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Five-Dimensional BF Theory and Four-Dimensional Current Algebra

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Abstract. We consider the relation between the five-dimensional BF model and a four-dimensional local current algebra from the point of view of perturbative local quantum field theory. We use an axial gauge fixing procedure and show that it allows for a well defined theory which actually can be solved exactly.

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1 Introduction

The relation between Topological Field Theories [1] and local Field Theories is well established for the three dimensional Chern-Simons (CS) model. Indeed, it is well known that the restriction of the latter on a two dimensional plane leads to physical observables, namely the two dimensional conserved chiral currents generating the Kac-Moody algebra of the Wess-Zumino type [2]. In a previous paper [3], we exhibit this fact using a very general procedure. More precisely, after having defined the model on a manifold with boundary, we chose to implement the effects of the latter by means of two requirements: a decoupling condition which forbids the existence of any interactions through the boundary, and a locality condition which states that away from it, the theory is the same as the one without boundary. It is remarkable that such conditions can be implemented directly at the level of the generating functional for the connected Green function. Therefore, it avoids the problem of dealing with surface terms which are *a priori* ill defined products of distributions at the same point and which would need to be regularized.

In [3] an axial gauge was chosen, which is a natural choice when considering a plane boundary in a plane space-time. Such a gauge fixing is however incomplete, invariance under residual gauge transformations being left. It is precisely the Ward identity expressing this residual invariance which has been interpreted as a current algebra on the boundary.

In this paper, we will apply the procedure of [3] in higher dimensions. More precisely, we shall start with a five-dimensional BF model defined on a manifold with a boundary of dimension 4, and look at the consequences of the Ward identities which correspond to the residual gauge invariances. At this point, let us note that the symmetry content of the BF system is greater than that of the CS theory. Indeed, such models are known to exhibit reducible symmetries [1, 4]. Therefore the question is to know whether such a difference breaks the full procedure. The answer is negative. The present theory possesses a four-dimensional current algebra which lies on the boundary. This algebra is the one which is generated through the residual Ward identity of the Yang-Mills symmetry. The one which corresponds to the reducible symmetry contains a hard breaking and cannot be interpreted as a current algebra.

It should be noticed that the derivation of the Ward identities for the residual gauge invariances suffers from an infrared (IR) problem, linked to the bad long distance behaviour associated with the particular geometry implemented, and particularly with the gauge condition given by the vanishing of the gauge field components which are orthogonal to the boundary plane. Therefore, one will have to make use of an infrared regularized gauge condition.

The paper is organized as follows. We first review some general facts about the five-dimensional BF theory in the axial gauge and the procedure followed for describing the effects of the boundary. Then we compute the propagators and the N-point Green functions as the general solution of the field equations. Finally we prove the residual Ward identity leading to the algebra of currents living on the boundary considered as a four-dimensional space-time. It is only for this last point that the infrared regularization is introduced, since the Green functions exist without such a regularization. We finish with some conclusions.

2 Five-Dimensional BF Theory in the Axial Gauge

The classical action of the BF model in 5 dimensions reads[1]

$$\Sigma_{\rm BF} = \int \mathrm{Tr}\ (B \wedge F) = \frac{1}{2\cdot 3!}\int d^5x\ \varepsilon^{\mu\nu\rho\sigma\tau}\mathrm{Tr}\ (B_{\mu\nu\rho}F_{\sigma\tau}) \tag{2.1}$$

where B is a three form and $F = dA + \frac{1}{2}[A, A]$ is the field strength of the gauge connection A. These fields, as well as all the one's encountered throughout this paper belong to the adjoint representation and are written as Lie algebra matrices $\varphi(x) = \varphi(x)^a\tau_a$ with

$$[\tau_a, \tau_b] = f_{ab}{}^c\tau_c, \qquad \mathrm{Tr}\,(\tau_a\tau_b) = \delta_{ab}.$$

This action is invariant under the usual Yang-Mills transformations δ_ω defined as

$$\begin{aligned} \delta_\omega A_\mu &= D_\mu\omega \\ \delta_\omega B_{\mu\nu\rho} &= [B_{\mu\nu\rho}, \omega] \end{aligned} \tag{2.2}$$

with $D_\mu\cdots = \partial_\mu\cdots + [A_\mu,\cdots]$. Furthermore, it is also invariant under the so-called reducible transformations δ_ψ defined as

$$\begin{aligned} \delta_\psi A_\mu &= 0 \\ \delta_\psi B_{\mu\nu\rho} &= -(D_\mu\psi_{\nu\rho} + \text{cyclic permutations}). \end{aligned} \tag{2.3}$$

where ω and ψ are forms of degree 0 and 2 respectively.

As usual, one has to fix the gauge. The first point is to determine the number of degrees of freedom of the field B. Using (2.3), one sees that all the ψ which can be written as[2] $\psi = D\psi'$ where ψ' is a 1-form, are irrelevant on shell, i.e. for fields solutions of their equation of motion. Indeed the transformation (2.3) then reads

$$\delta_\psi B_{\mu\nu\rho} = -(F_{\mu\nu}\psi'_\rho + \text{cyclic permutations})$$

[1]**Conventions**: $\mu,\nu,\cdots = 0,1,2,3,4$, $g_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1,-1)$, $\varepsilon^{\mu\nu\rho\sigma\tau} = \varepsilon_{\mu\nu\rho\sigma\tau} = \varepsilon^{[\mu\nu\rho\sigma\tau]}$, $\varepsilon_{01234} = 1$.

[2]The action of the covariant derivative over a form Ω is given by $D\Omega = dx^\mu D_\mu\Omega$.

and $F_{\mu\nu} = 0$ is the field equation for A_μ. By repeating the same argument for $\psi' = D\psi''$, where ψ'' is now a 0-form, one obtain that the number of degrees of freedom of B we have to fix is equal to $10 - (5-1) = 6$. On the other hand, the field A has one gauge degree of freedom to be fixed, as usual.

