method and the choice of 't Hooft's gauge-averaging term lead to (Allen 1985, Buccella *et al* 1992)

$$V(r, \hat{\mathbf{\Phi}}) = \frac{15}{64\pi^2} \left\{ Q + \frac{1}{3}\Bigl(1 - \log(r^2 M_X^2)\Bigr) \right\} R\, g^2 \| \hat{\mathbf{\Phi}} \|$$

$$+ \left\{ \frac{9}{128\pi^2}\Bigl(1 - \log(r^2 M_X^2)\Bigr) - \frac{21}{320\pi^2}\widetilde{\Lambda} \right\} g^4 \| \hat{\mathbf{\Phi}} \|^2$$

$$+ \frac{15}{128\pi^2} \left\{ \frac{12}{5}\widetilde{\Lambda} + \Bigl(1 - \log(r^2 M_X^2)\Bigr) \right\} g^4 \sum_{i=1}^{5} {\varphi_i}^4$$

$$-\frac{3}{16\pi^2 r^4}\sum_{i,j=1}^{5}\mathcal{A}\left[\frac{r^2 g^2}{2}(\varphi_i-\varphi_j)^2\right]+V_0 \qquad (1.1)$$

where Q has been defined in equation (2.6) of Buccella *et al* 1992, and we here denote by $\widetilde{\Lambda}$ the parameter defined in equation (2.7) of Buccella *et al* 1992, to avoid confusion with the cosmological constant Λ. Note also that $\mathcal{A}$ denotes the special function defined in equation (A.1) of the appendix, $r=\sqrt{3/\Lambda}$ is the four-sphere radius, and the constant V_0 is equal to the desired $V(\mathbf{\Phi}=0)$ value.

A naturally-occurring question is whether further restrictions on the $SU(5)$ broken-symmetry phases can be derived within the framework of inflationary cosmology. This paper is devoted to the study of such a problem, and is thus organized as follows. Section 2 performs the Hamiltonian analysis of the Riemannian (i.e. Euclidean-time) version of a de Sitter background coupled to the $SU(5)$ model, following Dirac's theory of constrained systems. Section 3 studies the 1-loop effective potential in the $SU(3)\times SU(2)\times U(1)$ and $SU(4)\times U(1)$ broken-symmetry phases. Section 4 shows that to be compatible with the prescriptions of the electroweak standard model and with the inflationary scheme, the early universe can only reach the $SU(3)\times SU(2)\times U(1)$ broken phase and that during all the de Sitter era, possible tunneling processes towards the $SU(4)\times U(1)$ invariant phase are energetically forbidden. Section 5 describes the numerical integration of the corresponding field equations for the four-sphere radius and the components of the Higgs field. Concluding remarks are presented in section 6.

2. Hamiltonian analysis

The effective Lagrangian for the Riemannian version of an exact de Sitter background with S^4 topology coupled to the $SU(5)$ model is given by (up to a multiplicative constant)

$$L = \left[-\frac{3r^2}{4\pi G} + r^4 \left(\sum_{i=1}^{5} \frac{\dot{\varphi}_i^2}{2} + V(r, \hat{\mathbf{\Phi}}) \right) \right] \tag{2.1}$$

where the derivatives $\dot{\varphi}_i$ are taken with respect to the Euclidean time τ and the spatial gradient of the Higgs field has been assumed to be negligible. Note that, after having integrated over the gauge-bosons degrees of freedom, the effective potential only involves the Higgs field. According to the simplified argument usually presented in the literature, the field equations obtained by varying the action with respect to r and $\hat{\mathbf{\Phi}}$ are then (Allen 1983)

$$r^2 = \frac{3}{8\pi G \left[\sum_{i=1}^{5} \dot{\varphi}_i^2/2 + V(r, \hat{\mathbf{\Phi}}) \right]} \tag{2.2a}$$

$$\ddot{\varphi}_i = \frac{\delta V(r, \hat{\mathbf{\Phi}})}{\delta \varphi_i} \qquad \forall i = 1, .., 5 \quad . \tag{2.2b}$$

However, this approach does not take into account the full Hamiltonian treatment of the problem. In other words, from equation (2.1) one derives the primary constraint $p_r \approx 0$, where p_r is the momentum conjugate to the four-sphere radius, and $\approx$ is the symbol of weak equality (i.e. an equality which only holds on the constraint surface). This primary constraint should be preserved using the technique described for example in Dirac 1964 and Esposito 1992. The corresponding Hamiltonian analysis is as follows.

