Quantum Potential Approach to Quantum Cosmology

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Abstract

In this paper we discuss the quantum potential approach of Bohm in the context of quantum cosmological model. This approach makes it possible to convert the wavefunction of the universe to a set of equations describing the time evolution of the universe. Following Ashtekar et. al., we make use of quantum canonical transformation to cast a class of quantum cosmological models to a simple form in which they can be solved explicitly, and then we use the solutions do recover the time evolution.

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1 Introduction

The problem of the origin of the universe is perhaps as old as the mankind, but only in the recent years our understanding of the nature had become sufficiently deep to address it in the framework of physics. The theories we have in hands nowadays make it possible to trace the evolution of the universe starting from tiny fraction of a second after the mysterious Big Bang. This is, without doubts, one of the greatest achievements of the modern science. However, during these investigations it became more and more clear that our present knowledge is not sufficiently developed to answer questions concerning the Big Bang itself and what happened before. Actually this last question may happen to be not well posed, as there are indications that "before Big Bang" time did not exist at all.

This circle of problems is the subject of a newly developed branch of physics which incorporates recent developments in high energy physics, quantum field theory, and quantum theory of gravity, and is called quantum cosmology. Even if in the recent years there is growing activity in this field, there is a number of problems which not only are unsolved, but concerning which there is even no consensus, as to which of many possible questions are relevant. Let us list some of these problems.

- It is natural to attack the problems of quantum cosmology by making use of some hybrid theory constructed from quantum field theory and the theory of gravity. Such a theory is certainly not known, but whatever final shape it would take, it will provide us with the class of "wavefunctions of the universe" as solutions. The problem is how should we interpret such wavefunctions. In quantum mechanics, a wavefunction provides us with the probability of finding a system in some configuration. But this is not really what we expect of quantum cosmology. The problem of major interest is how universe evolved and what is the class of initial conditions leading to the universe we observe. One can also pose a question as to whether there are any alternatives, i.e., are there any alternative consistent universes.

- It is well known that the fundamental symmetry of general relativity is the general coordinate invariance. This leads, through the Dirac procedure, to the quantum theory which has reparametrization invariance built-in by default. This means, in particular, that the theory is not sensitive to any time reperametrizations, and therefore happens to be completely time-independent. Then how we can talk about 'evolution'? This is the problem of time, which is probably the most important conceptual problem of quantum gravity [1].

- Equally well known is the fact that since the equations governing quantum mechanical evolution are linear, any linear combination of their solutions is a solution itself. But how could we interpret such a solution consisting of, say, a sum of two different wave functions. Should we interpret this solution as describing two universes, or maybe hope that for some mysterious reasons, only one of them is relevant for our observations of the universe? This is the problem of decoherence (for review, see [1], section 5.5 and [2] and references therein.)

In the present paper we describe the quantum potential approach to quantum theories proposed by David Bohm [3] and developed in the context of quantum gravity in [4], and for cosmology in [5] and [6]. This approach makes it possible not only to identify trajectories associated with 'wave function of the universe', but also to circumvent both the time and decoherence problems mentioned above. By making use of the important technical result of Ashtekar, Tate, and Uggla [7], [8], who were able to transform the Hamiltonian constraint of some interesting cosmological models to the form of free two or three dimensional wave equation, we are able to find a large class of exact wave functions for these models and to identify the corresponding time evolutions.

Our investigations are undertaken within the framework of minisuperspace approximation to quantum gravity. This means that out of the infinite number of degrees of freedom of the full theory, we choose to investigate the dynamics of only finite number of modes. This approach is frequently criticized because there is no apparent reason why such a handicapped theory is to describe the evolution of real universe. However, one may argue that such a reduced model can result from taking into account the solutions of the full quantum theory with high degree of symmetry, exactly in the same way as one derives the Friedmann–Robertson–Walker cosmological model from classical general relativity. In fact, there is no apparent reason whatsoever for the FRW class of solutions of Einstein equations to describe the physical universe, but surprisingly enough it does with great accuracy. There is, no doubt, a great mystery in the fact that our (classical) universe is homogeneous and isotropic, but there is no clear reason why this high degree of symmetry should not survive quantization. If the quantum universe also possesses a high degree of symmetry, the minisuperspace approach may be adequate after all.