The choice of the gauge fixing condition is naturally related to the geometry of the problem under interest. The space-time we will consider here is $\mathbb{R}^5$, and the boundary $\mathcal{B}$ is the plane defined by $x^4 = 0$. Therefore, the axial gauge

$$\begin{aligned} n^\rho A_\rho &= 0, \\ n^\rho B_{\mu\nu\rho} &= 0 \end{aligned} \tag{2.4}$$

with $n^\mu = (0,0,0,0,1)$ is the natural choice[3].

The gauge fixed action then reads

$$\begin{aligned} \Sigma = \mathrm{Tr} \int d^5x \quad & \left\{ \tilde{B}^q \left(\partial_4 A_q - \partial_q A_4 + [A_4, A_q]\right) + \right. \\ & \left. + \tfrac{1}{2}\varepsilon^{mnpq} B_{mn4} \left(\partial_p A_q + A_p A_q\right) + \pi A_4 + \tfrac{1}{2}\pi^{mn} B_{mn4} \right\} \end{aligned} \tag{2.5}$$

where $\tilde{B}^q$ is the four-dimensional dual of B_{mnp} defined as

$$\tilde{B}^q = \frac{1}{3!}\varepsilon^{qmnp} B_{mnp}$$

together with $\varepsilon^{qmnp4} = \varepsilon^{qmnp}$. We have not introduced the Faddeev-Popov ghosts since, in the axial gauge, they are decoupled and thus they are not needed.

It is easy to check that the gauge fixing given above is sufficient, the Lagrange multiplier fields π, π^{mn} fix the seven gauge degrees of freedom we have counted above.

The full gauge fixed action (2.5) still possesses invariances. The gauge fixing terms are responsible for the breaking of (2.2), (2.3) as well as for the breaking of the five-dimensional Poincaré invariance. Nevertheless, we stay with four-dimensional Poincaré invariance in the coordinates transverse to the boundary, together with two residual gauge invariances whose transformation laws have the same form as (2.2) and (2.3) but where the gauge parameters do not depend of x^4:

$$\omega = \omega(x^0, x^1, x^2, x^3), \tag{2.6}$$

$$\psi_{mn} = \psi_{mn}(x^0, x^1, x^2, x^3). \tag{2.7}$$

Let us now introduce the boundary. Following the procedure given in [3], the effects of the boundary $\mathcal{B}$ are specified by two conditions. The first one is a decoupling condition which states that $\mathcal{B}$ separates our space $\mathbb{R}^5$ in two half-spaces

[3]The transverse coordinates with respect to $\mathcal{B}$ are denoted by x^{tr} and are labelled by $m, n, p, \cdots = 0, 1, 2, 3$.

labeled by $+$ and $-$ corresponding to the sign of the x^4 component. We also impose a locality condition which states that the behavior away from the boundary is the same as the one of the theory without boundary.

The simplest way to fill these conditions, *i.e.*, to describe the effect of the boundary, is by working directly at the level of the generating functional of the connected Green functions $Z_{\rm c}[J^q, \tilde{J}_q, J_4, J^{mn4}, J_\pi, J_{\pi^{mn}}]$, whose arguments are the sources of the fields A_q, $\tilde{B}^q$, A_4, B_{mn4}, π and π^{mn} respectively. Indeed, the decoupling condition corresponds to the decomposition of $Z_{\rm c}$ into two parts

$$Z_{\rm c}(J_\varphi) = Z_{{\rm c}+}(J_\varphi) + Z_{{\rm c}-}(J_\varphi) \ , \tag{2.8}$$

which implies that an n-point Green function will be written as

$$\langle \varphi_{i_1}(x_1)\cdots\varphi_{i_n}(x_n)\rangle = \theta_+ \langle \varphi_{i_1}(x_1)\cdots\varphi_{i_n}(x_n)\rangle_+ + \theta_- \langle \varphi_{i_1}(x_1)\cdots\varphi_{i_n}(x_n)\rangle_- \tag{2.9}$$

with

$$\theta_\pm = \theta(\pm x_1^4)\theta(\pm x_2^4)\cdots\theta(\pm x_n^4)$$

In particular the propagators take the form

$$\Delta^{\varphi\varphi'}(x,x') = \theta_+\Delta_+^{\varphi\varphi'}(x,x') + \theta_-\Delta_-^{\varphi\varphi'}(x,x') \tag{2.10}$$

where $\theta_\pm = \theta(\pm x^4)\theta(\pm x'^4)$.

Then, the locality condition allows us to calculate the Green functions $\langle\cdots\rangle_+$ and $\langle\cdots\rangle_-$ as the solutions of the equations of motion of the theory without boundary.

The locality condition implying that the effects of the boundary are local, the latter will be described by terms in $\delta(x^4)$. These terms will be constrained by dimensional – we want to preserve scale invariance – and other symmetry arguments. Therefore, the equations of motion of the theory with boundary will be of the form

$$\begin{aligned}
&-\partial_4\tilde{B}^q + \left[J_\pi, \tilde{B}^q\right] + \varepsilon^{mnpq}\partial_p J_{\pi^{mn}} + \tfrac{1}{2}\varepsilon^{mnpq}\left[A_p, J_{\pi^{mn}}\right] + J^q &&= \delta(\pm x^4)\lambda_\pm \tilde{B}^q_\pm \\
&\partial_4 A_q - \left[J_\pi, A_q\right] + \partial_q J_\pi + \tilde{J}_q &&= \delta(\pm x^4)\lambda_\pm A_{q\pm} \\
&\partial_q\tilde{B}^q + \left[A_q, \tilde{B}^q\right] + \pi + J^4 &&= 0 \\
&\varepsilon^{mnpq}\left(\partial_p A_q + A_p A_q\right) + \pi^{mn} + J^{mn4} &&= 0 \\
&A_4 + J_\pi &&= 0 \\
&B_{mn4} + J_{\pi^{mn}} &&= 0
\end{aligned} \tag{2.11}$$