The canonical Hamiltonian H_c, defined as the Legendre transform of the Lagrangian L in equation (2.1), takes the form

$$H_c = r^{-4} \sum_{i=1}^{5} \frac{p_{\varphi_i}^2}{2} + br^2 - r^4 V(r, \hat{\mathbf{\Phi}}) \tag{2.3}$$

where we have defined $b \equiv 3/(4\pi G)$. Thus, the effective Hamiltonian $\widetilde{H}$ defined on the whole phase space becomes

$$\widetilde{H} \equiv H_c + \lambda(r, \hat{\mathbf{\Phi}}, p_r, p_{\hat{\mathbf{\Phi}}})\ p_r \quad . \tag{2.4}$$

The preservation of the primary constraint $V_1 \equiv p_r$ yields

$$\dot{p}_r \equiv \left\{p_r, \widetilde{H}\right\} \approx 2r^{-5} \sum_{i=1}^{5} p_{\varphi_i}^2 - 2br + 4r^3 V(r, \hat{\mathbf{\Phi}}) + r^4 \frac{\delta V}{\delta r} \tag{2.5}$$

where $\left\{\ ,\ \right\}$ denote the Poisson brackets. One therefore finds in our model the secondary constraint

$$V_2 \equiv 2r^{-5} \sum_{i=1}^{5} p_{\varphi_i}^2 - 2br + 4r^3 V(r, \hat{\mathbf{\Phi}}) + r^4 \frac{\delta V}{\delta r} \quad . \tag{2.6}$$

Further constraints are not found, since the preservation of V_2 leads to the condition

$$0 = \dot{V}_2 \equiv \left\{V_2, \widetilde{H}\right\} \approx \left\{V_2, H_c\right\} + \lambda \left\{V_2, V_1\right\} \tag{2.7}$$

which can be solved for λ as

$$\lambda = -\frac{\left\{V_2, H_c\right\}}{\left\{V_2, V_1\right\}} \quad . \tag{2.8}$$

This solution can be obtained since the constraints V_1 and V_2 are second-class, and the Poisson brackets appearing in the formula for λ are found to be

$$\Big\{V_2, H_c\Big\} = -\frac{4}{r}\sum_{i=1}^{5}\Big\{p_{\varphi_i}^2, V(r,\hat{\mathbf{\Phi}})\Big\} - \frac{1}{2}\sum_{i=1}^{5}\Big\{p_{\varphi_i}^2, \frac{\delta V}{\delta r}\Big\}$$

$$= \frac{8}{r}\sum_{i=1}^{5} p_{\varphi_i}\frac{\delta V}{\delta \varphi_i} + \sum_{i=1}^{5} p_{\varphi_i}\frac{\delta^2 V}{\delta\varphi_i \delta r} \tag{2.9}$$

$$\Big\{V_2, V_1\Big\} = -10r^{-6}\sum_{i=1}^{5} p_{\varphi_i}^2 - 2b + 12r^2 V(r,\hat{\mathbf{\Phi}}) + r^4\frac{\delta^2 V}{\delta r^2} + 8r^3\frac{\delta V}{\delta r} \quad . \tag{2.10}$$

Moreover, since V_1 and V_2 are second-class, they can be set strongly to zero using Dirac brackets (Dirac 1964, Esposito 1992), hereafter denoted by $\{\ ,\ \}^*$. The corresponding field equations are

$$\dot{r} \approx \Big\{r, \widetilde{H}\Big\}^* \approx \lambda \tag{2.11}$$

$$\dot{\varphi}_i \approx \Big\{\varphi_i, \widetilde{H}\Big\}^* \approx r^{-4} p_{\varphi_i}\left(1 - \frac{2}{r}\frac{V_2}{\{V_1, V_2\}}\right) \tag{2.12}$$

$$\dot{p}_r \approx \Big\{p_r, \widetilde{H}\Big\}^* \approx \Big\{p_r, H_c\Big\} - \Big\{p_r, V_l\Big\} C_{lm}^{-1}\Big\{V_m, \widetilde{H}\Big\} \approx 0 \tag{2.13}$$

$$\dot{p}_{\varphi_i} \approx \Big\{p_{\varphi_i}, \widetilde{H}\Big\}^* \approx \Big\{p_{\varphi_i}, H_c\Big\} - \Big\{p_{\varphi_i}, V_l\Big\} C_{lm}^{-1}\Big\{V_m, \widetilde{H}\Big\}$$

$$\approx r^4\frac{\delta V}{\delta\varphi_i} + \left(4r^3\frac{\delta V}{\delta\varphi_i} + r^4\frac{\delta^2 V}{\delta r\delta\varphi_i}\right)\frac{V_2}{\{V_1, V_2\}} \tag{2.14}$$

where C_{lm} is the matrix of Poisson brackets of second-class constraints. Note that, since $V_2 = 0$ when Dirac brackets are used, equations (2.12) and (2.14) can be written as

$$\dot{\varphi}_i \approx r^{-4} p_{\varphi_i} \tag{2.15}$$

$$\dot{p}_{\varphi_i} \approx r^4 \frac{\delta V}{\delta \varphi_i} \quad . \tag{2.16}$$

For the purpose of numerical integration, the most convenient form of these equations is

$$\frac{d}{d\tau}\left(r^4 \dot{\varphi}_i\right) \approx r^4 \frac{\delta V}{\delta \varphi_i} \tag{2.17}$$

$$\dot{r} \approx -\frac{r^3 \left(8 \sum_{i=1}^5 \dot{\varphi}_i \frac{\delta V}{\delta \varphi_i} + r \sum_{i=1}^5 \dot{\varphi}_i \frac{\delta^2 V}{\delta \varphi_i \delta r}\right)}{\left(-10 r^2 \sum_{i=1}^5 \dot{\varphi}_i^2 - 2b + 12 r^2 V + r^4 \frac{\delta^2 V}{\delta r^2} + 8 r^3 \frac{\delta V}{\delta r}\right)} \quad . \tag{2.18}$$