The plan of the paper is as follows. In section 2 we discuss the general features of the quantum potential approach. In section 3 we introduce the class of metrics which will be of our interest and derive the form of classical and quantum Hamiltonian constraints. In the next section, following [8], we analyze the canonical transformations which drastically simplify the form of Hamiltonian constraint, in fact in all considered cases this constraint reduces to the two- or three-dimensional wave equation.

Having solutions of equations is, as it is well known, not sufficient. One must decide which of the solutions are 'physical'. The simplest criterion is to demand that the resulting wave function is normalizable. Therefore, in section 5 we address the question of what the inner product is. Section 6 is devoted to description of a simple class of solutions of resulting equations and to the derivation of the time evolutions corresponding to the so obtained 'wave functions of the universe.'

2 The quantum potential

In this section we will explain how the quantum potential approach works in the context of quantum cosmology. The general features of this approach (in particular, applied to the standard quantum mechanics and quantum field theory) are discussed in details in [3] and we recommend the reader to check these papers for general background and discussion of conceptual issues.

For the cases which will be the object of our investigations in the sections to come, it is sufficient to consider a model for which the whole of quantum dynamics resides in the single equation[1]

$$\mathsf{H}\,\psi(x^i) = \left(\frac{1}{2} g^{ij}\nabla_i\nabla_j - V(x^i)\right)\,\psi(x^i) = \left(\frac{1}{2}\Box - V(x^i)\right)\,\psi(x^i) = 0, \qquad (1)$$

[1]In what follows all quantum mechanical operators will be written in sans serif type face: a, b, ..., A, B etc.

where g^{ij} may be x-dependent. From now on we will call ψ the wave function of the universe. Let us assume that ψ has the following polar decomposition

$$\psi = R(x^i)\exp\left(\frac{i}{\hbar}S(x^i)\right) \tag{2}$$

with both R and S real. Inserting (2) into (1), we obtain two equations corresponding to real and imaginary part, respectively. These equations read

$$\mathcal{H}[S(x)] = \frac{1}{2}g_{ij}\frac{\partial S}{\partial x^i}\frac{\partial S}{\partial x^j} + V(x^i) = \frac{\hbar^2}{2}\frac{1}{R}\Box R, \tag{3}$$

$$R\Box S + 2g^{ij}\frac{\partial S}{\partial x^i}\frac{\partial R}{\partial x^j} = 0. \tag{4}$$

Equation (4) will not concern us anymore. In this paper we assume that the wave function ψ is a solution of equation (1), and thus this equation is identically satisfied (even though we may not know what the explicit form of the wavefunction is.) On the other hand, equation (3) is of crucial importance. This equation can be used to derive the time dependence and then serves as the evolutionary equation in the formalism.

For, let us introduce time t through the following equation

$$\frac{dx^i}{dt} = g^{ij}\frac{\delta\mathcal{H}[S(x)]}{\delta(\partial S/\partial x^j)}. \tag{5}$$

This equation defines the trajectory $x^i(t)$ in terms of the phase of the wavefunction S. Now we can substitute back equation (5) to (3). Assuming that the matrix g^{ij} has the inverse g_{ij}, we find ($\dot{x}^i = \frac{dx^i}{dt}$)

$$\frac{1}{2}g_{ij}\dot{x}^i\dot{x}^j + V(x^i) = \frac{\hbar^2}{2}\frac{1}{R}\Box R. \tag{6}$$

We see therefore that the quantum evolution differs from the classical one only by the presence of the quantum potential term

$$-V_{quant}(x^i) = \frac{\hbar^2}{2}\frac{1}{R}\Box R$$

on the right hand side of equation of motion. Since we assume that the wave function is known, the quantum potential term is known as well.

Equation (6) is not in the form which is convenient for our further investigations. To obtain the desired form we define classical momenta

$$p_i = \frac{\delta \mathcal{H}[S(x)]}{\delta(\partial S/\partial x^i)} = g_{ij}\dot{x}^j$$

and cast equation (6) to the form

$$\mathcal{H} \equiv \frac{1}{2} g^{ij} p_i p_j + V(x^i) - \frac{\hbar^2}{2}\frac{1}{R}\Box R = 0. \tag{7}$$

We regard $\mathcal{H}$ as the generator of dynamics acting through the Hamilton equations

$$\begin{aligned} \dot{p}_i &= -\frac{\partial \mathcal{H}}{\partial x^i} \\ \dot{x}^i &= \frac{\partial \mathcal{H}}{\partial p_i}. \end{aligned} \tag{8}$$

Thus we developed the scheme in which we can identify the time evolution corresponding to the wavefunction of the universe. This time evolution is governed by equations (8) subject to the constraint for initial conditions (7). This completes the technical part of quantum potential program. However a number of remarks is in order.

