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The Calculation of the Two-Loop Spin Splitting Functions $P_{ij}^{(1)}(x)$

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Abstract

We present the calculation of the two-loop spin splitting functions $P_{ij}^{(1)}(x)$ $(i,j=q,g)$ contributing to the next-to-leading order corrected spin structure function $g_1(x,Q^2)$. These splitting functions, which are presented in the $\overline{\rm MS}$ scheme, are derived from the order α_s^2 contribution to the anomalous dimensions γ_{ij}^m $(i,j=q,g)$. The latter correspond to the local operators which appear in the operator product expansion of two electromagnetic currents. Some of the properties of the anomalous dimensions will be discussed. In particular we find that in order α_s^2 the supersymmetric relation $\gamma_{qq}^m+\gamma_{gq}^m-\gamma_{qg}^m-\gamma_{gg}^m=0$ is violated.

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1 Introduction

During the last few years there has been a great deal of activity in the area of polarized lepton-hadron physics both from the experimental as well as the theoretical side. This interest started with the discovery of the EMC-experiment [1] that the Ellis-Jaffe sum rule [2], which represents the first moment of the spin structure function $g_1(x, Q^2)$, was violated by the combined SLAC-EMC data [1, 3]. This discrepancy between theory and experiment, also called the "spin crisis", came as a great surprise because one expected that sum rules derived in the context of the constituent quark model, which is valid at low energy scales, should also hold at large energy scales characteristic of the current quark (parton) regime. In particular the constituent quark model assumes that the spin of the proton can be mainly attributed to its valence quarks and the sea quark contribution is negligible small. This assumption leads to a value of the Ellis-Jaffe sum rule which is appreciably larger than the one found by experiment. Although more recent experiments [4, 5, 6] lead to a result which is closer to the theoretical prediction the discrepancy is still large enough to warrant explanation.

Many theorists have tried to explain the above discrepancy (for recent reviews see [7]) in the framework of perturbative and also non-perturbative QCD. From this theoretical work one can draw the conclusion that the interpretation of the spin structure function $g_1(x, Q^2)$, using the ideas of the operator product expansion (OPE) and the QCD improved parton model, is not as simple as that given to the structure functions which show up in unpolarized lepton-hadron scattering. In particular the axial vector operator is renormalized due to the Adler-Bell-Jackiw anomaly. Therefore the interpretation that the polarized parton densities represent the spin carried by the corresponding partons does not hold anymore. Fortunately this operator cancels in the Bjorken sum rule [8] so that the latter has a more reliable theoretical basis. It is therefore no surprise that its result is in agreement with recent data [4, 5, 6]. The above theoretical work also led to many different parametrizations of the parton densities in terms of which the spin structure function $g_1(x, Q^2)$ can be expressed. One of the key issues is the role of the gluon density which can account for the negative contribution to the Ellis-Jaffe sum rule depending on the chosen scheme. However if one wants to give a complete next-to-leading order (NLO) description of $g_1(x, Q^2)$, and not only its first moment, one needs a full knowledge of the order α_s coefficient functions, which are known (see e. g. [9, 10, 11]) and the order α_s^2 corrected Altarelli-Parisi (AP) spin splitting functions P_{ij} $(i, j = q, g)$. The lowest order AP-splitting functions $P_{ij}^{(0)}$ have been calculated in [12] and [13] respectively using different methods. In [12] the operator product expansion (OPE) techniques are applied to obtain the anomalous dimensions of the composite operators appearing in the spin dependent part of the current-current correlation function. The latter appears in the expression for the deep inelastic cross section. The authors in [13] have used the parton model approach. The NLO (order α_s^2) splitting functions $P_{qq}^{(1),S}$ and $P_{qg}^{(1)}$ have been computed in [9] using the standard techniques of perturbative QCD. They emerge while performing mass factorization on the order α_s^2 corrected parton cross sections of the processes $\gamma^* q$ and $\gamma^* g$ which contribute to the deep inelastic spin structure function. Unfortunately the remaining splitting functions $P_{gq}^{(1)}$ and $P_{gg}^{(1)}$ could not be obtained in this way since they do not show up in the mass factorization of order α_s^2 corrected parton cross sections. This can be traced back to the phenomenon that there is no direct coupling of the virtual photon γ^* or any other electroweak vector boson to the gluon. Therefore $P_{gq}^{(1)}$ and $P_{gg}^{(1)}$ will appear in the mass factorizaton of the order α_s^3 corrected parton cross sections which are very difficult to calculate. In order to avoid the above complication we will resort to the standard OPE techniques to calculate the missing splitting functions which are derived from the inverse Mellin transform of the anomalous dimensions of composite operators.

