THERMODYNAMIC BETHE ANSATZ AND DILOGARITHM IDENTITIES I.

EDWARD FRENKEL AND ANDRÁS SZENES

1. Introduction

In the decade that has passed since the seminal work [?] by A.A. Belavin, A.P. Polyakov, and A.B. Zamolodchikov, a lot of progress has been made in understanding Conformal Field Theories (CFTs) in two dimensions. The success in the study of CFTs is due to their invariance with respect to the Virasoro algebra, or, more generally, extended conformal algebras. This property allows one to describe CFTs in terms of representation theory of infinite-dimensional Lie algebras or vertex operator algebras, and algebraic geometry of complex curves.

In [?] A.B. Zamolodchikov introduced an interesting class of 2D quantum field theories – perturbations of CFTs by relevant operators. These theories lack conformal invariance, but possess some other remarkable algebraic structures, which are yet to be fully understood from the mathematical point of view. One of the properties is the existence of infinitely many local integrals of motion in involution. This was conjectured in [?] (see also [?]) and proved in [?]. Thus, a perturbation of a CFT is an integrable 2D quantum field theory, and as such, it is governed by a purely elastic S–matrix, which satisfies various algebraic constraints [?]. These constraints are so strong that knowing the spins of local integrals of motion one can often conjecture the S–matrix and hence determine the theory completely, see [?, ?] and references therein.

The Thermodynamic Bethe Ansatz (TBA) is a method of verifying these conjectures, which was first applied in this context by Al.B. Zamolodchikov [?]. One starts with an integrable field theory conjectured to be the perturbation of a CFT $\mathcal{T}$, and studies its ultraviolet (UV) behavior. A theory on an infinitely long cylinder of circumference R is described by a system of integral equations called the TBA equations. To write down this system explicitly, let us assume that the theory has N species of particles with masses $m_a, a = 1, \ldots, N$. One is interested

Partially supported by NSF Grants.

in the functions $\epsilon_a(\theta)$, which are called the spectral densities of particles of species a, see e.g. [**?**]. These are functions of the rapidity θ (recall that rapidity is related to the energy E and the momentum p by the formulas $E = m\cosh\theta, p = m\sinh\theta$). The TBA equations on the functions $\epsilon_a(\theta)$ read:

m(1.1)

$$m_a R\cosh\theta = \epsilon_a(\theta) + \frac{1}{2\pi}\sum_{b=1}^{N}\int_{-\infty}^{\infty} d\theta' \ \phi_{ab}(\theta-\theta') \ \log\left(1+Y_a(\theta')\right),$$

where $Y_a(\theta) = e^{-\epsilon_a(\theta)}$, $\phi_{ab}(\theta) = -i\dfrac{\partial \log S_{ab}}{\partial\theta}$, and $S_{ab}(\theta)$ is the S–matrix.

The TBA equations are usually hard to solve, but one can extract a lot of information from them even without solving them explicitly. The ground state energy of the theory is given by

m(1.2) $$E(R) = -\frac{1}{2\pi}\sum_{a=0}^{N}\int_{-\infty}^{\infty} d\theta \ m_a R\cosh\theta \ \log\left(1+Y_a(\theta)\right).$$

In the UV limit $R \to 0$, in which one is supposed to recover the initial CFT $\mathcal{T}$, we should have $E(R) \simeq -\pi\widetilde{c}(R)/6R$, where $\widetilde{c}(R) \sim \widetilde{c} + O(R)$. From energy one finds using the TBA equations, see e.g. [**?**]:

m(1.3) $$\frac{\pi^2}{6}\widetilde{c} = \sum_{a=1}^{N} L\left(\frac{1}{1+y_a}\right),$$

where $L(z)$ is the Rogers dilogarithm function [**?**]:

m(1.4)

$$L(z) = \frac{1}{2}\int_0^z \left(\log w \, d\log(1-w) - \log(1-w)\, dw\right), \quad 0 \leq z \leq 1,$$

and $y_a = \lim_{R\to 0} Y_a(\theta)$. The numbers y_a satisfy the system of algebraic equations

m(1.5) $$y_a = \prod_{b=1}^{N}\left(1+\frac{1}{y_b}\right)^{N_{ab}},$$

where N_{ab} is the number of poles of $S_{ab}(\theta)$ in the upper half plane; in particular, they do not depend on θ.

If the conjectural description of the perturbation of the CFT $\mathcal{T}$ is correct, the number $\widetilde{c}$ in the left hand side of the formula di should coincide with the effective central charge of $\mathcal{T}$. But in that case formula di can be considered as a dilogarithm identity, which relates a rational number $\widetilde{c}$ to the algebraic numbers y_a's.

Many dilogarithm identities have been discovered this way in recent years. While the TBA method has not yet been made rigorous, the identities have been proved rigorously by other methods, see [?, ?, ?]. Mathematically, the dilogarithm identities manifest a connection between 2D quantum field theory on the one hand and algebraic K–theory and number theory on the other, see [?]. We hope that better understanding of the TBA will enable us to gain new insights into this connection.