These equations have been written in a functional way, with the notation

$$\varphi(x) = \frac{\delta Z_{\rm c}}{\delta J_\varphi(x)} \ , \qquad \varphi = A_m, \tilde{B}^q, A_4, B_{mn4}, \pi, \pi^{mn} \tag{2.12}$$

and where $\varphi_\pm(z)$ means the insertion of the field $\varphi(x)$ on the right, respectively on the left of the boundary $\mathcal{B}$:

$$\varphi_\pm(z) = \lim_{x^4\to\pm 0} \frac{\delta Z_{\rm c}}{\delta J_\varphi(x)} \ . \tag{2.13}$$

The last two equations of (2.11) are the gauge conditions (2.4) written in term of $Z_{\rm c}$.

Let us remark that the parameter $\lambda_\pm$ of the boundary terms in the right-hand sides of the first two field equations has been set equal for both in order to assure their mutual consistency. A motivation for the form of these boundary terms may be found in the remark that they could formally be inferred from a surface term in the action of the form

$$\mathrm{Tr} \int_{\mathcal{B}} d^4x^{\rm tr}\, \tilde{B}^q A_q = \mathrm{Tr} \int d^5x\, \delta(x^4) \tilde{B}^q A_q \tag{2.14}$$

The arbitrariness of the coefficients $\lambda_\pm$ would then follow from ambiguities caused by the multiplication of distributions at the same point.

The gauge invariances (2.2), (2.3) of the theory with boundary also lead to functional identities. These identities, due to the decoupling of the Faddeev-Popov ghosts, take the form of two local Ward identities:

$$\sum_\varphi [J_\varphi, \varphi] - \partial_q J^q - \partial_4 J^4 - \partial_4 \pi = -\delta(\pm x^4)\lambda_\pm \partial_q \tilde{B}^q_\pm \tag{2.15}$$

$$\begin{aligned} &[J^{mn4}, A_4] - \varepsilon^{mnpq}[\tilde{J}_q, A_p] + [J^\pi, \pi^{mn}] + \varepsilon^{mnpq}\partial_p \tilde{J}_q - \partial_4 \pi^{mn} - \\ &-\partial_4 J^{mn4} = -\delta(\pm x^4)\lambda_\pm \varepsilon^{mnpq}\left(\partial_p A_{q\pm} + \tfrac{1}{2}\left[A_{p\pm}, A_{q\pm}\right]\right) \end{aligned} \tag{2.16}$$

3 The Free Propagators

We apply now the procedure described above to the computation of the propagators. The locality condition implies the results shown in Table (1). We have set

$$T_\xi(x,x') = \left[\theta(x^4 - x'^4) + \xi\right] \delta^{(4)}(x^{\rm tr} - x'^{\rm tr})$$

and $(\xi, \Delta^1_1, \Delta^1_2, \Delta^2_1, \Delta^3_1, \Delta^3_2)$ are just integration "constants". Note that, whereas ξ is just a number, the Δ^i_j are arbitrary functions of the transverse coordinates, more precisely of $(x^{\rm tr} - x'^{\rm tr})^2$ due to the four-dimensional Poincaré invariance.

However, further restrictions on the propagators will follow from the decoupling condition. Indeed, the form (2.10), due to the presence of the θ-functions, will

$$
\begin{pmatrix}
\begin{array}{l}\delta_{pq}\Delta_1^1 \\ +\partial_p\partial_q\Delta_2^1\end{array} & \begin{array}{l}T_\xi(x',x)\delta_q^p \\ +\partial_q\partial^p\Delta_1^2\end{array} & \begin{array}{l}\partial_q T_\xi(x',x) \\ +\partial_q\partial^2\Delta_1^2\end{array} & \varepsilon^{rstp}\partial_t\Delta_1^1\delta_{pq} \\[2ex]
\begin{array}{l}T_\xi(x,x')\delta_p^q \\ +\partial_p\partial^q\Delta_1^2\end{array} & \begin{array}{l}\delta^{pq}\Delta_1^3 \\ +\partial^p\partial^q\Delta_2^3\end{array} & \begin{array}{l}\partial^q\Delta_1^3 \\ +\partial^q\partial^2\Delta_2^3\end{array} & \begin{array}{l}\varepsilon^{rstq}\partial_q\left\{T_\xi(x,x')\delta_t^p\right. \\ \left.+\ \partial_t\partial^p\Delta_1^2\right\}\end{array} \\[2ex]
\begin{array}{l}-\partial_p T_\xi(x,x') \\ -\partial_p\partial^2\Delta_1^2\end{array} & \begin{array}{l}-\partial^p\Delta_1^3 \\ -\partial^p\partial^2\Delta_2^3\end{array} & \begin{array}{l}-\partial^2\Delta_1^3 \\ -\partial^2\partial^2\Delta_2^3\end{array} & 0 \\[2ex]
\varepsilon^{mnrq}\partial_q\Delta_1^1\delta_{rp} & \begin{array}{l}\varepsilon^{mnqr}\partial_q\left\{T_\xi(x',x)\delta_r^p\right. \\ \left.+\partial_r\partial^p\Delta_1^2\right\}\end{array} & 0 & \varepsilon^{mnpq}\varepsilon^{rstl}\partial_p\partial_l\Delta_1^1\delta_{tq}
\end{pmatrix}
$$