The paper is organized as follows. In section 2 we introduce our notations and present a short discussion of the composite twist-2 operators contributing to the spin structure function $g_1(x,Q^2)$. Here we also derive the general form of the renormalized and unrenormalized operator matrix elements (OME) where the operators are sandwiched between polarized quark and gluon states. The calculation of the OME's is presented in section 3, from which one extracts the anomalous dimensions and the AP splitting functions which are presented in the $\overline{\rm MS}$ scheme. Further we give the lowest order coefficient functions of $g_1(x,Q^2)$ in the same scheme. The properties of the anomalous dimensions are discussed in section 4. In Appendix A one can find the operator vertices needed for the computation of the operator matrix elements in section 3. The tensorial reduction of the Feynman integrals which show up in the calculation is discussed in Appendix B.

2 Operators contributing to the spin structure function $g_1(x,Q^2)$

In this section we specify the composite operators which appear in the light-cone expansion of two electromagnetic currents. Furthermore we present the operator matrix elements (OME's) as a power series in the strong coupling constant. The coefficients of the perturbation series are determined by the renormalization group (Callan-Symanzik) equations. We will write the OME's in the most general way so that they can be used to extract the anomalous dimensions of the composite operators. The light-cone expansion of two electromagnetic currents is given in [12] and reads as follows

$$
\begin{aligned}
J_\mu(z)J_\nu(0) \quad \overset{z^2\to 0}{\simeq} \quad & (-g_{\mu\nu}\Box + \partial_\mu\partial_\nu)\frac{1}{z^2 - i\varepsilon z_0}\sum_{m=0}^{\infty}\sum_i C_{i,1}^m(z^2 - i\varepsilon z_0, \mu^2, g) \\
& z_{\mu_1}\cdots z_{\mu_m} O_i^{\mu_1\cdots\mu_m}(0) - (g_{\mu\mu_1}g_{\nu\mu_2}\Box - g_{\mu\mu_1}\partial_\nu\partial_{\mu_2} - g_{\nu\mu_2}\partial_\mu\partial_{\mu_1} \\
& + g_{\mu\nu}\partial_{\mu_1}\partial_{\mu_2})\sum_{m=2}^{\infty}\sum_i C_{i,2}^m(z^2 - i\varepsilon z_0, \mu^2, g) z_{\mu_3}\cdots z_{\mu_m} O_i^{\mu_1\cdots\mu_m}(0) \\
& - i\epsilon_{\mu\nu\lambda\mu_1}\partial^\lambda \frac{1}{z^2 - i\varepsilon x_0}\sum_{m=1}^{\infty}\sum_i E_{i,1}^m(z^2 - i\varepsilon z_0, \mu^2, g) \\
& \qquad \times z_{\mu_2}\cdots z_{\mu_m} R_i^{\mu_2\cdots\mu_m}(0). \qquad (2.1)
\end{aligned}
$$

In the above we only consider the contribution of twist-2 operators. The index i of the locally gauge invariant operators $O_i^{\mu_1\cdots\mu_m}$ and $R_i^{\mu_2\cdots\mu_m}$ stands for the representation of the flavour group $SU(n_f)$. Notice that the operators are also irreducible representations of the Lorentz group which means that they are traceless and symmetric in the Lorentz indices $\mu_1\cdots\mu_m$. The Wilson coefficient functions, denoted by $C_{i,k}^m$ ($k=1,2$) and $E_{i,1}^m$, can be expressed into a perturbation series in the gauge (strong) coupling constant g. Notice that all the above quantities are renormalized which is indicated by the renormalization scale μ. The product of the two electromagnetic currents appear in the hadronic tensor defined in polarized deep inelastic lepton-hadron scattering which is given by

$$\begin{aligned} W_{\mu\nu}(p,q,s) &= \frac{1}{4\pi}\int d^4z\, e^{iqz}\langle p,s|J_\mu(z)J_\nu(0)|p,s\rangle \\ &= W^S_{\mu\nu}(p,q)+iW^A_{\mu\nu}(p,q,s). \end{aligned} \tag{2.2}$$