1. Equation (8) with the hamiltonian $\mathcal{H}$ defined by (7) is identical with the classical equations of motion with the only difference that in the hamiltonian the classical potential is appended by the quantum term V_{quant} which is a quantity of order $\hbar^2$. Observe that in the course of deriving the quantum potential equations of motion, we did not make any approximations and therefore the resulting theory is completely equivalent to the original quantum system. What makes this approach different from the standard quantum mechanics, is that the wave function whose interpretation is rather obscured, especially in the case of quantum gravity and quantum cosmology, is now replaced by trajectories with well understood physical meaning.

2. The quantum potential interpretation may be used to obtain a well defined semi-classical approximation to quantum theory. Indeed, it

can be said that the system enters the (semi-) classical regime if the quantum potential is much smaller than the regular potential term. This observation has been extensively used in the paper [5], where we were interested in the various mixed regimes (quantum matter and classical gravity etc.)

3. One of the major adventages of the quantum potential approach is that it provides one with an affective and simple way of introducing time even if the system under consideration has a hamiltonian as one of the constraints. In particular this approach serves as a possible route to final understanding of the problem of time in quantum gravity. We do not intend to suggest that this is the only possible approach to this problem, however, to our taste, this is the best one available now.

4. Related to this is the problem of interpretation of wave functions which are real. This problem has been a subject of numerous investigations, but from the point of view of quantum potential the resolution of it is quite simple. We just say that real wavefunctions (of the universe) represent a model without time evolution (and therefore time) at all. Recall that the standard (time-dependent) Schrödinger equation does not have any real solutions. It is clear from the formalism: $\dot{x}$ is just equal to zero, so nothing evolves and therefore there is no clock to measure time. On the other hand, equation (6) means that for the real wave function the system settles down to the configuration for which the total potential (i.e., classical plus quantum) is equal to zero.

5. There is one particular example of equation (1) which will be important for us later on. Consider the situation when the classical potential is absent. Since the kinetic term $\Box$ is a hyperbolic differential operator, the wave function is a sum of function corresponding to 'retarded' and 'advanced' waves, to wit

$$\psi = \psi_1(u) + \psi(v).$$

Now the quantum potential is defined in terms of the same operator $\Box$ acting on the modulus of the *total* wave function. We see therefore that if we want the quantum potential not to vanish, we must keep both components of the wavefunction decomposition above.

6. The definition of time by equation (5) is, of course, not unique. In fact, we can can use a more general expression

$$\dot{x}^i = N(x) \frac{\delta \mathcal{H}[S(x)]}{\delta(\partial S/\partial x^i)}, \tag{9}$$

where, in the case of gravity, the function N is to be identified with the lapse function of ADM formalism.

This completes our formal investigations of the quantum potential approach. Let us now turn to cosmology.

3 Classical cosmological models

This section concerns a class of homogeneous cosmological models, called the diagonal, intristically multiply transitive models, whose quantum evolution we will investigate in the following sections. Our presentation essentially follows the paper [8] and we recommend the reader to consult this paper for more details and references.

A spacially homogeneous 4-metric can be expressed in the form

$$ds^2 = -N^2(t)\, dt^2 + \sum_{i=1}^{3} g_{ii}(\omega^i)^2, \tag{10}$$

where $N(t)$ is the lapse function and ω^i are some appropriate 1-forms on the 3-space. To make this metric compatible with the non-evolutionary Einstein equations, ω^i must satisfy certain conditions whose explicit form will not concern us. Particular forms of ω^i are classified and called Bianchi types.