Table 1: *The propagators $\Delta_\pm(x,x')$. The table is ordered according to the sequence $A_{q(p)}, \tilde{B}^{q(p)}, \pi, \pi^{mn(rs)}$ for the columns (resp.lines). The gauge indices have been dropped out since the propagators are diagonal in the group space.*

generate a boundary term when substituted into (2.11). Therefore, at the limit $x^4 \to \pm 0$, one gets the following constraints between the parameters $\lambda_\pm$ and the integration constants:

$$
\begin{array}{ll}
(1-\lambda_+)\,\Delta_1^1 = 0 & (1+\lambda_-)\,\Delta_1^1 = 0 \\
(1-\lambda_+)\,\Delta_2^1 = 0 & (1+\lambda_-)\,\Delta_2^1 = 0 \\
(1-\lambda_+)\,\Delta_1^2 = 0 & (1+\lambda_-)\,\Delta_1^2 = 0 \\
(1-\lambda_+)\,(1+\xi) = 0 & (1+\lambda_-)\,\xi = 0 \\
(1+\lambda_+)\,\Delta_1^3 = 0 & (1-\lambda_-)\,\Delta_1^3 = 0 \\
(1+\lambda_+)\,\Delta_2^3 = 0 & (1-\lambda_-)\,\Delta_2^3 = 0 \\
(1+\lambda_+)\,\Delta_1^2 = 0 & (1-\lambda_-)\,\Delta_1^2 = 0 \\
(1+\lambda_+)\,\xi = 0 & (1-\lambda_-)\,(1+\xi) = 0\ .
\end{array}
\tag{3.1}
$$

The system (3.1) admit two solutions for each side of $\mathcal{B}$:

$$
\begin{array}{rllll}
\mathrm{I}_+ : & \lambda_+ = -1 & \xi = -1 & \Delta_1^1 = \Delta_2^1 = \Delta_1^2 = 0 & \Delta_1^3,\ \Delta_2^3 \text{ arb.} \\
\mathrm{I}_- : & \lambda_- = +1 & \xi = 0 & \Delta_1^1 = \Delta_2^1 = \Delta_1^2 = 0 & \Delta_1^3,\ \Delta_2^3 \text{ arb.} \\
\mathrm{II}_+ : & \lambda_+ = +1 & \xi = 0 & \Delta_1^3 = \Delta_2^3 = \Delta_1^2 = 0 & \Delta_1^1,\ \Delta_2^1 \text{ arb.} \\
\mathrm{II}_- : & \lambda_- = -1 & \xi = -1 & \Delta_1^3 = \Delta_2^3 = \Delta_1^2 = 0 & \Delta_1^1,\ \Delta_2^1 \text{ arb.}
\end{array}
\tag{3.2}
$$

These solutions are labelled according to the boundary conditions implied by (3.1) on the fields at $\mathcal{B}$: $\mathrm{I}_\pm$ corresponds to Dirichlet conditions (i.e. vanishing of the fields on the corresponding side of the boundary) for A_q, π^{mn} whereas $\mathrm{II}_\pm$ corresponds to Dirichlet conditions for $\tilde{B}^q$, π.

Finally, it follows directly from the gauge conditions, i.e. from the last two equations of (2.11), that the Green functions containing A_4 and/or B_{mn4} are all zero, except

$$
\begin{aligned}
&\Delta_{\pi A_4}(x,x') = -\theta_\pm \delta^{(5)}(x-x') \\
&\Delta_{\pi^{mn} B_{mn4}}(x,x') = -\theta_\pm \delta^{(5)}(x-x') \ .
\end{aligned}
\tag{3.3}
$$

Let us display, for further use, the propagators corresponding to the solution I_+:

$$
\begin{pmatrix}
0 & -T(x,x')\delta^p_q & -\partial_q T(x,x') & 0 \\
-T(x',x)\delta^q_p & \delta^{pq}\Delta^3_1 + \partial^p\partial^q\Delta^3_2 & \partial^q\Delta^3_1 + \partial^q\partial^2\Delta^3_2 & \varepsilon^{rspq}\partial_q T(x',x) \\
\partial_p T(x',x) & -\partial^p\Delta^3_1 - \partial^p\partial^2\Delta^3_2 & -\partial^2\Delta^3_1 - \partial^2\partial^2\Delta^3_2 & 0 \\
0 & \varepsilon^{mnqp}\partial_q T(x,x') & 0 & 0
\end{pmatrix}
\tag{3.4}
$$

where $T(x,x') = \theta(x^4 - x'^4)\delta^{(4)}(x^{\rm tr} - x^{\rm tr\prime})$ and the conventions are the same as in Table (1). The integration "constants" Δ^3_1, Δ^3_2 are arbitrary function of the transverse coordinates $(x^{\rm tr} - x^{\rm tr\prime})^2$.

4 General Solution

We want now to discuss the general solution of the field equations (2.11) for the n-point Green functions of the theory. We shall restrict our analysis starting from the solution I for the free propagators, an analogous discussion been possible starting from solution II. We choose the solution I_+, the physical implications of I_- being identical. In this context, the equations of motion are given by (2.11), with $\lambda_+ = -1$.

As in [3], the third and fourth equations in (2.11) could suffer from ill-defined products of fields. More precisely, they may generate divergent loops through the nonlinear terms $[A_q, \tilde{B}^q]$ and $\varepsilon^{mnpq}A_pA_q$. These diagrams would contribute to the Green functions of the Lagrange multiplier field[4] π. Nevertheless, these *a priori* UV divergent contributions shown in Fig. 1 factorize into a divergent x^4-independent part which can be regularized [3], and an x^4-dependent part of the form $\theta(x_1^4 - x_2^4)\theta(x_2^4 - x_3^4)\cdots\theta(x_n^4 - x_1^4)$ which is zero by itself. This shows the absence of any radiative corrections and thus allows one to neglect the nonlinear terms in the third and fourth equations (2.11).

The first two equations in (2.11) give rise to recursion relations for the Green functions. Such relations are generated by differentiation with respect to the sources,

[4]The vanishing of the propagator $\langle AA \rangle$ for this solution forbids the existence of loops for π^{mn}.

