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INFINITE-DIMENSIONAL ALGEBRAS IN DIMENSIONALLY REDUCED STRING THEORY

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Abstract

We examine 4-dimensional string backgrounds compactified over a two torus. There exist two alternative effective Lagrangians containing each two $SL(2)/U(1)$ sigma-models. Two of these sigma-models are the complex and Kähler structures on the torus. The effective Lagrangians are invariant under two different $O(2,2)$ groups and by the successive applications of these groups the affine $\widehat{O}(2,2)$ Kac-Moody algebra is emerged. The latter has also a non-zero central term which generates constant Weyl rescalings of the reduced 2-dimensional background. In addition, there exists a number of discrete symmetries relating the field content of the reduced effective Lagrangians.

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It is known that higher-dimensional gravitational theories exhibit unexpected new symmetries upon reduction [1]. Dimensional reduction of the string background equations [2] with dilaton and antisymmetric field also exhibit new symmetries as for example dualities [3]–[5]. However, the exact string symmetries will necessarily be subgroups or discrete versions of the full symmetry group of the string background equations and thus, a study of the latter would be useful. The empirical rule is that the rank of the symmetry group increases by one as the dimension of the space-time is decreased by one after dimensional reduction [6]. However, the appearance of non-local currents in two-dimensions in addition to the local ones, turns the symmetry group infinite dimensional. Let us recall the $O(8,24)$ group of the heterotic string after reduction to three dimensions [7] which turns out to be the affine $\widehat{O}(8,24)$ algebra by further reduction to two dimensions [8] or the $\widehat{O}(2,2)$ algebra after the reduction of 4-dimensional backgrounds [9]. The latter generalizes the Geroch group of Einstein gravity [10]–[12]. We will examine here the "affinization" of the symmetry group of the string background equations for 4-dimensional space-times with two commuting Killing vectors and we will show the emergence of a central term. Generalization to higher dimensions is straightforward.

The Geroch group is the symmetry group which acts on the space of solutions of the Einstein equations [10]. Its counterpart in string theory, the "string Geroch group", acts, in full analogy, on the space of solutions of the one-loop beta functions equations [9]. The Geroch group, as well as its string counterpart, results by dimensional reducing four-dimensional backgrounds with zero cosmological constant over two commuting, orthogonal transitive, Killing vectors or, in other words, by compactifing M_4 to $M_2 \times T^2$. In dimensional reduced Einstein gravity, there exist two $SL(2,\mathbb{R})$ groups (the Ehlers' and the Matzner-Misner groups [13]) acting on the space of solutions, the interplay of which produce the infinite dimensional Geroch group. In the string case, we will see that apart from the Ehlers and Matzner-Misner groups acting on the pure gravitational sector, there also exist two other $SL(2,\mathbb{R})$ groups, one of which generates the familiar S-duality, acting on the antisymmetric-dilaton fields sector.

The Geroch group was also studied in the Kaluza-Klein reduction of supergravity theories [1]. It was B. Julia who showed that the Lie algebra of the Geroch group in Einstein gravity is the affine Kac-Moody algebra $\widehat{sl}(2)$ and he pointed out the existence of a central term [13]. We will show here that in the string case, after the reduction to $M_2 \times T^2$, there exist four $SL(2,\mathbb{R})$ groups, the interplay of which produce the infinite dimensional Geroch group. However, there is also a central term which rescales the metric of M_2 so that the Lie algebra of the string Geroch group turns out to be the $\widehat{sl}(2) \times \widehat{sl}(2) \simeq \widehat{o}(2,2)$ affine

Kac-Moody algebra. The appearance of a non-zero central term already at the tree-level is rather surprising since usually such terms arise as a concequence of quantization [15]. Here however, the central term acts non-trivially even at the "classical level" by constant Weyl rescalings of the reduced two-dimensional space M_2.