Such a system is here solved choosing the following initial conditions:

$$r(0) \equiv r_0 = \sqrt{3/8\pi G V_0} \tag{2.19}$$

$$\varphi_i(0) = \varphi_i^0 \tag{2.20}$$

$$p_r(0) = 0 \tag{2.21}$$

$$p_{\varphi_i}(0) = 0 \tag{2.22}$$

where in equation (2.19) the value chosen for r_0 leads to a suitable cosmological constant for the inflationary era, and in equation (2.22) we have neglected for simplicity initial-kinetic-energy effects (cf end of section 5). Of course, the φ_i^0 values should obey the constraint $V_2(\tau = 0) = 0$, i.e.

$$0 = -2br_0 + 4(r_0)^3 V(r_0, \hat{\mathbf{\Phi}}_0) + (r_0)^4 \left.\frac{\delta V}{\delta r}\right|_{r_0, \hat{\mathbf{\Phi}}_0} \quad . \tag{2.23}$$

From now on it is useful to use dimensionless units. For this purpose, we define $\sigma \equiv \sqrt{2G/3\pi} = \sqrt{2/(3\pi M_P^2)}$ and make the rescalings $r \to \sigma\, r$, $\tau \to \sigma\, \tau$, $\varphi_i \to \phi_i/(\pi\sigma\sqrt{2})$, and we also define $2\pi^2\sigma^4 V(\sigma r, \phi_i/(\pi\sigma\sqrt{2})) \equiv \widetilde{V}(r, \phi_i)$. The dimensionless field equations corresponding to equations (2.17)-(2.18) are then found to be

$$\frac{d}{d\tau}\left(r^4\dot{\phi}_i\right) \approx r^4 \frac{\delta\widetilde{V}}{\delta\phi_i} \tag{2.24}$$

$$\dot{r} \approx -\frac{r^3\left(8\sum_{i=1}^5 \dot{\phi}_i \frac{\delta\widetilde{V}}{\delta\phi_i} + r\sum_{i=1}^5 \dot{\phi}_i \frac{\delta^2\widetilde{V}}{\delta\phi_i\delta r}\right)}{\left(-10r^2\sum_{i=1}^5 \dot{\phi}_i^2 - 2 + 12r^2\widetilde{V} + r^4\frac{\delta^2\widetilde{V}}{\delta r^2} + 8r^3\frac{\delta\widetilde{V}}{\delta r}\right)} \quad . \tag{2.25}$$

Moreover, the initial conditions here chosen take the form

$$r(0) \equiv r_0 = \sqrt{\frac{1}{2\widetilde{V}_0}} \cong \frac{3M_P^2}{4M_X^2} \tag{2.26}$$

$$\phi_i(0) = \phi_i^0 \tag{2.27}$$

$$p_r(0) = 0 \tag{2.28}$$

$$p_{\varphi_i}(0) = 0 \tag{2.29}$$

where the ϕ_i^0 values obey, for a given r_0 value, the constraint

$$0 = -2r_0 + 4(r_0)^3\widetilde{V}(r_0, \phi_i^0) + (r_0)^4 \left.\frac{\delta\widetilde{V}}{\delta r}\right|_{r_0, \phi_i^0} \quad . \tag{2.30}$$

Note that the choice $V_0 = M_X^4$ fixes reasonably the critical temperature for the phase transitions to be of order $T_C \cong M_X$ (M_X, approximately equal to 10^{15} Gev, is the typical order of magnitude of the unification mass in the minimal $SU(5)$ model).

3. 1-loop effective potential

We here study $\hat{\phi}$ in the forms invariant under the subgroups $SU(3)\times SU(2)\times U(1)$ and $SU(4)\times U(1)$ respectively, i.e.

$$\hat{\phi}_{\mathbf{321}} = \frac{\|\hat{\phi}_{\mathbf{321}}\|^{1/2}}{\sqrt{30}}\mathrm{diag}(2,2,2,-3,-3) \tag{3.1}$$

$$\hat{\phi}_{\mathbf{41}} = \frac{\|\hat{\phi}_{\mathbf{41}}\|^{1/2}}{\sqrt{20}}\mathrm{diag}(1,1,1,1,-4) \tag{3.2}$$

since it has been shown in Allen 1985 and Buccella *et al* 1992 that these are the only subgroups relevant to the $SU(5)$ symmetry-breaking pattern. From now on, we denote by $\gamma(\tau)$ the norm of $\hat{\phi}_{\mathbf{321}}$ or of $\hat{\phi}_{\mathbf{41}}$, which is the only variable characterizing these broken-symmetry phases.