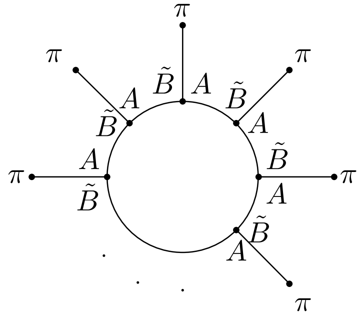

Figure 1: *Loop contributions to the Green functions of the field* π*.*

i.e., by means of a general functional operator

$$\Xi = \frac{\delta^{L+M+N+P}}{(\delta J^A)^L \left(\delta J^{\tilde{B}}\right)^M (\delta J^\pi)^N (\delta J^{\pi^{mn}})^P} \tag{4.1}$$

We begin the analysis outside from the boundary, i.e. we consider first the equations for the component $\langle \cdots \rangle_+$ of the decomposition (2.9). This gives

$$\begin{aligned} \partial_4 \left\langle \tilde{B}^{q\,a}(x) X \right\rangle_+ &= \sum_{i=1}^N f^{aa_i e} \delta^{(5)}(x - x_i) \left\langle \tilde{B}^{q\,e}(x_i) X \backslash \pi^{a_i}(x_i) \right\rangle_+ \\ &\quad + \sum_{j=1}^P f^{ab_j e} \varepsilon^{m_j n_j pq} \delta^{(5)}(x - y_j) \left\langle A_p^e(y_j) X \backslash \pi^{m_j n_j\, b_j}(y_j) \right\rangle_+ \\ &\quad + \delta_{X, A_q^a(y)} \delta^{(5)}(x - y) + \delta_{X, \pi^{mn\,a}(z)} \delta^{(5)}(x - z) \end{aligned} \tag{4.2}$$

$$\begin{aligned} \partial_4 \left\langle A_q^a(x) X \right\rangle_+ &= \sum_{i=1}^N f^{aa_i e} \delta^{(5)}(x - x_i) \left\langle A_q^e(x_i) X \backslash \pi^{a_i}(x_i) \right\rangle_+ \\ &\quad + \delta_{X, \tilde{B}^{q\,a}(y)} \delta^{(5)}(x - y) + \delta_{X, \pi^a(z)} \delta^{(5)}(x - z) \end{aligned} \tag{4.3}$$

where

$$X = \left(\prod_1^L A_{m_i}^{a_i}(w_i) \right) \left(\prod_1^M \tilde{B}^{m_i\, b_i}(z_i) \right) \left(\prod_1^N \pi^{c_i}(x_i) \right) \left(\prod_1^P \pi^{m_i n_i\, d_i}(y_i) \right) \tag{4.4}$$

and where $X \backslash \varphi$ means the omission of the field φ from the string X. The solution for $L + M + N + P \geq 2$ has the form of the following recursion relations:

$$\left\langle \tilde{B}^{q\,a}(x) X \right\rangle_+ = -\sum_{i=1}^N f^{ac_i e} \theta(x_i^4 - x^4) \delta^{(4)}(x^{\rm tr} - x_i^{\rm tr}) \left\langle \tilde{B}^{q\,e}(x_i) X \backslash \pi^{c_i}(x_i) \right\rangle_+$$

$$
-\sum_{j=1}^{P} f^{ad_j e}\varepsilon^{m_j n_j pq}\,\theta(y_j^4-x^4)\delta^{(4)}(x^{\rm tr}-y_j^{\rm tr})\left\langle A_p^e(y_j)X\backslash\pi^{m_j n_j\,d_j}(y_j)\right\rangle_+
$$

$$
+\left\langle \tilde{B}^{q\,a}(x^{\rm tr})X\right\rangle_+ \tag{4.5}
$$

$$
\left\langle A_q^a(x)X\right\rangle_+ = \sum_{i=1}^{N} f^{ac_i e}\,\theta(x^4-x_i^4)\delta^{(4)}(x^{\rm tr}-x_i^{\rm tr})\left\langle A_q^e(x_i)X\backslash\pi^{c_i}(x_i)\right\rangle_+
$$

$$
+\left\langle A_q^a(x^{\rm tr})X\right\rangle_+
$$

Bose statistics and the consistency of the procedure – a same Green function can be determined in various ways by the recursion, and the 2-point function $\left\langle A(x)\tilde{B}(x')\right\rangle_+$ *does* depend on x^4 and x'^4 – fix the "integration constants" $\left\langle \tilde{B}^{q\,a}(x^{\rm tr})X\right\rangle_+$ and $\left\langle A_q^a(x^{\rm tr})X\right\rangle_+$ to zero, except for the case of the Green functions of the fields A and $\tilde{B}$ alone (corresponding to $N=P=0$ in (4.4)). The only information that we have for the latter up to this point is that they depend only on the transverse coordinates. As we did for the 2-point functions, we have to take into account the effect of the boundary. The complete equations for these Green functions, following from the second of the field equations (2.11), read (with $\lambda_+=-1$)

$$
\partial_{x^4}\left\langle A(x)\prod_1^L A(w_i^{\rm tr})\prod_1^M \tilde{B}(z_i^{\rm tr})\right\rangle = -\delta(x^4)\left\langle A(x)\prod_1^L A(w_i^{\rm tr})\prod_1^M \tilde{B}(z_i^{\rm tr})\right\rangle \tag{4.6}
$$

whereas the decoupling condition expressed by (2.9) yields the same equation but with the opposite sign for the right-hand side. On the other hand, use of the first of the field equations (2.11) does not lead to a contradictory sign. Hence the result

$$
\begin{aligned}
&\left\langle \prod_1^L A(w_i^{\rm tr})\prod_1^M \tilde{B}(z_i^{\rm tr})\right\rangle = 0 \quad \text{for} \quad L\neq 0\\
&\left\langle \prod_1^M \tilde{B}(z_i^{\rm tr})\right\rangle \quad \text{arbitrary}
\end{aligned} \tag{4.7}
$$