String propagation in a critical background $\mathcal{M}$, parametrized with coordinates (x^M) and metric $G_{MN}(x^M)$, is described by a two-dimensional sigma-model action

$$S = \frac{1}{4\pi\alpha'}\int d^2z\,(G_{MN}+B_{MN})\,\partial x^M\bar{\partial}x^N - \frac{1}{8\pi}\int d^2z\phi R^{(2)}\,, \tag{1}$$

where B_{MN}, ϕ are the antisymmetric and dilaton fields, respectively. The conditions for conformal invariance at the 1-loop level in the coupling constant α' are

$$\begin{aligned} R_{MN} - \frac{1}{4}H_{MK\Lambda}H_N{}^{K\Lambda} - \nabla_M\nabla_N\phi &= 0 \\ \nabla^M(e^\phi H_{MNK}) &= 0 \\ -R + \frac{1}{12}H_{MNK}H^{MNK} + 2\nabla^2\phi + (\partial_M\phi)^2 &= 0\,, \end{aligned} \tag{2}$$

and the above equations may be derived from the Lagrangian [16]

$$\mathcal{L} = \sqrt{-G}e^\phi(R - \frac{1}{12}H_{MNK}H^{MNK} + \partial_M\phi\partial^M\phi), \tag{3}$$

where $H_{MN\Lambda} = \partial_M B_{N\Lambda} + cycl.perm.$ is the field strength of the antisymmetric tensor field B_{MN}.

The right-hand side of the last equation in eq. (2) has been set to zero assuming that the central charge deficit δc is of order α'^2 (no cosmological constant). We will also assume that the string propagates in $M_4 \times K$ with $c(M_4) = 4 + \mathcal{O}(\alpha'^2)$ and that the dynamics is completely determined by M_4 while the dynamics of the internal space K is irrelevant for our purposes. Thus, we will discuss below general 4-dimensional curved backgrounds in which $H_{\mu\nu\rho}$ can always be expressed as the dual of H^M

$$H_{MN\Lambda} = \frac{1}{2}\sqrt{-G}\,\eta_{MN\Lambda K}H^K, \tag{4}$$

with $\eta_{1234} = +1$ and $M, N, ... = 0,1,2,3$. The Bianchi identity $\partial_{[K}H_{MN\Lambda]} = 0$ gives the constraint

$$\nabla_M H^M = 0\,, \tag{5}$$

which can be incorporated into (3) as $b\nabla_M H^M$ by employing the Lagrange multiplier b so that (3) turns out to be

$$\mathcal{L} = \sqrt{-G}e^\phi(R - \frac{1}{2}sH_MH^M + \epsilon^{-\phi}b\nabla_MH^M + \partial_M\phi\partial^M\phi). \tag{6}$$

$s = \pm 1$ for spaces of Euclidean or Lorentzian signature, respectively and we will assume that $s = -1$ since the results may easily be generalized to include the $s = +1$ case as well. We may now eliminate H_M by using its equation of motion

$$H_M = e^{-\phi}\partial_M b\,, \tag{7}$$

and the Lagrangian (6) turns out to be

$$\mathcal{L} = \sqrt{-G}e^{\phi}(R - \frac{1}{2}e^{-2\phi}\partial_M b\partial^M b + \partial_M\phi\partial^M\phi). \tag{8}$$

Let us now suppose that the space-time M_4 has an abelian space-like isometry generated by the Killing vector $\xi_1 = \frac{\partial}{\partial\theta_1}$ so that the metric can be written as

$$ds^2 = G_{11}d\theta_1^2 + 2G_{1\mu}d\theta_1 dx^\mu + G_{\mu\nu}dx^\mu dx^\nu\,, \tag{9}$$

where $\mu, \nu, ... = 0, 2, 3$ and G_{11}, $G_{1\mu}$, $G_{\mu\nu}$ are functions of x^μ. We may express the metric (9) as

$$ds^2 = G_{11}(d\theta_1 + 2A_\mu dx^\mu)^2 + \gamma_{\mu\nu}dx^\mu dx^\nu\,, \tag{10}$$

where

$$\begin{aligned}\gamma_{\mu\nu} &= G_{\mu\nu} - \frac{G_{1\mu}G_{1\nu}}{G_{11}}\,,\\ A_\mu &= \frac{G_{1\mu}}{G_{11}}\,.\end{aligned} \tag{11}$$