The form of the dimensionless effective potential in the 321 and 41 directions (i.e. when $\hat{\phi}=\hat{\phi}_{\mathbf{321}}$ or $\hat{\phi}=\hat{\phi}_{\mathbf{41}}$) is obtained inserting equations (3.1)-(3.2) into equation (1.1), which yields

$$\begin{aligned}\widetilde{V}(r,\hat{\phi}_{\mathbf{321}})|_{\|\hat{\phi}\|^{1/2}=\gamma} &\equiv \widetilde{V}_{321}(r,\gamma)\\ &= \frac{45\alpha}{4\pi r^2}\left\{Q+\frac{1}{3}\left[1-\log\left(\frac{2M_X^2}{3\pi M_P^2}r^2\right)\right]\right\}\gamma^2\\ &+\frac{25\alpha^2}{32\pi^2}\left[1-\log\left(\frac{2M_X^2}{3\pi M_P^2}r^2\right)\right]\gamma^4\\ &-\frac{9}{2r^4}\mathcal{A}\left[\frac{5\alpha}{6\pi}r^2\gamma^2\right]+\widetilde{V}_0\end{aligned} \tag{3.3}$$

$$\begin{aligned}\widetilde{V}(r,\hat{\phi}_{\mathbf{41}})|_{\|\hat{\phi}\|^{1/2}=\gamma} &\equiv \widetilde{V}_{41}(r,\gamma)\\ &= \frac{45\alpha}{4\pi r^2}\left\{Q+\frac{1}{3}\left[1-\log\left(\frac{2M_X^2}{3\pi M_P^2}r^2\right)\right]\right\}\gamma^2\\ &+\frac{75\alpha^2}{16\pi^2}\left\{\frac{1}{4}\left[1-\log\left(\frac{2M_X^2}{3\pi M_P^2}r^2\right)\right]+\frac{\widetilde{\Lambda}}{5}\right\}\gamma^4\\ &-\frac{3}{r^4}\mathcal{A}\left[\frac{5\alpha}{4\pi}r^2\gamma^2\right]+\widetilde{V}_0 \end{aligned} \tag{3.4}$$

where $\widetilde{V}_0 \equiv (8M_X^4)/(9M_P^4)$ and $\alpha \equiv g^2/4\pi$. It is also useful to derive the $r \to \infty$ limit of these potentials (i.e. their flat-space limit) as

$$\widetilde{V}(\hat{\phi}_{\mathbf{321}})|_{\|\hat{\phi}\|^{1/2}=\gamma}(r\to\infty) = \widetilde{V_0} + \frac{25\alpha^2}{32\pi^2}\gamma^4\left[\log\left(\frac{5\alpha M_P^2}{4M_X^2}\gamma^2\right)-\frac{1}{2}\right] \tag{3.5}$$

$$\widetilde{V}(\hat{\phi}_{\mathbf{41}})|_{\|\hat{\phi}\|^{1/2}=\gamma}(r\to\infty) = \widetilde{V_0} + \frac{15\alpha^2}{16\pi^2}\widetilde{\Lambda}\gamma^4 + \frac{75\alpha^2}{64\pi^2}\gamma^4\left[\log\left(\frac{15\alpha M_P^2}{8M_X^2}\gamma^2\right)-\frac{1}{2}\right] \quad . \tag{3.6}$$

The potentials (3.5) and (3.6) evaluated at their minima, denoted for simplicity by γ_m, turn out to be

$$\widetilde{V}(\hat{\phi}_{\mathbf{321}})|_{\|\hat{\phi}\|^{1/2}=\gamma_m}(r\to\infty) = \widetilde{V}_0 - \frac{1}{4\pi^2}\frac{M_X^4}{M_P^4} \tag{3.7}$$

$$\widetilde{V}(\hat{\phi}_{\mathbf{41}})|_{\|\hat{\phi}\|^{1/2}=\gamma_m}(r\to\infty) = \widetilde{V}_0 - \frac{1}{6\pi^2}\frac{M_X^4}{M_P^4}\exp\left[-\frac{8}{5}\widetilde{\Lambda}\right] \quad . \tag{3.8}$$

The experimental evidence for the $SU(3)\times SU(2)\times U(1)$ gauge symmetry at energy E greater than or of order 100 GeV requires for the scalar potential that the absolute minimum of $\widetilde{V}(\hat{\phi}_{\mathbf{321}})(r\to\infty)$ should remain below the absolute minimum of $\widetilde{V}(\hat{\phi}_{\mathbf{41}})(r \to$

∞) and below $\widetilde{V}_0$. One thus finds the condition on the bare parameters Λ_2 and Λ_4 (Allen 1985) of the scalar potential

$$\frac{3}{5}\Lambda_4 - \Lambda_2 > -\frac{75\alpha^2}{32}\log\left(\frac{3}{2}\right) \quad . \tag{3.9}$$

From now on, we restrict the choice of Λ_2 and Λ_4 to the region defined by the inequality (3.9). Under the assumptions described so far, the initial conditions (2.27) and (2.29) take the form

$$\gamma(0) = \gamma_0 \tag{3.10}$$

$$\dot{\gamma}(0) = 0 \quad . \tag{3.11}$$

The γ_0 value and the corresponding residual symmetry are obtained by solving separately the constraint equations (cf equation (2.23))

$$0 = -2r_0 + 4(r_0)^3\widetilde{V}_{321}(r_0,\gamma_0) + (r_0)^4 \left.\frac{\delta\widetilde{V}_{321}}{\delta r}\right|_{r_0,\gamma_0} \tag{3.12}$$

$$0 = -2r_0 + 4(r_0)^3\widetilde{V}_{41}(r_0,\gamma_0) + (r_0)^4 \left.\frac{\delta\widetilde{V}_{41}}{\delta r}\right|_{r_0,\gamma_0} \quad . \tag{3.13}$$