Let us come back to the Green functions involving the Lagrange fields. Taking into account the expression (3.4) for the propagators of I_+, we can write the recursion relations for these Green functions in the following form:

$$
\begin{aligned}
\left\langle \tilde{B}^{q\,a}(x)X\right\rangle_+ \;&=\; \sum_{i=1}^{N} f^{e_i c_i e}\left\langle \tilde{B}^{q\,a}(x)A_{q_i}^{e_i}(x_i)\right\rangle_+\left\langle \tilde{B}^{q_i\,e}(x_i)X\backslash\pi^{c_i}(x_i)\right\rangle_+ \\
&\quad +\sum_{j=1}^{P}\varepsilon^{m_j n_j pq} f^{e_j d_j e}\left\langle \tilde{B}^{q\,a}(x)A_{q_j}^{e_j}(y_j)\right\rangle_+\left\langle A_{q_j}^{e}(y_j)X\backslash\pi^{m_j n_j\,d_j}(y_j)\right\rangle_+
\end{aligned} \tag{4.8}
$$

$$
\left\langle A_q^a(x)X\right\rangle_+ \;=\; -\sum_{i=1}^{N} f^{e_i c_i e}\left\langle A_q^a(x)\tilde{B}^{q_i\,e_i}(x_i)\right\rangle_+\left\langle A_{q_i}^{e}(x_i)X\backslash\pi^{a_i}(x_i)\right\rangle_+ \tag{4.9}
$$

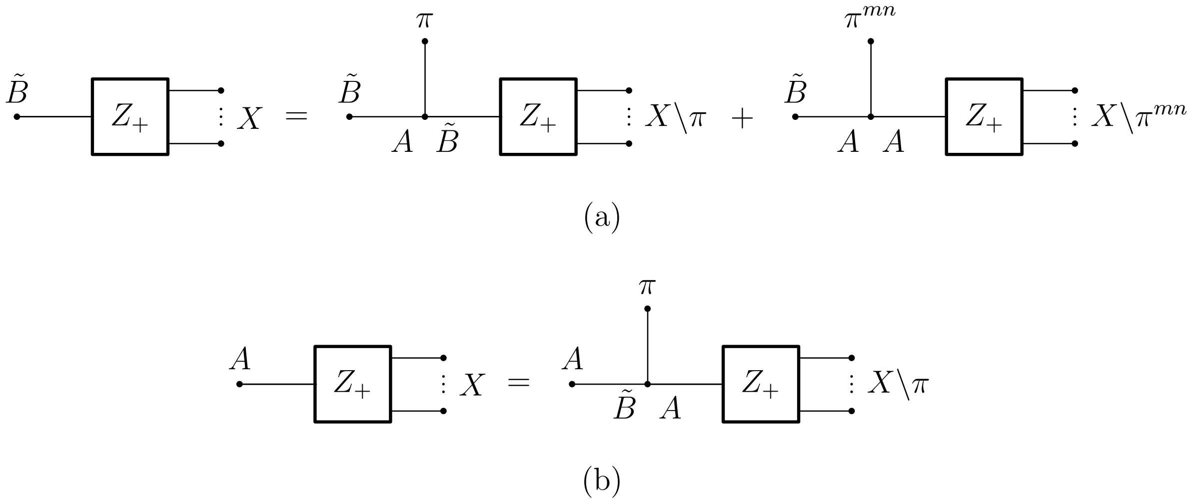

Figure 2: *Diagrammatic representation of* (4.8) (a) *and* (4.9) (b).

and this allows for the diagrammatical representation shown in Fig. 2.

The non-vanishing Green functions generated by this procedure divide in two classes.

The first class is made of the Green functions of the type

$$\begin{gathered}
\left\langle (A)(\tilde{B})(\pi)^N \right\rangle \ , \quad \left\langle (\tilde{B})(\pi)^N(\pi^{mn}) \right\rangle \ , \quad \left\langle (\tilde{B})^2(\pi)^N(\pi^{mn}) \right\rangle \ , \\
\left\langle (A)(\pi)^N \right\rangle \ , \quad \left\langle (\tilde{B})(\pi)^N \right\rangle \qquad (N \text{ arbitrary})
\end{gathered} \tag{4.10}$$

One sees that they are completely determined by the recursion relations and the two-point functions. They correspond in fact to the tree graphs generated by the Feynman rules defined by the propagators (3.4) and the BF vertex read off from the action. A typical example is shown in Fig. 3

Remark:

As we will see in Section 5, the residual Ward identities will impose some constraints on these Green functions, more precisely a transversality condition for the propagator $\left\langle \tilde{B}\tilde{B} \right\rangle$ *(see (5.10)). One of the consequences is that the propagator* $\left\langle \pi\tilde{B} \right\rangle$ *(3.4) will vanish. Therefore, all the Green functions of the type* $\left\langle (\tilde{B})(\pi)^N \right\rangle$ *– shown in Fig. 3 – will be zero, once the residual gauge invariance is taken into account, since they involve this propagator.*

The second class is made of the Green functions

$$\left\langle (\tilde{B})^M(\pi)^N \right\rangle \qquad (M \geq 2, \ N \text{ arbitrary}) \tag{4.11}$$

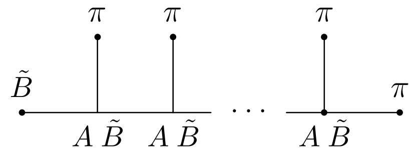

Figure 3: *Diagrammatic representation of* $\left\langle(\tilde{B})(\pi)^N\right\rangle$.