The metric (10) indicates the $M_3 \times S^1$ topology of M_4 and $\gamma_{\mu\nu}$ may be considered as the metric of the 3-dimensional space M_3. Space-times of this form have extensively been studied in the Kaluza-Klein reduction where A_μ is considered as a U(1)–gauge field. The scalar curvature R for the metric (10) turns out to be

$$R = R(\gamma) - \frac{1}{4}G_{11}F_{\mu\nu}F^{\mu\nu} - \frac{2}{{G_{11}}^{1/2}}\nabla^2 {G_{11}}^{1/2}\,, \tag{12}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $\nabla^2 = \frac{1}{\sqrt{-\gamma}}\partial_\mu\sqrt{-\gamma}\gamma^{\mu\nu}\partial_\nu$. By replacing (12) into (3) and integrating by parts we get the reduced Lagrangian

$$\mathcal{L} = \sqrt{-\gamma}{G_{11}}^{1/2}e^{\phi}\left(R(\gamma) - \frac{1}{4}G_{11}F_{\mu\nu}F^{\mu\nu} + \frac{1}{G_{11}}\partial_\mu G_{11}\partial^\mu\phi - \frac{1}{4}\frac{1}{G_{11}}H_{\mu\nu}H^{\mu\nu} + \partial_\mu\phi\partial^\mu\phi\right) \tag{13}$$

where $H_{\mu\nu} = H_{\mu\nu 1} = \partial_\mu B_{\nu 1} - \partial_\nu B_{\mu 1}$. (A general discussion on the dimensional reduction of various tensor fields can be found in [17]). We have taken $H_{\mu\nu\rho} = 0$ since in three dimensions

$B_{\mu\nu}$ has no physical degrees of freedom. Let us note that the Lagrangian (13) is invariant under the transformation

$$
\begin{aligned}
G_{11} &\rightarrow \frac{1}{G_{11}}\,, \\
H_{\mu\nu} &\rightarrow F_{\mu\nu}\,, \\
\phi &\rightarrow \phi - \ln G_{11}\,, \\
\gamma_{\mu\nu} &\rightarrow \gamma_{\mu\nu}\,,
\end{aligned} \tag{14}
$$

which, in terms of G_{11}, $G_{1\mu}$, $G_{\mu\nu}$, $B_{1\mu}$ and ϕ may be written as

$$
\begin{aligned}
G_{11} \rightarrow \frac{1}{G_{11}} \quad &, \quad B_{\mu 1} \rightarrow \frac{G_{\mu 1}}{G_{11}}\,, \\
G_{1\mu} \rightarrow \frac{B_{\mu 1}}{G_{11}} \quad &, \quad G_{\mu\nu} \rightarrow G_{\mu\nu} - \frac{{G_{1\mu}}^2 - {B_{\mu 1}}^2}{G_{11}}\,, \\
\phi &\rightarrow \phi - \ln G_{11}\,,
\end{aligned} \tag{15}
$$

and it is easily be recognized as the abelian duality transformation.

Let us further assume that M_3 has also an abelian spece-like isometry generated by $\xi_2 = \frac{\partial}{\partial\theta_2}$ so that $M_3 = M_2 \times S^1$. We will further assume that the two Killings (ξ_1, ξ_2) of M_4 are orthogonal to the surface M_2. Thus, the metric (9) can be written as

$$
ds^2 = G_{11} d\theta_1^2 + 2G_{12} d\theta_1 d\theta_2 + G_{22} d\theta_2^2 + G_{ij} dx^i dx^j\,, \tag{16}
$$

where $i, j, ... = 0, 3$ and G_{11}, G_{12}, G_{22}, G_{ij} are functions of x^i only. We may write the metric above as

$$
ds^2 = G_{11}(d\theta_1 + A d\theta_2)^2 + V d\theta_2^2 + G_{ij} dx^i dx^j\,, \tag{17}
$$

where

$$
A = \frac{G_{12}}{G_{11}} \quad , \quad V = \frac{G_{11}G_{22} - G_{12}^2}{G_{11}}\,. \tag{18}
$$