Since the metric coefficients g_{ii} are positive, they can be represented in the form

$$g_{ii} = e^{2\beta^i}$$

and it turns out that the following parametrization, due to Misner, is very convenient

$$\begin{pmatrix} \beta^1 \\ \beta^2 \\ \beta^3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & \sqrt{3} \\ 1 & 1 & -\sqrt{3} \\ 1 & -2 & 0 \end{pmatrix} \begin{pmatrix} \beta^0 \\ \beta^+ \\ \beta^- \end{pmatrix},$$

$$(x^0, x^+, x^-) = \frac{1}{\sqrt{3}}(2\beta^0 - \beta^+, -\beta^0 + 2\beta^+, \sqrt{3}\beta^-).$$

Defining the momenta associated with the variables x^i

$$\{x^i, p_j\} = \delta^i_j$$

and turning to the so called Taub time gauge which fixes the lapse function N to be

$$N_T = 12\exp(3\beta^0)$$

we finally arrive at the following simple form of the single remaining Einstein equation (in phase space)

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_+ + \mathcal{H}_- = 0, \tag{11}$$

$$\mathcal{H}_0 = -\frac{1}{2}p_0^2 + k_0 e^{2\sqrt{3}x^0}, \tag{12}$$

$$\mathcal{H}_+ = \frac{1}{2}p_+^2 + k_+ e^{-4\sqrt{3}x^+}, \tag{13}$$

$$\mathcal{H}_- = \frac{1}{2}k_- p_0^2, \tag{14}$$

where the coefficients $k_0, k_\pm$ for various models Bianchi are given in the table above.

Before going any further let us make an important remark. As it is well known the Einstein lagrangian for cosmological models has the generic form

$$\frac{1}{M_P^2}\int dt(p\dot{x} - N(t)\mathcal{H}).$$

Setting $M_P = 1$, as it is usually done, means that all quantities are scaled by the Planck mass, length etc. Thus if in the numerical computations some quantity is equal 1, this quantity is 1 times an appropriate power of M_P and this means that 1 is just the border between the "classical" and the "quantum" world.

There is one more model, the so called Bianchi type V model for which the scheme is applicable as well; for this model, the x and β variables are identical and the coefficients are given by $k_0 = 72$, $k_+ = 0$, $k_- = 1$.

All the models listed above have the important property of being separable which makes it possible to find a canonical transformation leading to the potential-free theory.

	I	II	VIII	IX	KS	III
k_0	0	0	24	-24	-24	24
k_+	0	6	6	6	0	0
k_-	1	1	0	0	0	0

Table 1: Coefficients of hamiltonian constraint for various Bianchi models

4 Classical and quantum canonical transformations

It is well known that canonical transformations are a very convenient tool of classical mechanics. If applicable, they are used to simplify complicated mechanical systems to the ones which are easy to deal with. It turns out however that in the case of quantum mechanics our understanding of canonical transformations is surprisingly poor. To our knowledge the only strong result is due to Berezin [9] and covers only linear canonical transformations. Therefore, in a non-linear case we cannot refer to any theory, but rather we must just check if a classical canonical transformation can be elevated to the quantum regime. The problem is that we must make sure that all the peculiarities of quantum theory (like, for example, operator ordering problem) do not spoil the classical transformation. This is exactly what we will do in this section in the context of models described in section 3.

The generic piece of the hamiltonian constraint can be written as

$$\epsilon\frac{1}{2}(p^2 + a\exp bx),$$

where $\epsilon = \pm 1$, and a, b are coefficients which can be read off from the table in the previous section. The obvious goal will be therefore to replace this term, after canonical trasformation, by a square of pure momentum. Thus the canonical transformation must read

$$P = \sqrt{p^2 + a\exp 2bx}. \tag{15}$$

To define the new variable X we must solve the equation $\{Q, P\} = 1$, Now we must distinguish two cases depending on the sign of the coefficicent a above.

$$X = \frac{1}{b}\left(\log[-p + \sqrt{p^2 + ae^{2bx}}] - \log[\sqrt{a}e^{bx}]\right), \quad a > 0, \tag{16}$$

$$X = \frac{1}{b}\left(\log[p - \sqrt{p^2 + ae^{2bx}}] - \log[\sqrt{-a}e^{bx}]\right), \quad a < 0. \tag{17}$$

The inverse canonical transformation reads

$$\sqrt{a}e^{bx} = \frac{P}{\cosh bX}, \quad p = -P \tanh bX, \quad a > 0 \tag{18}$$

$$\sqrt{-a}e^{bx} = \frac{P}{\sinh bX}, \quad p = P \coth bX, \quad a < 0. \tag{19}$$

Let us observe that the momentum P defined by (15) is positive. This does not pose any problems in classical theory, but we will have to struggle a little with this condition in the quantum case below. In the paper [8] one can find a detailed discussion of topology of the phase space of the models after canonical transformation. It should be stressed that since the canonical transformations lead to a free theory, this topology encodes all the nontrivial information concerning the dynamics of the model.