For $N=0$, they are the arbitrary Green functions of $\tilde{B}$ (see second line of (4.7)). Those for $N\geq 1$ are determined from the former through the recursion relations (4.8). They correspond to the tree graphs generated by the same set of Feynman rules, but starting from a trunk given by one of the Green functions $\left\langle(B)^M\right\rangle$.

A priori, we still have to examine the consequences of the equations of motion for the Lagrange multiplier fields. Indeed, the fourth of the field equations (2.11) allows to compute directly $\langle\pi^{mn}\varphi\cdots\varphi\rangle$ in terms of Green functions with the field π^{mn} replaced by a derivative of the field A, and the third one leads to a similar dependance for $\langle\pi\varphi\cdots\varphi\rangle$. However, the latter Green functions – except the ones involving only the Lagrange multiplier fields – have already been generated by the recursion relations (4.8) and (4.9) following from the first two equations (2.11). Therefore, we have to address the problem of consistency between these two procedures. For this purpose, let us rewrite (5.1) as a functional operator $W(x)$

$$W(x)Z_+\equiv\left\{\partial_q\frac{\delta}{\delta\tilde{J}_{q+}}-\int_0^{+\infty}\sum_\varphi\left[J^\varphi\frac{\delta}{\delta J^\varphi}\right]\right\}Z_+ \tag{4.12}$$

and, define the linearized operators M^π, $M^{\pi^{mn}}$ corresponding to the third and fourth equations (2.11) as

$$\begin{aligned}
M^\pi Z_+ &\equiv \left\{\frac{\delta}{\delta J_\pi}+\partial_q\frac{\delta}{\delta\tilde{J}_q}+\left[\frac{\delta Z_+}{\delta J^q},\frac{\delta}{\delta\tilde{J}_q}\right]+\left[\frac{\delta Z_+}{\delta\tilde{J}_q},\frac{\delta}{\delta J^q}\right]\right\}Z_+\\
M^{\pi^{mn}}Z_+ &\equiv \left\{\frac{\delta}{\delta J_{\pi^{mn}}}+\varepsilon^{mnpq}\partial_p\frac{\delta}{\delta J^q}+2\varepsilon^{mnpq}\left[\frac{\delta Z_+}{\delta J^p},\frac{\delta}{\delta J^q}\right]\right\}Z_+
\end{aligned}$$

Thus, it is easy to check that

$$\begin{aligned}
\left[M^{\pi\,a}(x),W^b(y)\right]&=0\\
\left[M^{\pi^{mn}\,a}(x),W^b(y)\right]&=0
\end{aligned} \tag{4.13}$$

which is nothing else than a consistency relation between the two procedures. This concludes the analysis of the solution I_+.

5 Ward Identities and Current Algebra

In order to get the Ward identities expressing the residual gauge invariance of the theory, i.e. the invariance under the gauge transformations (2.2) and (2.3), one has to integrate the local Ward identities (2.15), (2.16) over x^4. This step suffers from a long distance problem inherent to the choice of an axial gauge. Indeed, from a naive computation, these residual Ward identities would have the form

$$\int_{-\infty}^{+\infty} dx^4 \left\{ \sum_{\varphi} [J_\varphi, \varphi] - \partial_q J^q \right\} = -\lambda_\pm \partial_q \tilde{B}^q_\pm \tag{5.1}$$

$$\begin{aligned} \int_{-\infty}^{+\infty} dx^4 \left\{ [J^{mn4}, A_4] - \varepsilon^{mnpq} [\tilde{J}_q, A_p] + [J^\pi, \pi^{mn}] + \varepsilon^{mnpq} \partial_p \tilde{J}_q \right\} = \\ = -\lambda_\pm \varepsilon^{mnpq} \left(\partial_p A_{q\pm} + \tfrac{1}{2} \left[A_{p\pm}, A_{q\pm} \right] \right) \end{aligned} \tag{5.2}$$

Comparing with the local Ward identities (2.15), (2.16), one sees that the validity of (5.1), (5.2) is equivalent to the conditions

$$\int_{-\infty}^{+\infty} dx^4 \, \partial_4 \left\langle \pi \varphi \cdots \varphi \right\rangle = 0 \tag{5.3}$$

$$\int_{-\infty}^{+\infty} dx^4 \, \partial_4 \left\langle \pi^{mn} \varphi \cdots \varphi \right\rangle = 0 \tag{5.4}$$

However these conditions are not fulfilled for neither of the solutions I or II discussed in Section 3. This is essentially due to the fact that, for any of these solutions, at least some of the propagators $\langle \varphi(x) \varphi'(x') \rangle$ do not vanish at infinite x^4. For the solution I_+, for instance (see (3.4)), this is the case for $\varphi(x) = A(x)$ and for $\varphi(x) = \pi^{mn}(x)$, and it is not difficult to see, by examining examples involving low-point functions, e.g. $\left\langle \pi A \tilde{B} \right\rangle$ or $\left\langle \pi^{mn} \tilde{B} \right\rangle$, that this leads to a violation of both conditions (5.3), (5.4).

In order to cure this pathological feature of the axial gauge, we may introduce an infrared regularization. It turns out that an appropriate way is to replace the gauge terms in (2.5) by

$$\Sigma^\varepsilon_{\mathrm{gf}} = \mathrm{Tr} \int d^5x \, e^{\varepsilon (x^4)^2} \left\{ \pi A_4 + \frac{1}{2} \pi^{mn} B_{mn4} \right\} \tag{5.5}$$

where the IR cut-off ε is a positive number. The resulting modification of the field equations (2.11) consists of the substitutions

$$\begin{array}{ll} \pi \longrightarrow e^{\varepsilon (x^4)^2} \pi & \pi^{mn} \longrightarrow e^{\varepsilon (x^4)^2} \pi^{mn} \\ J_\pi \longrightarrow e^{-\varepsilon (x^4)^2} J_\pi & J_{\pi^{mn}} \longrightarrow e^{-\varepsilon (x^4)^2} J_{\pi^{mn}} \end{array} \tag{5.6}$$

This leads to a damping factor $\varepsilon^{-N\varepsilon (x^4)^2}$ in front of each Green function, where N is its number of fields π and π^{mn}. This is sufficient for guarantying the validity of the conditions (5.3), (5.4), hence of the residual gauge Ward identities (5.1) and (5.2). We don't need to write the latter again since they don't depend explicitly on ε.