After doing this, one compares the $\widetilde{V}_{321}(r_0,\gamma_0)$ and $\widetilde{V}_{41}(r_0,\gamma_0)$ values, requiring that the correct initial condition should lead to the minimum value of the effective potential. Once the correct initial condition has been picked out in this way, the system (2.24)-(2.25) expressed in terms of γ becomes

$$\frac{d}{d\tau}\left(r^4\dot{\gamma}\right) \approx r^4\frac{\delta\widetilde{V}}{\delta\gamma} \tag{3.14}$$

$$\dot{r} \approx - \frac{r^3 \dot{\gamma} \left(8 \frac{\delta \widetilde{V}}{\delta \gamma} + r \frac{\delta^2 \widetilde{V}}{\delta \gamma \delta r} \right)}{\left(-10 r^2 \dot{\gamma}^2 - 2 + 12 r^2 \widetilde{V} + r^4 \frac{\delta^2 \widetilde{V}}{\delta r^2} + 8 r^3 \frac{\delta \widetilde{V}}{\delta r} \right)} \tag{3.15}$$

jointly with the constraint (cf equation (2.6))

$$V_2(\tau) \equiv 2r^3\dot{\gamma}^2 - 2r + 4r^3\widetilde{V}(r,\gamma) + r^4 \frac{\delta \widetilde{V}}{\delta r} \tag{3.16}$$

which should vanish $\forall \tau$.

4. Absolute minimum

By virtue of equations (A.7)-(A.19) of the appendix, the asymptotic form of the constraints (3.12)-(3.13) is

$$\frac{25\alpha^2}{32\pi^2}\gamma_0^4 \left[\log\left(\frac{5\alpha M_P^2}{4M_X^2}\gamma_0^2 \right) - \frac{1}{2} \right] + \mathrm{O}\left(r_0^{-2} \right) = 0 \tag{4.1}$$

$$\frac{25\alpha^2}{32\pi^2}\gamma_0^4 \left[\frac{3}{2} \log\left(\frac{15\alpha M_P^2}{8M_X^2}\gamma_0^2 \right) - \frac{3}{4} + \frac{6}{5}\widetilde{\Lambda} \right] + \mathrm{O}\left(r_0^{-2} \right) = 0 \quad . \tag{4.2}$$

The numerical solution of equation (4.1) yields $\gamma_0 \cong 1.42 \cdot 10^{-3}$, where the NAG-library routine C05ADF has been used in double-precision. Such a value of γ_0 is compatible with the asymptotic formulae appearing in the appendix. Interestingly, if the inequality (3.9) holds, one finds $\forall \tau$

$$\widetilde{V}_{41}(\tau) - \widetilde{V}_{321}(\tau) \sim \frac{25\alpha^2}{32\pi^2}\gamma^4 \left[\frac{3}{2}\log\left(\frac{3}{2}\right) - \frac{1}{4} + \frac{6}{5}\widetilde{\Lambda} \right] > \frac{25\alpha^2}{32\pi^2}\gamma^4 \left[\frac{3}{4}\log\left(\frac{3}{2}\right) - \frac{1}{4} \right] > 0 \quad . \tag{4.3}$$

This result ensures that at $\tau = 0$, if r_0 is taken to be of order 10^8, so that the asymptotic formulae of the appendix can be applied, the only possible residual symmetry is $SU(3) \times SU(2) \times U(1)$. Remarkably, because equation (4.1) is, at least up to first order in the r_0-expansion, independent of the bare parameters $(\widetilde{\Lambda}, Q)$ of the scalar potential and only dependent on M_X, the value found for γ_0 is basically model-independent. If the alternative symmetry $SU(4) \times U(1)$ were chosen, the inequality (4.3) would lead to a tunneling of the Higgs field towards the energetically more favourable phase. Moreover, our result (4.3) ensures that, once the initial $SU(3) \times SU(2) \times U(1)$ symmetry is chosen, the Higgs field remains in this broken-symmetry phase during the whole de Sitter phase of the early universe.

5. Numerical analysis

The numerical integration of the system (3.14)-(3.15) can be performed after reduction to first-order form, and using the second-class constraint (3.16) and equations (2.15)-(2.16). Thus, defining

$$\dot{\gamma}(\tau) \equiv \eta(\tau) \tag{5.1}$$

this leads to

$$\dot{\eta}(\tau) \approx \frac{\delta \widetilde{V}}{\delta \gamma} - \frac{4\eta^2 r^2 \left(8 \frac{\delta \widetilde{V}}{\delta \gamma} + r \frac{\delta^2 \widetilde{V}}{\delta \gamma \delta r}\right)}{\left(12 - 32 r^2 \widetilde{V} - 13 r^3 \frac{\delta \widetilde{V}}{\delta r} - r^4 \frac{\delta^2 \widetilde{V}}{\delta r^2}\right)} \tag{5.2}$$

$$\dot{r}(\tau) \approx \frac{r^3 \eta \left(8 \frac{\delta \widetilde{V}}{\delta \gamma} + r \frac{\delta^2 \widetilde{V}}{\delta \gamma \delta r} \right)}{\left(12 - 32 r^2 \widetilde{V} - 13 r^3 \frac{\delta \widetilde{V}}{\delta r} - r^4 \frac{\delta^2 \widetilde{V}}{\delta r^2} \right)} \quad . \tag{5.3}$$