Let us now turn to the quantum picture. As we mentioned above there is no complete theory of quantum canonical transformations, and all particular (nonlinear) cases must be discussed separately. In what follows, we will discuss the case of positive a, the negative a case can be discussed similarly.

We look for the linear transformation which maps wave functions $\psi(x)$ into wave functions $\Psi(X)$. In fact the canonical transformation in our case is nothing but the transformation to the basis in which the operators $\mathsf{p}_i^2 + a_i \exp b_i \mathsf{x}_i$ are diagonal. It follows from the basic principles of quantum mechanics that such a (unitary and linear) transformation exists. This transformation must have the property that solutions of constraint equations before and after the canonical transformation are mapped into each other. Moreover we would like this transformation to be unitary; this, however requires the knowledge of the inner products. Since the inner product in the free case is easy to construct we use the transformation to *define* the inner product in the space of original wave functions.

Thus we start with writing

$$\psi(x) = \int_{-\infty}^{+\infty} dX \ < x|X > \ \Psi(X).$$

To find the kernel $< x|X >$ satisfying the requirement above we investigate the action of the quantum operators corresponding to (18) on both sides of the above equation. To this end, we must choose some particular operator ordering, and it turns out that only the ordering "PQ" leads to the desired result.

We therefore assume that

$$\mathsf{p}\psi(x) = -\int_{-\infty}^{+\infty} dX \ < x|X > \ \mathsf{P} \tanh b\mathsf{X}\Psi(X), \tag{20}$$

$$\sqrt{a}\exp(b\,\mathsf{x})\psi(x) = \int_{-\infty}^{+\infty} dX \ < x|X > \ \mathsf{P}\frac{1}{\cosh b\mathsf{X}}\Psi(X), \tag{21}$$

From these two conditions we have (in position representation) after integrating by parts

$$-i\frac{\partial}{\partial x} < x|X >= -i\frac{\partial}{\partial X}(< x|X >)\tanh bX,$$

and

$$\sqrt{a}\exp(bx) < x|X >= i\frac{\partial}{\partial X}(< x|X >)\frac{1}{\cosh bX},$$

which can be solved to give

$$< x|X >= \exp\left(-i\frac{\sqrt{a}}{b}e^{bx}\sinh(bX)\right) \tag{22}$$

In general two or three dimensional case the kernel has the form

$$< x_1, x_2, x_3|X_1, X_2, X_3 >= \exp\left(-i\sum_{i=1}^{3}\frac{\sqrt{a}_i}{b_i}e^{bx_i}\sinh(bX_i)\right).$$

It should be observed that in order that the procedure described above make sense and that solutions of constraints are mapped to solutions, the wave function $\Psi(X)$ must be integrable with the kernel (22) at least together with its first derivative. By changing variables in the integral $\sinh bX = Y$, we see that

$$\psi(x) = \int_{-\infty}^{+\infty} dY \ \exp\left(-i\frac{\sqrt{a}}{b}e^{bx}\,Y\right)\frac{1}{b\sqrt{1+Y^2}}\Psi(X(Y))$$

and the integral exists (in the sense of principal value) if $\Psi(Y)$ tends to a constant when $Y \to \pm\infty$. Of course, $\Psi(Y)$ cannot have any nasty singularities in finite interval.

Therefore given any solution of simple free system we can, in principle, construct a solution of the original theory. The problem is that the integral transform defined above is very complex because the kernel oscillates very rapidly for large X. For that reason it is not only hopeless to find the integral in terms of tabulated functions but it is even difficult to compute the transform numerically. However some approximate methods (for large x, for example) may work for some $\Psi(X)$.