Here p and s denote the momentum and spin of the hadron respectively and q stands for the virtual photon momentum. The symmetric part of the hadronic tensor is given by

$$W^S_{\mu\nu}(p,q) = (-g_{\mu\nu}+\frac{q_\mu q_\nu}{q^2})F_1(x,Q^2)+(p_\mu-\frac{p\,q}{q^2}q_\mu)(p_\nu-\frac{p\,q}{q^2}q_\nu)\frac{F_2(x,Q^2)}{p\,q}, \tag{2.3}$$

while the antisymmetric part is equal to

$$W^A_{\mu\nu}(p,q,s) = -\frac{m}{p\,q}\,\epsilon_{\mu\nu\alpha\beta}q^\alpha\left[s^\beta g_1(x,Q^2)+(s^\beta-\frac{s\,q}{p\,q}p^\beta)\,g_2(x,Q^2)\right], \tag{2.4}$$

with the properties $s\cdot p=0$, $s^2=1$ and m denotes the mass of the hadron. The Bjorken scaling variable is given by $x=Q^2/(2p\,q)$ and $Q^2=-q^2>0$. The spin averaged structure functions are denoted by $F_k(x,Q^2)(k=1,2)$. In polarized electroproduction one has in addition the longitudinal spin structure function $g_1(x,Q^2)$ and the transverse spin structure function $g_2(x,Q^2)$. The twist-2 operators $O_i^{\mu_1\cdots\mu_m}(0)$ corresponding to the spin averaged structure functions are given in the literature and their anomalous dimensions have been calculated up to two-loop order [14]–[17]. The twist-2 operators contributing to the spin structure functions are given by [12]

$$\begin{aligned} R^{\mu_1\cdots\mu_m}_{NS,q}(z) &= i^n\;S\;\left\{(\bar\psi(z)\gamma_5\gamma^{\mu_1}D^{\mu_2}\cdots D^{\mu_m}\frac{1}{2}\lambda_i\psi(z)-(\mathrm{traces})\right\}, &(2.5)\\ R^{\mu_1\cdots\mu_m}_{S,q}(z) &= i^n\;S\;\left\{(\bar\psi(z)\gamma_5\gamma^{\mu_1}D^{\mu_2}\cdots D^{\mu_m}\psi(z)-(\mathrm{traces})\right\}, &(2.6)\\ R^{\mu_1\cdots\mu_m}_{S,g}(z) &= i^n\;S\;\left\{\frac{1}{2}\epsilon^{\mu_1\alpha\beta\gamma}Tr(F_{\beta\gamma}(z)D^{\mu_2}\cdots D^{\mu_{m-1}}F_\alpha^{\mu_m}(z))-(\mathrm{traces})\right\}. &(2.7) \end{aligned}$$

The symbol S in front of the curly brackets stands for the symmetrization of the indices $\mu_1\cdots\mu_m$ and λ_i is the flavour group generator of $SU(n_f)$. The quark and the gluon field tensor are given by $\psi(z)$ and $F^a_{\mu\nu}(z)$ respectively and $F_{\mu\nu}=F^a_{\mu\nu}T^a$ where T^a stands for the generator of the colour group $SU(N)$ $(N=3)$. The covariant derivative is given by $D_\mu=\partial_\mu+igT^aA^a_\mu(x)$ where $A^a_\mu(z)$ denotes the gluon field. From eqs. (2.5)–(2.7) one infers that with respect to the flavour group one can distinguish the local operators in a non-singlet part represented by $R_{NS,q}$ and in a singlet part consisting of $R_{S,q}$ and $R_{S,g}$.