By taking $r_0 \cong 10^8$, which is a typical value for GUT models in a de Sitter universe, the system (5.1)-(5.3) has been solved within the framework of the Runge-Kutta-Merson method (NAG-library routine D02BAF). The results of our numerical analysis are shown in figures $1-3$, which present the Euclidean-time evolution of γ, $\dot{\gamma}$ and r respectively. These plots clearly show the existence of an *almost exact* de Sitter phase (i.e. when the four-sphere radius remains approximately constant) with typical time τ, in dimensionless units, of order $3 \cdot 10^4$, whereas the duration of the exponentially-expanding phase may be taken to be of order $7 \cdot 10^4$. Interestingly, this result is practically independent of the bare parameters in the $SU(5)$ potential we have chosen if the inequality (3.9) is satisfied. Further restrictions are then given by particle physics, i.e. proton-lifetime experiments and renormalization-group equations for the coupling constants and $(\sin\theta_W)^2$ (Buccella *et al* 1989, Becker-Szendy *et al* 1990, Amaldi *et al* 1992).

If $\tau \geq 7 \cdot 10^4$, figure 3 shows a rapid variation of the four-sphere radius, so that the early universe is no longer well described by a de Sitter or exponentially-expanding model. Note also that the numerical analysis here presented rules out the occurrence of tunneling effects between broken-symmetry phases. Moreover, it should be emphasized that, setting to zero the initial kinetic energy (cf equation (3.11)) one obtains the most favourable initial conditions for a long de Sitter phase of the early universe, whereas values of $\dot{\gamma}(0) \neq 0$ may be shown to lead to a much more rapid variation of the four-sphere radius r.

6. Concluding remarks

This paper has studied the $SU(5)$ symmetry-breaking pattern in a de Sitter universe from a Hamiltonian point of view. As a result of this analysis, one finds that the model is characterized by two second-class constraints. Interestingly, the secondary second-class constraint has been used to prove that the early universe can only reach the $SU(3) \times SU(2) \times U(1)$ broken-symmetry phase if we require the correct low-energy limit for the GUT theory. Furthermore, under this obvious requirement, the $SU(3) \times SU(2) \times U(1)$ invariant direction turns out to be energetically more favourable than $SU(4) \times U(1)$. Thus, during the whole inflationary era, possible tunneling effects between the two broken-symmetry phases are forbidden.

This conclusion supersedes earlier work on the $SU(5)$ symmetry-breaking pattern in de Sitter cosmologies appearing in Allen 1985 and Buccella *et al* 1992. It also provides a relevant example of cosmological restrictions on GUT models (cf Collins and Langbein 1992).

Moreover, the resulting field equations have been solved numerically. Standard methods for the numerical integration of first-order systems of ordinary differential equations show that the early universe starts out in a de Sitter state. The exact de Sitter state is then replaced by a more general exponentially-expanding universe corresponding to the slow-rolling-over phase, as one would expect. In dimensionless units, the total duration of these two phases has been found to be of order $7 \cdot 10^4$.

From a field-theoretical point of view, the 1-loop effective potential of equation (1.1), originally derived in Allen 1985, and also used in Buccella *et al* 1992 and in this paper, might be subject to criticisms, since topological methods have been used to prove the general non-existence of a gauge choice for a theory with gauge group $SU(N)$, $N \geq 3$, over the manifolds S^3 and S^4 (Jungman 1992 and references therein). However, this difficulty holds for a classical non-Abelian gauge-field theory. By contrast, we have studied the quantization of a Yang-Mills-Higgs theory where gauge-averaging terms are included in the Wick-rotated path integral using the Faddeev-Popov technique and a specific choice first proposed by 't Hooft (Allen 1985, Buccella *et al* 1992). Although the choice of a gauge-averaging term is clearly suggested by what would be set to zero at the classical level, there is by now some evidence that the resulting quantum theory is not equivalent to the model where only physical degrees of freedom are quantized after fixing the gauge (Esposito 1992 and references therein). It therefore appears that the approach used in Allen 1985 to evaluate the 1-loop effective potential for $SU(5)$ theory in de Sitter cosmologies remains a useful tool to improve our understanding of the symmetry-breaking pattern of non-Abelian gauge theories in curved backgrounds.