5 The inner product

We succeeded therefore to reduce our problem to the problem of free scalar particle in two or three dimensions. However not all solutions of the constraint equation

$$\Box\Psi(X) = 0 \tag{23}$$

can be interpreted as a physical wavefunction. First of all, in order to make contact with the original theory, we must make sure that the integral transform defined in the previous section exists. Secondly, the wavefunction is supposed to describe physical probabilities, thus it must be normalizable. Thus we must define an inner product in the space of solutions of equation (23).

The problem of defining the inner product for Dirac quantization scheme is unsolved in general (see however [10]). In the case in hand, however, it suffices just to construct the inner product.

To this end, let us observe that any solution of equation (23) can be, in momentum representation, written in the form

$$\Psi(P) = \delta(\eta^{ij}P_iP_j)\tilde{\Psi}(P),$$

where $\tilde{\Psi}(P)$ is an arbitrary function of P sufficiently well behaving at $\eta^{ij}P_iP_j = 0$. In any other representation one must replace $\delta(\eta^{ij}P_iP_j)$ with the delta function of the corresponding operator

$$\Psi = \delta(\eta^{ij}\mathsf{P}_i\mathsf{P}_j)\tilde{\Psi} \equiv \int_{-\infty}^{+\infty} d\lambda\, \exp(i\lambda\eta^{ij}\mathsf{P}_i\mathsf{P}_j)\tilde{\Psi}. \tag{24}$$

Now we can define the inner product on the space of solutions of costraint equation to be

$$\parallel \Psi \parallel^2 = \int \tilde{\Psi}^\star\delta(\eta^{ij}\mathsf{P}_i\mathsf{P}_j)\tilde{\Psi}.$$

This inner product is not very convenient for our future investigations. The reason is that the wave function being a solution of (two dimensional) wave operator has a general form

$$\Psi(X_0, X_1) = \int_{-\infty}^{+\infty} dk A_k e^{ikU} + \int_{-\infty}^{+\infty} dl B_l e^{ilV},$$

where $U = X_0 + X_1$ and $V = X_0 - X_1$. For that reason we would like to have an equivalent definition of the inner product for wave functions in position representation.

Following [10] we proceed as follows. We define the inner product to be

$$\parallel \Psi(X) \parallel^2 = \int_{-\infty}^{+\infty} d^2X \Psi^\star(X) \hat{\mu} \Psi(X),$$

where $\hat{\mu}$ is a gauge fixing operator. In the case in hand this operator is chosen to be $\frac{\mathsf{X}_0}{2\mathsf{P}_0}$.[2] This operator acts in a simple way in momentum representation. We have therefore[3]

$$\begin{aligned} & \parallel \Psi(X) \parallel^2 = \\ = & \int_{-\infty}^{+\infty} d^2X' d^2P d\lambda \Psi^\star(X') < X'|P > \exp\left(-\lambda \frac{1}{2P_0} \frac{\partial}{\partial P_0}\right) < P|X > \Psi(X) = \\ = & \int_{-\infty}^{+\infty} d^2X' d^2P d\lambda \Psi^\star(X') e^{-i(X_0' P_0 + X_1' P_1)} e^{i((X_0(P_0 - \frac{\lambda}{2P_0}) + X_1 P_1)} \Psi(X_0, X_1). \end{aligned}$$

Integrating over P_1 and X_1', we get

$$\int_{-\infty}^{+\infty} dX_0' dX_1 dX_0 dP_0 d\lambda \Psi^\star(X_0', X_1) e^{-iX_0' P_0} e^{i(X_0(P_0 - \frac{\lambda}{2P_0})} \Psi(X_0, X_1) =$$

$$\int_{-\infty}^{+\infty} dX_0' dX_1 dX_0 dP_0 \Psi^\star(X_0', X_1) e^{iP_0(X_0 - X_0')} \delta\left(\frac{X_0}{2P_0}\right) \Psi(X_0, X_1),$$

which after integration over X_0 gives

$$\int_{-\infty}^{+\infty} dX_0' dX_1 dP_0 \Psi^\star(X_0', X_1) e^{-iP_0 X_0'} 2P_0 \Psi(0, X_1) =$$

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[2] In general case when the comutator of gauge fixing operator and gauge constraint is not constant, the formula above must be generalized, see [10].