In the Bjorken limit ($Q^2\to\infty, x=Q^2/(2p\,q)$ fixed) the current-current correlation function in (2.2) is dominated by the light cone $z^2=0$ so that it is justified to make a light cone expansion for the product of the two electromagnetic currents. When $Q^2\to\infty$ the leading contribution of $g_1(x,Q^2)$ consists of the twist-2 operators listed in (2.5)–(2.7) whereas $g_2(x,Q^2)$ also receives contributions of twist-3 operators which are not given in the expansion in eq. (2.1). Since we are only interested in the longitudinal spin structure function $g_1(x,Q^2)$ we can limit ourselves to the renormalization of the twist-2 operators mentioned in (2.5)–(2.7). Inserting the light cone expansion for $J_\mu(z)J_\nu(0)$ in (2.2) one can derive the following relation

$$\int_0^1 dx\;x^{m-1}g_1(x,Q^2)=\sum_i A^m_i(p^2,\mu^2,g)\tilde E^m_{i,1}(Q^2,\mu^2,g)\,;\qquad m\;\;\mathrm{odd}. \tag{2.8}$$

The left-hand side of the above equation stands for the Mellin transform of $g_1(x,Q^2)$ and the right-hand side is given by the operator matrix element (OME).

$$\langle p,s|R_i^{\mu_1\cdots\mu_m}(0)|p,s\rangle = i^m A_i^m(p^2,\mu^2,g) S\left\{(s^{\mu_1}p^{\mu_2}\cdots p^{\mu_m}) - (\text{traces})\right\}, \tag{2.9}$$

with $i=$NS,S, and $\tilde{E}_i^m$ stands for the coefficient function.

$$\begin{aligned}\tilde{E}_i^m(Q^2,\mu^2,g) &= -\frac{1}{4}(Q^2)^m \\ &\times\left(\frac{\partial}{\partial q^2}\right)^{m-1}\int d^4z e^{iqz}\frac{1}{z^2-i\varepsilon z_0}E_{i,1}^m(z^2-i\varepsilon z_0,\mu^2,g).\end{aligned} \tag{2.10}$$

The Q^2-evolution of the spin structure function is determined by the anomalous dimensions of the composite operators in eqs. (2.5)–(2.7). They are obtained from the renormalized partonic OME's

$$\langle j,p,s|R_{k,i}^{\mu_1\cdots\mu_m}|j,p,s\rangle = A_{k,ij}^m(p^2,\mu^2,g)S\left\{(s^{\mu_1}p^{\mu_2}\cdots p^{\mu_{m_1}}) - (\text{traces})\right\}, \tag{2.11}$$

where now the quark and gluon operators are sandwiched between quark and gluon states. The indices in (2.11) stand for $k=NS,S$ and $i=q,g$; $j=q,g$. The $A_{k,ij}^m$ are derived from the Fourier transform into momentum space of the connected Green's functions

$$\langle 0|T(\bar{\phi}_j(x)R_{k,i}^{\mu_1\cdots\mu_m}(0)\phi_j(y)|0\rangle_c, \tag{2.12}$$

where the external lines are amputated. The fields $\phi_i(x)$ stand either for the quark fields $\psi(x)$ or for the gluon fields $A_\mu^a(x)$. The renormalized partonic OME's satisfy the Callan-Symanzik equations

$$\left[\mu\frac{\partial}{\partial\mu}+\beta(g)\frac{\partial}{\partial g}+\delta(\alpha,g)\frac{\partial}{\partial\alpha}+\gamma_{NS,qq}^m(g)\right]A_{NS,qq}^{(m)}(p^2,\mu^2,g,\alpha)=0, \tag{2.13}$$

and

$$\left[(\mu\frac{\partial}{\partial\mu}+\beta(g)\frac{\partial}{\partial g}+\delta(\alpha,g)\frac{\partial}{\partial\alpha})\delta_{ij}+\gamma_{S,ij}^m(g)\right]A_{S,jk}^{(m)}(p^2,\mu^2,g,\alpha)=0. \tag{2.14}$$