Appendix

The function $\mathcal{A}(z)$ appearing in equation (1.1), and its derivatives, are given by (Allen 1985)

$$\mathcal{A}(z) \equiv \frac{z^2}{4} + \frac{z}{3} - \int_2^{\frac{3}{2}+\sqrt{\frac{1}{4}-z}} y\Big(y - \frac{3}{2}\Big)\Big(y - 3\Big)\psi(y)\; dy$$

$$- \int_{1}^{\frac{3}{2}-\sqrt{\frac{1}{4}-z}} y\Big(y-\frac{3}{2}\Big)\Big(y-3\Big)\psi(y)\ dy \tag{A.1}$$

$$\mathcal{A}'(z) = -\frac{1}{2}(z+2)\left[\psi\left(\frac{3}{2}+\sqrt{\frac{1}{4}-z}\right)+\psi\left(\frac{3}{2}-\sqrt{\frac{1}{4}-z}\right)\right]+\frac{z}{2}+\frac{1}{3} \tag{A.2}$$

$$\mathcal{A}''(z) = \frac{1}{2}-\frac{1}{2}\left[\psi\left(\frac{3}{2}+\sqrt{\frac{1}{4}-z}\right)+\psi\left(\frac{3}{2}-\sqrt{\frac{1}{4}-z}\right)\right]$$
$$+\frac{1}{4}\frac{(z+2)}{\sqrt{\left(\frac{1}{4}-z\right)}}\left[\psi'\left(\frac{3}{2}+\sqrt{\frac{1}{4}-z}\right)-\psi'\left(\frac{3}{2}-\sqrt{\frac{1}{4}-z}\right)\right] \quad . \tag{A.3}$$

At large z, the following asymptotic expansions hold (Allen 1985):

$$\mathcal{A}(z) \sim -\left(\frac{z^2}{4}+z+\frac{19}{30}\right)\log(z)+\frac{3}{8}z^2+z+\text{const.}+\mathrm{O}\left(z^{-1}\right) \tag{A.4}$$

$$\mathcal{A}'(z) \sim -\frac{1}{2}(z+2)\log(z)+\frac{z}{2}+\mathrm{O}\left(z^{-1}\right) \tag{A.5}$$

$$\mathcal{A}''(z) \sim -\frac{1}{2}\log(z)+\mathrm{O}\left(z^{-1}\right) \quad . \tag{A.6}$$

In section 4, we rely on the following exact formulae for the derivatives of the dimensionless effective potential in the $SU(3)\times SU(2)\times U(1)$ and $SU(4)\times U(1)$ broken-symmetry phases:

$$\frac{\delta\widetilde{V}_{321}}{\delta r}(r,\gamma) = -\frac{45\alpha}{2\pi r^3}\left\{Q+\frac{1}{3}\left[2-\log\left(\frac{2M_X^2}{3\pi M_P^2}r^2\right)\right]\right\}\gamma^2-\frac{25\alpha^2}{16\pi^2 r}\gamma^4$$
$$+\frac{18}{r^5}\mathcal{A}\left[\frac{5\alpha}{6\pi}r^2\gamma^2\right]-\frac{15\alpha}{2\pi r^3}\gamma^2\mathcal{A}'\left[\frac{5\alpha}{6\pi}r^2\gamma^2\right] \tag{A.7}$$

$$\frac{\delta^2 \widetilde{V}_{321}}{\delta r^2}(r,\gamma) = \frac{135\alpha}{2\pi r^4}\left\{Q + \frac{1}{3}\left[\frac{8}{3} - \log\!\left(\frac{2M_X^2}{3\pi M_P^2}r^2\right)\right]\right\}\gamma^2 + \frac{25\alpha^2}{16\pi^2 r^2}\gamma^4$$
$$- \frac{90}{r^6}\mathcal{A}\left[\frac{5\alpha}{6\pi}r^2\gamma^2\right] + \frac{105\alpha}{2\pi r^4}\gamma^2\mathcal{A}'\left[\frac{5\alpha}{6\pi}r^2\gamma^2\right]$$
$$- \frac{25\alpha^2}{2\pi^2 r^2}\gamma^4\mathcal{A}''\left[\frac{5\alpha}{6\pi}r^2\gamma^2\right] \qquad (A.8)$$

$$\frac{\delta^2 \widetilde{V}_{321}}{\delta\gamma\delta r}(r,\gamma) = -\frac{15\alpha}{\pi r^3}\left\{3Q + 2 - \log\!\left(\frac{2M_X^2}{3\pi M_P^2}r^2\right)\right\}\gamma - \frac{25\alpha^2}{4\pi^2 r}\gamma^3$$
$$+ \frac{15\alpha}{\pi r^3}\gamma\mathcal{A}'\left[\frac{5\alpha}{6\pi}r^2\gamma^2\right] - \frac{25\alpha^2}{2\pi^2 r}\gamma^3\mathcal{A}''\left[\frac{5\alpha}{6\pi}r^2\gamma^2\right] \qquad (A.9)$$

$$\frac{\delta \widetilde{V}_{321}}{\delta\gamma}(r,\gamma) = \frac{45\alpha}{2\pi r^2}\left\{Q + \frac{1}{3}\left[1 - \log\!\left(\frac{2M_X^2}{3\pi M_P^2}r^2\right)\right]\right\}\gamma$$
$$+ \frac{25\alpha^2}{8\pi^2}\left[1 - \log\!\left(\frac{2M_X^2}{3\pi M_P^2}r^2\right)\right]\gamma^3 - \frac{15\alpha}{2\pi r^2}\gamma\mathcal{A}'\left[\frac{5\alpha}{6\pi}r^2\gamma^2\right] \qquad (A.10)$$