[3] $< \mid >$ is the inner product of the original Hilbert (or, more generally Krein) space of the model. Thus $< P|X >= e^{iPX}$.

$$-2\int_{-\infty}^{+\infty} dX_1 \, \frac{\partial}{\partial X_0'} \Psi^\star(X_0', X_1)\Big|_{X_0'=0} \Psi(0, X_1).$$

This inner product can be also written as

$$\int_{-\infty}^{+\infty} dX dY \; \Psi^\star(Y,X) \, \overset{\leftrightarrow}{\partial}_Y \, \Psi(Y,X)\Big|_{Y=0} \quad (25)$$

This is in fact the standard inner product for massless spin zero particle, as it could have been expected. Observe that we clearly have here normalizable negative norm states. This fact is well known, and there is an open problem what is the physical meaning of such states, and/or if their existance indicate the need of third quantization.

6 A simple model

In this section we make use of the machinery built above to discuss a simplest possible nontrivial model, namely

$$\Psi(X_0, X_1) = e^{i(k+l)U} + e^{i(k-l)V}, \quad (26)$$

where $U, V = X_0 \pm X_+$ as before. The above wave function is certainly not physical (it is not normalizable), however it has the virtue to analogous to the plane wave states in the standard quantum field theory. For that reason all the results below should be taken with the grain of salt, however it does not seem unlikely that some physical prediction of this simple model may hold for the full theory which will be discussed elsewhere.

Given the wavefunction (26), one can readily find the quantum potential. Equation (7) reads in our case

$$-P_0^2 + P_+^2 + (k^2 - l^2) = 0$$

and from the hamiltonian equation of motion we find easily that

$$P_0 = k \qquad X^0 = -kt \quad (27)$$

$$P_+ = l \qquad X^+ = lt + t_0 \quad (28)$$

From the condition for the canonical transformation to be meaningful, we see that both k and l must be positive. Now we are ready to turn to the starting variables. To this end we choose to work with the Bianchi type IX model, and using equation (18), we find

$$\sqrt{48}e^{2\sqrt{3}x^0} = \frac{k}{\cosh 2\sqrt{3}kt}, \quad p_0 = -l\tanh 2\sqrt{3}kt, \tag{29}$$

$$\sqrt{12}e^{-4\sqrt{3}x^+} = \frac{l}{\cosh 4\sqrt{3}(lt+t_0)}, \quad p_+ = l\tanh 4\sqrt{3}(lt+t_0). \tag{30}$$

Looking at the Misner parametrization, we see that the metric depends on t through the combination

$$\beta^0 = \frac{1}{3}\log\left(\frac{k}{4\sqrt{3}}\left(\frac{2\sqrt{3}}{l}\right)^{1/4}\frac{\cosh^{1/4}(4\sqrt{3}lt+t_0)}{\cosh(2\sqrt{3}kt)}\right)$$

$$\beta^+ = \frac{1}{6}\log\left(\frac{k}{2l}\frac{\cosh(4\sqrt{3}lt+t_0)}{\cosh(2\sqrt{3}kt)}\right)$$

Now we can discuss the characteristic features of the model. It is described by the metric

$$ds^2 = -N_T^2dt^2 + e^{2\beta^0}\left(e^{2\beta^+}(\omega^1)^2 + e^{2\beta^+}(\omega^2)^2 + e^{-4\beta^+}(\omega^3)^2\right),$$

where $e^{2\beta^0}$ is the scale factor and $e^{2\beta^+}$ is a measure of the anisotropy. The physical time is given by the monotoneous function of the parameter t,

$$\tau \sim \int^\tau \frac{\cosh^{1/4}(4\sqrt{3}lt+t_0)}{\cosh(2\sqrt{3}kt)}\,dt$$

and for large t, τ behaves as $\tau \sim \frac{1}{l-2k}e^{\sqrt{3}t(l-2k)}$. The scale factor is a symmetric function of t with one extremum, which is a minimum for $l > 2k$, and a maximum for $l < 2k$. Similarly, the anisotropy factor has only one maximum for $2l > k$ and only one minimum for $2l < k$. Thus we have four qualitatively different classes of models, depending on different values of k, l. Observe that the classical behaviour corresponds to $k = l$ (in this case the quantum potential vanishes.)

The model above should be viewed only as an illustration of application of our method to quantum cosmological models. As it was mentioned at the beginning of this section its physical value is diminished by the fact that the corresponding wavefunction is not normalizable. The more realistic models will be considered in the separate paper.

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