By further reducing (13) with respect to ξ_2 and using the fact that the only non-vanishing components of $F_{\mu\nu}$ and $H_{\mu\nu}$ are

$$
\begin{aligned}
F_{i2} &= \partial_i A\,, \\
H_{i2} &= \partial_i B\,,
\end{aligned} \tag{19}
$$

with $B = B_{21}$, we get

$$
\begin{aligned}
\mathcal{L} = & \sqrt{-G^{(2)}}{G_{11}}^{1/2} V^{1/2} e^{\phi} \Big(R(G^{(2)}) - \frac{1}{2}(\partial A)^2 \frac{G_{11}}{V} - \frac{1}{8}(\partial \ln \frac{G_{11}}{V})^2 \\
& -\frac{1}{2}(\partial B)^2 \frac{1}{G_{11}V} - \frac{1}{8}(\partial \ln G_{11}V)^2 + (\partial\tilde{\phi})^2 \Big)\,,
\end{aligned} \tag{20}
$$

where $\tilde{\phi} = \phi + \frac{1}{2}\ln G_{11}V$ and $(\partial\phi)^2 = \partial_i\phi\partial^i\phi$. Let us now introduce the two complex coordinates $\tau\,,\rho$ [18] defined by

$$\tau = \tau_1 + i\tau_2 = \frac{G_{12}}{G_{11}} + i\frac{\sqrt{G}}{G_{11}}\,, \tag{21}$$

$$\rho = \rho_1 + i\rho_2 = B_{21} + i\sqrt{G}\,, \tag{22}$$

where $G = G_{11}G_{22} - {G_{12}}^2$ is the determinant of the metric on the 2-torus $T^2 = S^1 \times S^1$, so that $\tau\,,\rho$ turn out to be the complex and Kähler structure on T^2. In terms of $\tau\,,\rho$, the Lagrangian (20) is written as

$$\mathcal{L} = \sqrt{-G^{(2)}}e^{\tilde{\phi}} \quad \left(\quad R(G^{(2)}) + 2\frac{\partial\tau\partial\bar{\tau}}{(\tau-\bar{\tau})^2} + 2\frac{\partial\rho\partial\bar{\rho}}{(\rho-\bar{\rho})^2} + (\partial\tilde{\phi})^2 \right)\,, \tag{23}$$

where $R(G^{(2)})$ is the curvature scalar of M_2. The Lagrangian above is clearly invariant under the $SL(2,I\!R) \times SL(2,I\!R) \simeq O(2,2,I\!R)$ transformation

$$\begin{aligned} \tau &\to \tau' = \frac{a\tau + b}{c\tau + d} \quad , \quad ad - bc = 1, \\ \rho &\to \rho' = \frac{\alpha\rho + \beta}{\gamma\rho + \delta} \quad , \quad \alpha\delta - \gamma\beta = 1\,. \end{aligned} \tag{24}$$

There also exist discrete symmetries acting on the $(\tau,\,\rho)$ space which leave $\tilde{\phi}$ invariant. One of these interchanges the complex and Kähler structures

$$D: \quad \tau \leftrightarrow \rho \quad , \tilde{\phi} \to \tilde{\phi}\,. \tag{25}$$

In terms of the fields $G_{11}\,, G_{12}\,, G_{22}$, and B_{12} the above transformation is written as

$$\begin{aligned} G_{11} &\overset{D}{\to} \frac{1}{G_{11}} \quad , \quad G_{12} \overset{D}{\to} \frac{B_{21}}{G_{11}}\,, \\ B_{21} &\overset{D}{\to} \frac{G_{12}}{G_{11}} \quad , \quad G_{22} \overset{D}{\to} G_{22} - \frac{{G_{12}}^2 - {B_{21}}^2}{G_{11}}\,, \end{aligned} \tag{26}$$

which may be recognized as the factorized duality.

Other discrete symmetries are [4]

$$W: \quad (\tau,\rho) \leftrightarrow \quad (\tau, -\bar{\rho}) \quad , \tilde{\phi} \to \tilde{\phi}\,, \tag{27}$$

as well as

$$R: \quad (\tau,\rho) \leftrightarrow \quad (-\bar{\tau}, \rho) \quad , \tilde{\phi} \to \tilde{\phi}\,, \tag{28}$$

with $R = DWDW$. The $W\,,R$ discrete symmetries leave invariant the fields G_{ij}, G_{11}, G_{22} and ϕ while

$$G_{12} \stackrel{W}{\to} G_{12} \quad , \quad B_{21} \stackrel{W}{\to} -B_{21}\,,$$
$$G_{12} \stackrel{R}{\to} -G_{12} \quad , \quad B_{21} \stackrel{R}{\to} -B_{21}\,. \tag{29}$$