$$\frac{\delta \widetilde{V}_{41}}{\delta r}(r,\gamma) = -\frac{45\alpha}{2\pi r^3}\left\{Q + \frac{1}{3}\left[2 - \log\!\left(\frac{2M_X^2}{3\pi M_P^2}r^2\right)\right]\right\}\gamma^2 - \frac{75\alpha^2}{32\pi^2 r}\gamma^4$$
$$+ \frac{12}{r^5}\mathcal{A}\left[\frac{5\alpha}{4\pi}r^2\gamma^2\right] - \frac{15\alpha}{2\pi r^3}\gamma^2\mathcal{A}'\left[\frac{5\alpha}{4\pi}r^2\gamma^2\right] \qquad (A.11)$$

$$\frac{\delta \widetilde{V}_{41}}{\delta\gamma}(r,\gamma) = \frac{45\alpha}{2\pi r^2}\left\{Q + \frac{1}{3}\left[1 - \log\!\left(\frac{2M_X^2}{3\pi M_P^2}r^2\right)\right]\right\}\gamma$$
$$+ \frac{75\alpha^2}{4\pi^2}\left\{\frac{1}{4}\left[1 - \log\!\left(\frac{2M_X^2}{3\pi M_P^2}r^2\right)\right] + \frac{\widetilde{\Lambda}}{5}\right\}\gamma^3$$
$$- \frac{15\alpha}{2\pi r^2}\gamma\mathcal{A}'\left[\frac{5\alpha}{4\pi}r^2\gamma^2\right] \quad . \qquad (A.12)$$

In light of equations (A.4)-(A.6), the asymptotic expansions of the dimensionless effective potential and of equations (A.7)-(A.12) are given by

$$\widetilde{V}_{321}(r,\gamma) \sim \widetilde{V}_0 + \frac{25\alpha^2}{32\pi^2}\gamma^4\left[\log\left(\frac{5\alpha M_P^2}{4M_X^2}\gamma^2\right) - \frac{1}{2}\right] + \frac{15\alpha}{4\pi r^2}\left[3Q + \log\left(\frac{5\alpha M_P^2}{4M_X^2}\gamma^2\right)\right]\gamma^2 + \mathrm{O}(r^{-4}) \qquad (A.13)$$

$$\frac{\delta\widetilde{V}_{321}}{\delta r}(r,\gamma) \sim -\frac{15\alpha}{2\pi r^3}\left[3Q + \log\left(\frac{5\alpha M_P^2}{4M_X^2}\gamma^2\right)\right]\gamma^2 + \mathrm{O}(r^{-5}) \qquad (A.14)$$

$$\frac{\delta^2\widetilde{V}_{321}}{\delta r^2}(r,\gamma) \sim \frac{135\alpha}{2\pi r^4}\left\{Q + \frac{1}{3}\left[-\frac{2}{3} + \log\left(\frac{5\alpha M_P^2}{4M_X^2}\gamma^2\right)\right]\right\}\gamma^2 + \mathrm{O}(r^{-6}) \qquad (A.15)$$

$$\frac{\delta^2\widetilde{V}_{321}}{\delta\gamma\delta r}(r,\gamma) \sim -\frac{15\alpha}{\pi r^3}\left[3Q + 2 + \log\left(\frac{5\alpha M_P^2}{4M_X^2}\gamma^2\right)\right]\gamma + \mathrm{O}(r^{-5}) \qquad (A.16)$$

$$\frac{\delta\widetilde{V}_{321}}{\delta\gamma}(r,\gamma) \sim \frac{25\alpha^2}{8\pi^2}\gamma^3\log\left(\frac{5\alpha M_P^2}{4M_X^2}\gamma^2\right) + \frac{15\alpha}{2\pi r^2}\left[3Q + 1 + \log\left(\frac{5\alpha M_P^2}{4M_X^2}\gamma^2\right)\right]\gamma + \mathrm{O}(r^{-4}) \qquad (A.17)$$

$$\frac{\delta\widetilde{V}_{41}}{\delta r}(r,\gamma) \sim -\frac{15\alpha}{2\pi r^3}\left[3Q + \log\left(\frac{15\alpha M_P^2}{8M_X^2}\gamma^2\right)\right]\gamma^2 + \mathrm{O}(r^{-5}) \qquad (A.18)$$

$$\frac{\delta\widetilde{V}_{41}}{\delta\gamma}(r,\gamma) \sim \frac{15\alpha^2}{4\pi^2}\gamma^3\left\{\frac{5}{4}\log\left(\frac{15\alpha M_P^2}{8M_X^2}\gamma^2\right) + \widetilde{\Lambda}\right\} + \mathrm{O}(r^{-2}) \quad . \qquad (A.19)$$

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Figure captions

Figure 1. The Euclidean-time evolution of the norm γ of the Higgs field is here plotted, after solving numerically the system (5.1)-(5.3).

Figure 2. The evolution of $\dot{\gamma}$ is here shown.

Figure 3. The numerical solution for the Euclidean-time evolution of the four-sphere radius r is here presented. If $\tau \in \left[0, 3 \cdot 10^4\right]$, our solution of the system (5.1)-(5.3) shows that r is approximately constant, as one would expect during the de Sitter phase of the early universe.