Here $\beta(g)$ denotes the β-function which in QCD is given by the following series expansion

$$\beta(g) = -\beta_0\frac{g^3}{16\pi^2}-\beta_1\frac{g^5}{(16\pi^2)^2}+\cdots \tag{2.15}$$

$$\beta_0 = \frac{11}{3}C_A-\frac{4}{3}T_f\,n_f, \tag{2.16}$$

$$\beta_1 = \frac{34}{3}C_A^2-4C_FT_fn_f-\frac{20}{3}C_AT_fn_f. \tag{2.17}$$

Further $\delta(\alpha,g)$ is the renormalization group function which controls the variation of the OME's under the gauge constant α. Choosing the general covariant gauge one obtains in QCD the following result

$$\delta(\alpha,g)=-\alpha z_\alpha\frac{g^2}{16\pi^2}+\cdots, \tag{2.18}$$

where z_α is given by

$$z_\alpha=\left(-\frac{10}{3}-(1-\alpha)\right)C_A+\frac{8}{3}T_f. \tag{2.19}$$

Furthermore the colour factors of SU(N) are defined by $C_A = N$, $C_F = (N^2-1)/(2N)$, $T_f = 1/2$ and n_f stands for the number of light flavours. The anomalous dimensions are given by the series expansion

$$\gamma_{k,ij}^m = \gamma_{k,ij}^{(0),m}\frac{g^2}{16\pi^2} + \gamma_{k,ij}^{(1),m}\left(\frac{g^2}{16\pi^2}\right)^2 + \cdots \tag{2.20}$$

Notice that for the subsequent part of this paper we do not need higher order terms in $\beta(g)$, $\delta(\alpha, g)$ and $\gamma_{k,ij}^m$. As an alternative to using the renormalized OME's the anomalous dimensions can also be derived from the operator renormalization constants $Z_{k,ij}^m$ which relate the bare operators $\hat{R}_{i,k}$ to the renormalized operators $R_{i,k}$ [1]. The renormalization of the non-singlet operator proceeds as

$$\hat{R}_{NS,q}^{\mu_1\cdots\mu_m}(z) \quad = \quad Z_{NS,qq}^m(\varepsilon, g) R_{NS,q}^{\mu_1\cdots\mu_m}(z). \tag{2.21}$$

Since the singlet operators in (2.6) and (2.7) mix among each other the operator renormalization constant becomes a matrix and we have

$$\hat{R}_{S,i}^{\mu_1\cdots\mu_m}(z) \quad = \quad Z_{S,ij}^m(\varepsilon, g) R_{S,j}^{\mu_1\cdots\mu_m}(z). \tag{2.22}$$

Now the anomalous dimensions also can be obtained from

$$\begin{aligned}
\gamma_{NS,qq}^m &= \beta(g,\varepsilon)\, Z_{NS,qq}^{-1}\frac{d}{dg} Z_{NS,qq}, \\
\gamma_{S,ij}^m &= \beta(g,\varepsilon)\,(Z_S^{-1})_{il}\,\frac{dZ_{S,lj}}{dg},
\end{aligned} \tag{2.23}$$

where

$$\beta(g,\varepsilon) = \frac{1}{2}\varepsilon g + \beta(g). \tag{2.24}$$

Here $\varepsilon = n - 4$ indicates that we will use n-dimensional regularization to regularize the ultraviolet singularities occurring in $Z_{k,ij}$ which are represented by pole terms of the type $1/\varepsilon^p$. The computation of the OME's proceeds in the following way. First one adds the operators (2.5)–(2.7) to the QCD effective lagrangian by multiplying them by sources $J_{\mu_1\cdots\mu_m}(z)$. The calculation simplifies considerably if the sources are chosen to be equal to $J_{\mu_1\cdots\mu_m}(z) = \Delta_{\mu_1}\cdots\Delta_{\mu_m}$ with $\Delta^2 = 0$. In this way one eliminates the trace terms on the right-hand side of eq. (2.11). The Feynman rules for the quark and gluon operator vertices are given in Appendix A. Starting from the bare lagrangian, which is expressed in the bare coupling constant and bare fields and operators, one obtains the following general form for the unrenormalized OME's. For the non-singlet OME we have