Let us note that there exists another Lagrangian which leads to the same equations as (23). In can be constructed by using the fact that in 3-dimensions, two-forms like $F_{\mu\nu}$ and $H_{\mu\nu}$ can be written as

$$\begin{aligned} F^{\mu\nu} &= \frac{1}{\sqrt{3}}\frac{\eta^{\mu\nu\rho}}{\sqrt{-\gamma}}F_\rho\,, \\ H^{\mu\nu} &= \frac{1}{\sqrt{3}}\frac{\eta^{\mu\nu\rho}}{\sqrt{-\gamma}}H_\rho\,. \end{aligned} \tag{30}$$

The Bianchi identities for $F_{\mu\nu}$, $H_{\mu\nu}$ are then imply

$$\nabla_\mu F^\mu = 0 \quad , \quad \nabla_\mu H^\mu = 0. \tag{31}$$

Thus, we may express (13) as

$$\begin{aligned} \mathcal{L}^* &= \sqrt{-\gamma}{G_{11}}^{1/2}e^\phi \Big(R + \frac{1}{2}G_{11}F_\mu F^\mu + {G_{11}}^{-1/2}\epsilon^{-\phi}\psi\nabla_\mu F^\mu \\ &\quad + \frac{1}{2}\frac{1}{G_{11}}H_\mu H_\mu + {G_{11}}^{-1/2}\epsilon^{-\phi}b\nabla_\mu H^\mu + \partial_\mu\phi\partial^\mu\phi\Big)\,, \end{aligned} \tag{32}$$

where the constraints (31) have been taken into account by employing the auxiliary fields $(b,\,\psi)$. The equations of motions for the H_μ, F_μ give

$$\begin{aligned} F_\mu &= {G_{11}}^{-3/2}e^{-\phi}\partial_\mu\psi\,, \\ H_\mu &= {G_{11}}^{1/2}e^{-\phi}\partial_\mu b\,, \end{aligned} \tag{33}$$

so that $\mathcal{L}^*$ is written as

$$\begin{aligned} \mathcal{L}^* &= \sqrt{-\gamma}{G_{11}}^{1/2}e^\phi \Big(R(\gamma) - \frac{1}{2}\frac{1}{{G_{11}}^2}e^{-2\phi}\partial_\mu\psi\partial^\mu\psi \\ &\quad - \frac{1}{2}e^{-2\phi}\partial_\mu b\partial^\mu b + \partial_\mu\phi\partial^\mu\phi\Big)\,. \end{aligned} \tag{34}$$

If we further reduce it with respect to ξ_2, we get

$$\begin{aligned} \mathcal{L}^* = \sqrt{-G^{(2)}}{G_{11}}^{1/2}V^{1/2}e^\phi \quad (\quad & R(G^{(2)}) + \frac{1}{2}\frac{\partial V}{V}\frac{\partial G_{11}}{G_{11}} - \frac{1}{2}\frac{1}{{G_{11}}^2}e^{-2\phi}(\partial\psi)^2 \\ & + \frac{1}{2}e^{-2\phi}(\partial b)^2 + (\partial\phi)^2\Big)\,. \end{aligned} \tag{35}$$

The two Lagrangians $\mathcal{L}$, $\mathcal{L}^*$ given by (20) (or (23)) and (35), respectively lead to the same equations of motions. $\mathcal{L}$ is invariant under $SL(2,I\!R)\times SL(2,I\!R)$ while the symmetries of $\mathcal{L}^*$ are less obvious. In order the invariance properties of both $\mathcal{L}$, $\mathcal{L}^*$ to become transparent, we adapt the following parametrization

$$G_{11} = e^{-\phi}\sigma \quad , \quad V = e^{-\phi}\frac{\mu^2}{\sigma} \tag{36}$$

$$G_{ij} = e^{-\phi}\frac{\lambda^2}{\sigma}\eta_{ij} \quad , \tag{37}$$

where $\eta_{ij} = (-1, 1)$. The metric (17) is then written as

$$ds^2 = e^{-\phi}\sigma(d\theta_1 + Ad\theta_2)^2 + e^{-\phi}\frac{1}{\sigma}(\mu^2 d\theta_2^2 + \lambda^2\eta_{ij}dx^i dx^j)\,. \tag{38}$$