Recently, F. Gliozzi and R. Tateo [?] made an important step in this direction. They found functional analogues of the identities di for a large class of theories, which are labeled by pairs (G, H) of Dynkin diagrams of types ADE and T (the latter is the diagram of type A with a loop attached to one of the end vertices). In such a theory, the species of particles are labeled by pairs of indices $a = 1, \dots, r_G$, and $b = 1, \dots, r_H$, where r_G and r_H are the numbers of vertices in the diagrams G and H, respectively.

The main fact, which is due to Al.B. Zamolodchikov [?] and, in the general case, to F. Ravanini, A. Valleriani and R. Tateo [?] is that any solution $\{Y_a^b(\theta)\}$ of the TBA equations tba corresponding to the (G, H) theory satisfies the following system of algebraic equations:

m(1.6)

$$Y_a^b\left(\theta + \frac{\pi I}{h_G^\vee}\right) Y_a^b\left(\theta - \frac{\pi I}{h_G^\vee}\right) = \prod_{c=1}^{r_G} \left(1 + Y_c^b(\theta)\right)^{G_{ac}} \prod_{d=1}^{r_H} \left(1 + \frac{1}{Y_a^d(\theta)}\right)^{-H_{bd}},$$

where $I = \sqrt{-1}$, (G_{ac}) and (H_{bd}) are the adjacency matrices of the diagrams G and H, respectively, and $h_G^\vee$ is the dual Coxeter number of G.

The Y*–system* Y and closely related to it T–system play an important role in quantum field theory and statistical mechanics [?, ?, ?, ?]. In [?] it was conjectured that certain solutions of this system are in one-to-one correspondence with the eigenvectors of integrals of motion of the corresponding integrable field theory.

Al.B. Zamolodchikov [?] has conjectured an important periodicity property of solutions of the system Y:

m(1.7)
$$Y_a^b\left(\theta + \pi I \frac{h_G^\vee + h_H^\vee}{h_G^\vee}\right) = Y_{\bar{a}}^{\bar{b}}(\theta),$$

where $h_H^\vee$ is the dual Coxeter number of H, and $\bar{a}, \bar{b}$ are the vertices conjugate to a, b, respectively. This periodicity property allows one to find the conformal dimension of the field responsible for the perturbation of the corresponding CFT, see [?].

Now we can write down the dilogarithm identities conjectured in [?]. Let $\{Y_a^b(\theta)\}$ is a solution of the Y–system Y. Fix θ and set

$$X_a^b(m) = \frac{Y_a^b\left(\theta + \pi I m/h_G^\vee\right)}{1 + Y_a^b\left(\theta + \pi I m/h_G^\vee\right)}.$$

Suppose that all $X_a^b(m)$ are real numbers between 0 and 1. Then

m(1.8) $$\sum_{a=1}^{r_G}\sum_{b=1}^{r_H}\sum_{m=1}^{h_G^\vee + h_H^\vee} L\left(X_a^b(m)\right) = \frac{\pi^2}{6} r_G r_H h_G^\vee.$$

Let $y_a^b = \lim_{\theta\to+\infty} Y_a^b(\theta)$. In the limit $\theta \to +\infty$ the system Y becomes a system of the type limiteq, and the identity gt becomes equivalent to the identity di corresponding to the UV limit of the (G, H) theory (in order to relate them, one has to use the Euler identity $L(z) + L(1 - z) = \pi^2/6$). Therefore the identities gt can be viewed as functional analogues of the known dilogarithm identities di. It is interesting that the identity corresponding to the (A_1, A_1) theory is the Euler identity above, and the identity corresponding to the (A_2, A_1) theory is the pentagon identity of the dilogarithm function, see [?].

There are many indications that analogues of the Y–system can be defined for other integrable field theories and that there are dilogarithm identities associated to them, see [?]. In [?] a geometric interpretation for these identities is suggested.

In this paper, we give a proof of the periodicity conjecture periodicity and the identities gt and their generalizations for the (A_n, A_1) theories. We also prove analogous identities for the Bloch-Wigner function, which is the imaginary counterpart of the Rogers dilogarithm. Our proof of these identities relies on a universal property of the dilogarithm functions, which for the Bloch-Wigner function was first proved by S. Bloch [?].

Our approach can be generalized to the identities corresponding to more general diagrams. We have already obtained a complete proof of periodicity and dilogarithm identities for the diagrams (A_n, A_2) and partial results in the general case. We will report on those results in a separate publication.

The paper is organized as follows. In Sect. 2 we prove the periodicity property of the Y–system. In Sect. 3 we give a general form of the dilogarithm identities for the Rogers and the Bloch-Wigner dilogarithms. In Sect. 4 we prove the identities gt of (A_n, A_1) type and their generalizations.