Remarks:

1. *The introduction of the infrared regularization does not change anything to the discussion and to the conclusions of Sections 3 and 4.*

2. *Since the infrared cut-off ε appears in the gauge fixing term, one expects the physical quantities not to depend on it. This will actually be the case for the current Green functions to be defined below.*

Let us now examine the consequences of the residual Ward identities, for each of the solution I and II, keeping ourselves on the + side.

Solution I_+

The Ward identity (5.1), which in terms of Green functions, reads

$$\int_0^{+\infty} dx^4 \left\{ \sum_\varphi f^{abc} \delta^{(5)}(x-y) \left\langle \varphi^c(y) X \backslash \varphi^b(y) \right\rangle^\varepsilon \right\} = \partial_q \left\langle \tilde{B}^{q^a}(x) X \right\rangle^\varepsilon \Big|_{x^4=+0} \tag{5.7}$$

– the upperscript ε reminds one that the theory is now regularized – gives essentially restrictions to the Green functions of the field $\tilde{B}$. For those involving only $\tilde{B}$, this gives the equations

$$\begin{aligned} &\partial_q \left\langle \tilde{B}^{q\,a}(x^{\mathrm{tr}}) \tilde{B}^{q_1\,a_1}(x_1^{\mathrm{tr}}) \cdots \tilde{B}^{q_N\,a_N}(x_N^{\mathrm{tr}}) \right\rangle = \\ &+ \sum_{i=1}^N f^{aa_ib} \delta^{(4)}(x^{\mathrm{tr}} - x_i^{\mathrm{tr}}) \left\langle \tilde{B}^{q_i\,b}(x^{\mathrm{tr}}) \tilde{B}^{q_1\,a_1}(x_1^{\mathrm{tr}}) \ldots \widehat{\tilde{B}^{q_i\,a_i}(x_i^{\mathrm{tr}})} \ldots \tilde{B}^{q_N\,a_N}(x_N^{\mathrm{tr}}) \right\rangle \end{aligned} \tag{5.8}$$

where the hat on an argument means the omission of the latter. We have taken into account the fact that the Green functions of $\tilde{B}$ depend only on the transverse coordinates, and we recall that they do not depend on the infrared cut-off.

For $N = 1$, in particular, we get the transversality condition

$$\partial_q \left\langle \tilde{B}^q \tilde{B}^p \right\rangle = 0 \tag{5.9}$$

which implies (see (3.4))

$$\Delta_1^3 = -\partial^2 \Delta_2^3 \tag{5.10}$$

and, subsidiarily, the vanishing of the propagators $\left\langle \pi \tilde{B} \right\rangle$ and $\langle \pi\pi \rangle$.

Now, defining the currents on the + side of the boundary $\mathcal{B}$ as

$$V^{p\,a}(x^{\mathrm{tr}}) = \lim_{x^4 \to +0} \tilde{B}^{p\,a}(x), \tag{5.11}$$

the Ward identities (5.8) imply the conservation law

$$\partial_q V^{q\,a}(x^{\rm tr}) = 0 \tag{5.12}$$

and the equal-time current algebra

$$\Big[V^{0\,a}(x^{\rm tr}), V^{p\,b}(x'^{\rm tr})\Big]_{x_0=y_0} = f^{abc}\delta^{(3)}(\mathbf{x}-\mathbf{x}')V^{p\,c}(x^{\rm tr}) \tag{5.13}$$

Turning now towards the Ward identity (5.2) we see that its right-hand side vanishes since the solution considered here corresponds to a Dirichlet condition for the field A. It therefore does not gives new information concerning the physics on $\mathcal{B}$.

Solution II_+

This solution corresponding to a Dirichlet condition for the field $\tilde{B}$, it is the Ward identity (5.1) which now becomes empty on the boundary.

On the other hand, the hard breaking of (5.2) caused by the presence of the boundary prevents us to interpret it as a current algebra. One sees actually that its interpretation, on the boundary, is simply the equation

$$F_{mn} = 0, \tag{5.14}$$

i.e. the vanishing of the Yang-Mills strength. This means that this solution gives a four-dimensional theory of topological type.

6 Conclusion

We have thus shown that, of the two possible solutions $\mathrm{I}_\pm$ and $\mathrm{II}_\pm$ of the field equations (2.11), only one, namely $\mathrm{I}_\pm$, generates a current algebra on $\mathcal{B}$ (5.13). This current algebra follows from the Ward identity describing the residual gauge invariance of the Yang-Mills type. The other residual Ward identity becomes empty, on $\mathcal{B}$, due to the Dirichlet boundary condition for the field A.

On the other hand, the solution $\mathrm{II}_\pm$ generates from the second residual Ward identity an identity which only reproduces on the boundary the field equation $F_{mn} = 0$, which was already known to hold outside from the boundary. It is the Yang-Mills residual Ward identity which is empty, in this case.

The infrared cut-off ε needed for the validity of the residual gauge invariance Ward identities does not affect the theory on the boundary which constitutes the physical output of the present considerations.

Finally, the presence of the boundary also leads to a theory which is free of radiative corrections. Indeed, it fixes the value of the integration constants in such a way that the only possible loops, which appear in the Green functions of the Lagrange multiplier field π, are products of θ-functions which close on themselves. Thus they are zero.

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