As a result, $\mathcal{L}$, $\mathcal{L}^*$ turn out to be

$$\begin{aligned}\mathcal{L} \;\; = \;\; & \mu\left(2\partial\ln\mu\partial(\frac{e^{-\phi/2}\lambda\mu^{1/2}}{\sigma^{1/2}}) - \frac{1}{2}\frac{\sigma^2}{\mu^2}(\partial A)^2 - \frac{1}{2}(\partial\ln\frac{\sigma}{\mu})^2\right. \\ & \left. -\frac{1}{2}\frac{e^{2\phi}}{\mu^2}(\partial B)^2 - \frac{1}{2}(\partial\ln e^{-\phi}\mu)^2\right) \,,\end{aligned} \tag{39}$$

and

$$\mathcal{L}^* = \mu\left(2\partial\mu\partial\ln\lambda - \frac{1}{2}\frac{1}{\sigma^2}(\partial\sigma)^2 - \frac{1}{2}\frac{1}{\sigma^2}(\partial\psi)^2 - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}e^{-2\phi}(\partial b)^2\right)\,. \tag{40}$$

Note that (A, ψ) and (B, b) are related through the relations

$$\partial_i A \;\; = \;\; -\frac{1}{\sqrt{3}}\varepsilon_{ij}\frac{\mu}{\sigma^2}\eta^{jk}\partial_k\psi\,, \tag{41}$$

$$\partial_i B \;\; = \;\; -\frac{1}{\sqrt{3}}\varepsilon_{ij}e^{-2\phi}\mu\eta^{jk}\partial_k b\,, \tag{42}$$

where $\varepsilon_{12} = 1$ is the antisymmetric symbol in two-dimensions.

Let us now define, in addition to the $(\tau,\,\rho)$ fields given in eqs. (21,22), the complex fields $(S,\,\Sigma)$

$$S = b + ie^{\phi} \quad , \quad \Sigma = \psi + i\sigma\,. \tag{43}$$

Then $\mathcal{L}, \mathcal{L}^*$ may be expressed as

$$\mathcal{L} \;\; = \;\; \mu\left(2\partial\ln\mu\partial(\frac{e^{-\phi/2}\lambda\mu^{1/2}}{\sigma^{1/2}}) + 2\frac{\partial\tau\partial\bar{\tau}}{(\tau-\bar{\tau})^2} + 2\frac{\partial\rho\partial\bar{\rho}}{(\rho-\bar{\rho})^2}\right) \tag{44}$$

$$\mathcal{L}^* \;\; = \;\; \mu\left(2\partial\mu\partial\ln\lambda + 2\frac{\partial S\partial\bar{S}}{(S-\bar{S})^2} + 2\frac{\partial\Sigma\partial\bar{\Sigma}}{(\Sigma-\bar{\Sigma})^2}\right)\,. \tag{45}$$

Thus, there exist four $SL(2,\mathbb{R})/U(1)$–sigma models, $\mathcal{L}$ is invariant under the $SL(2,\mathbb{R})\times SL(2,\mathbb{R})$ transformations (24) and $\mathcal{L}^*$ is invariant under

$$S\rightarrow\frac{kS+m}{nS+\ell}\quad,\quad\Sigma\rightarrow\frac{\kappa\Sigma+\eta}{\nu\Sigma+\theta}\,. \tag{46}$$

These transformation do not affect μ. There also exist discrete Z_2 transformations, besides those that have already been noticed in eqs. (25,27,28), namely

$$\begin{aligned} D':(S,\Sigma) &\leftrightarrow (\Sigma,S) &(47)\\ W':(S,\Sigma) &\leftrightarrow (S,-\bar{\Sigma}) &(48)\\ R':(S,\Sigma) &\leftrightarrow (-\bar{S},\Sigma)\,. &(49)\end{aligned}$$

Moreover, the transformations

$$\begin{aligned} N:(\tau,\rho)\leftrightarrow(S,\Sigma)\quad,\quad \lambda\leftrightarrow e^{-\phi/2}\frac{\mu^{1/2}}{\sigma^{1/2}}\lambda\,, &\qquad(50)\\ N':(\tau,\rho)\leftrightarrow(\Sigma,S)\quad,\quad \lambda\leftrightarrow e^{-\phi/2}\frac{\mu^{1/2}}{\sigma^{1/2}}\lambda\,, &\qquad(51)\end{aligned}$$

indentify the two Lagrangians and thus, may be considered as the string counterpart of the Kramer-Neugebauer symmetry [19]. Note that $\mathcal{L},\mathcal{L}^*$ may also be written as

$$\begin{aligned}\mathcal{L} &= \mu\left(2\partial\ln\mu\partial(\frac{e^{-\phi/2}\lambda\mu^{1/2}}{\sigma^{1/2}})-\frac{1}{4}Tr({h_1}^{-1}\partial h_1)^2-\frac{1}{4}Tr({h_2}^{-1}\partial h_2)^2\right) &(52)\\ \mathcal{L}^* &= \mu\left(2\partial\mu\partial\lambda-\frac{1}{4}Tr({g_1}^{-1}\partial g_1)^2-\frac{1}{4}Tr({g_2}^{-1}\partial g_2)^2\right)\,. &(53)\end{aligned}$$

where the 2×2 matrices h_1, h_2, g_1 and g_2 are

$$h_1=\begin{pmatrix}\frac{\sigma}{\mu} & \frac{\sigma}{\mu}A\\ \frac{\sigma}{\mu}A & \frac{\sigma}{\mu}A^2+\frac{\mu}{\sigma}\end{pmatrix}\quad,\quad h_2=\begin{pmatrix}\frac{e^{\phi}}{\mu} & \frac{e^{\phi}}{\mu}B\\ \frac{e^{\phi}}{\mu}B & \frac{e^{\phi}}{\mu}B^2+\frac{\mu}{e^{\phi}}\end{pmatrix}\,, \tag{54}$$

$$g_1=\begin{pmatrix}\frac{1}{\sigma} & \frac{1}{\sigma}\psi\\ \frac{1}{\sigma}\psi & \frac{1}{\sigma}\psi^2+\sigma\end{pmatrix}\quad,\quad g_2=\begin{pmatrix}e^{-\phi} & e^{-\phi}b\\ e^{\phi}b & e^{\phi}b^2+e^{-\phi}\end{pmatrix}\,. \tag{55}$$

The Lagrangian $\mathcal{L}$ is invariant under the infinitesimal transformations

$$\begin{aligned}\delta\sigma=\sqrt{2}\frac{1}{\sigma}A\epsilon_1^+-2\epsilon_1^0\quad&,\quad\delta A=-\frac{1}{\sqrt{2}}(\frac{\sigma^2}{\mu^2}-A^2)\epsilon_1^+-2A\epsilon_1^0+\sqrt{2}\epsilon_1^-\,,\\ \delta\phi=-\sqrt{2}B\epsilon_2^++2\epsilon_2^0\quad&,\quad\delta B=-\frac{1}{\sqrt{2}}(\frac{e^{2\phi}}{\mu^2}-B^2)\epsilon_2^+-2B\epsilon_2^0+\sqrt{2}\epsilon_2^-\,,\end{aligned} \tag{56}$$

while $\mathcal{L}^*$ is invariant under

$$\begin{aligned}\delta\sigma=-\sqrt{2}\psi\sigma\epsilon_3^++2\sigma\epsilon_3^0\quad&,\quad\delta\psi=-\frac{1}{\sqrt{2}}(\frac{1}{\sigma^2}-\psi^2)\epsilon_3^+-2\psi\epsilon_3^0+\sqrt{2}\epsilon_3^-\,,\\ \delta\phi=\sqrt{2}b\epsilon_4^+-2\epsilon_4^0\quad&,\quad\delta b=-\frac{1}{\sqrt{2}}(e^{2\phi}-b^2)\epsilon_4^+-2b\epsilon_4^0+\sqrt{2}\epsilon_4^-\,.\end{aligned} \tag{57}$$

The above infinitesimal transformations are generated by a set of four Killing vectors ($\mathbf{K}_a^{(i)}$, $a = 1, 2, 3,\ i = 1, 2, 3, 4$) which can easily be written down by recalling that the metric

$$ds^2 = dx^2 + e^{2x} dy^2 \tag{58}$$

has a three-parameter group of isometries generated by

$$\begin{aligned} K_+ &= -\sqrt{2} y \partial_x - \frac{1}{\sqrt{2}} (e^{-2x} - y^2) \partial_y \,, \\ K_0 &= 2(\partial_x - y\partial_y) \,, \\ K_- &= \sqrt{2} \partial_y \,, \end{aligned} \tag{59}$$

which satisfy the $SL(2)$ commutation relations

$$[K_+, K_0] = 2K_+ \ \ , [K_-, K_0] = -2K_- \ \ , [K_-, K_+] = -K_0 \,. \tag{60}$$

Among these Killing vectors, let us consider $K_0^{(3)}$ which scales both ψ and σ as

$$K_0^{(3)} : \ (\psi, \sigma) \to (\alpha\psi, \alpha\sigma). \tag{61}$$

In view of eq. (41), A is also scaled as

$$A \to \frac{1}{\alpha} A \,, \tag{62}$$

so that (A, σ) is transformed into $(\frac{1}{\alpha} A, \alpha\sigma)$ which is generated by $-K_0^{(1)}$. However, $\mathcal{L}$ is not invariant unless we also scale the conformal factor λ as $\sqrt{\alpha}\lambda$. Let us denote the generator of constant Weyl transformations by k. Then we have the relation

$$K_0^{(1)} + K_0^{(3)} = k \,. \tag{63}$$

In the same way, one may see that $K_0^{(2)}$, $K_0^{(4)}$ which transform (B, ϕ) and (b, ϕ) as $(e^{-\alpha} B, \phi + \alpha)$, $(e^{\alpha}, \phi + \alpha)$ respectively satisfy

$$K_0^{(2)} + K_0^{(4)} = k. \tag{64}$$

As a result, the algebra turns out to be

$$\begin{array}{lll} [K_+^{(1)}, K_0^{(1)}] = 2K_+^{(1)} \,, & [K_-^{(1)}, K_0^{(1)}] = -2K_-^{(1)} \,, & [K_-^{(1)}, K_+^{(1)}] = K_0^{(1)} \,, \\ [K_+^{(2)}, K_0^{(2)}] = 2K_+^{(2)} \,, & [K_-^{(2)}, K_0^{(2)}] = -2K_-^{(2)} \,, & [K_-^{(2)}, K_+^{(2)}] = K_0^{(2)} \,, \\ [K_+^{(3)}, k - K_0^{(1)}] = 2K_+^{(3)} \,, & [K_-^{(3)}, k - K_0^{(1)}] = -2K_-^{(3)} \,, & [K_-^{(3)}, K_+^{(3)}] = k - K_0^{(1)} \,, \\ [K_+^{(4)}, k - K_0^{(2)}] = 2K_+^{(4)} \,, & [K_-^{(4)}, k - K_0^{(2)}] = -2K_-^{(4)} \,, & [K_-^{(4)}, K_+^{(4)}] = k - K_0^{(2)} \end{array} \tag{65}$$

If we define the generators $(h_i,\, k_i,\, f_i)$ by

$$h_i = K_0^{(i)}\,,\ f_i = K_+^{(i)}\,,\ e_i = K_-^{(i)}\,, \tag{66}$$

then the algebra (65) may be writen as

$$\begin{aligned} [h_i, h_j] &= 0\,, \\ [h_i, e_j] &= A_{ij} e_j\,, \\ [h_i, f_j] &= -A_{ij}\,, \\ [e_i, f_j] &= \delta_{ij} h_j\,, \end{aligned} \tag{67}$$

where the Cartan martix A_{ij} is

$$A_{ij} = \begin{pmatrix} a_{ij} & 0 \\ 0 & a_{ij} \end{pmatrix} \quad , \quad a_{ij} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}. \tag{68}$$

In addition, one may verify the Serre relation

$$(ade_i)^{1-A_{ij}}(e_j) = 0 \quad , \quad (adf_i)^{1-A_{ij}}(f_j) = 0\,. \tag{69}$$

As a result, the algebra generated by the successive applications of the transformations (56,57) is the affine Kac-Moody algebra $\widehat{o}(2,2)$ with a central term corresponding to constant Weyl rescalings of the 2-dimensional background metric. The central term survives in higher dimensions as well, since its emergence is related to the existence of two alternative effective Lagrangians after reducing the 3-dimensional theory down to two dimensions over an abelian isometry. It is the interplay of the symmetries of these Lagrangians which produce the Kac-Moody algebra.

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