$$\begin{aligned}
\hat{A}_{NS,qq} \quad = \quad & 1 + \left(\frac{\hat{g}^2}{16\pi^2}\right) S_\varepsilon \left(\frac{-p^2}{\mu^2}\right)^{\varepsilon/2}\left[\gamma_{NS,qq}^{(0)}\frac{1}{\varepsilon} + a_{NS,qq}^{(1)} + \varepsilon\, b_{NS,qq}^{(1)}\right] \\
& + \left(\frac{\hat{g}^2}{16\pi^2}\right)^2 S_\varepsilon^2 \left(\frac{-p^2}{\mu^2}\right)^{\varepsilon}\left[\left\{\frac{1}{2}(\gamma_{NS,qq}^{(0)})^2 - \beta_0\gamma_{NS,qq}^{(0)}\right\}\frac{1}{\varepsilon^2}\right. \\
& + \left\{\frac{1}{2}\gamma_{NS,qq}^{(1)} - 2\beta_0\, a_{NS,qq}^{(1)} + \gamma_{NS,qq}^{(0)} a_{NS,qq}^{(1)} - \hat{\alpha}\left(\frac{d\, a_{NS,qq}^{(1)}}{d\hat{\alpha}}\right) z_\alpha\right\}\frac{1}{\epsilon} \\
& \left. + a_{NS,qq}^{(2)}\right]_{\hat{\alpha}=1}.
\end{aligned} \tag{2.25}$$

[1]Notice that in the subsequent part of the paper the unrenormalized quantities will be indicated by a hat.

Here S_ε is a factor which originates from n-dimensional regularization. It is defined by

$$S_\varepsilon = e^{\frac{\varepsilon}{2}(\gamma_e - \ln 4\pi)}. \tag{2.26}$$

The singlet quark OME can be written as

$$\hat{A}_{S,qq} = \hat{A}_{NS,qq} + \hat{A}_{PS,qq}, \tag{2.27}$$

where the pure singlet (PS) part is given by

$$\begin{aligned}\hat{A}_{PS,qq} &= \left(\frac{\hat{g}^2}{16\pi^2}\right)^2 S_\varepsilon^2 \left(\frac{-p^2}{\mu^2}\right)^\varepsilon \left[\frac{1}{2}\gamma_{S,qg}^{(0)}\gamma_{S,gq}^{(0)}\frac{1}{\varepsilon^2}\right.\\ &\left. + \left\{\frac{1}{2}\gamma_{PS,qq}^{(1)} + \gamma_{S,qg}^{(0)} a_{S,gq}^{(1)}\right\}\frac{1}{\varepsilon} + a_{PS,qq}^{(2)}\right], \end{aligned}\tag{2.28}$$

so that $\gamma_{S,qq}^{(1)} = \gamma_{NS,qq}^{(1)} + \gamma_{PS,qq}^{(1)}$.
The other OME's can be expressed in the renormalization group coefficients as

$$\begin{aligned}\hat{A}_{S,qg} &= \left(\frac{\hat{g}^2}{16\pi^2}\right) S_\varepsilon \left(\frac{-p^2}{\mu^2}\right)^{\varepsilon/2}\left[\gamma_{S,qg}^{(0)}\frac{1}{\varepsilon} + a_{S,qg}^{(1)} + \varepsilon\, b_{S,qg}^{(1)}\right]\\ &+ \left(\frac{\hat{g}^2}{16\pi^2}\right)^2 S_\varepsilon^2 \left(\frac{-p^2}{\mu^2}\right)^\varepsilon \left[\left\{\frac{1}{2}\,\gamma_{S,qg}^{(0)}\,\left(\gamma_{S,qq}^{(0)} + \gamma_{S,gg}^{(0)}\right) - \beta_0\,\gamma_{S,qg}^{(0)}\right\}\frac{1}{\varepsilon^2}\right.\\ &+ \left\{\frac{1}{2}\,\gamma_{S,qg}^{(1)} - 2\beta_0\, a_{S,qg}^{(1)} + \gamma_{S,qg}^{(0)}\, a_{S,gg}^{(1)} + \gamma_{S,qq}^{(0)}\, a_{S,qg}^{(1)} - \hat{\alpha}\left(\frac{da_{s,qg}^{(1)}}{d\hat{\alpha}}\right) z_\alpha\right\}\frac{1}{\varepsilon}\\ &\left. + a_{S,qg}^{(2)}\right]_{\hat{\alpha}=1}, \end{aligned}\tag{2.29}$$

$$\begin{aligned}\hat{A}_{S,gq} &= \left(\frac{\hat{g}^2}{16\pi^2}\right) S_\varepsilon \left(\frac{-p^2}{\mu^2}\right)^{\varepsilon/2}\left[\gamma_{S,gq}^{(0)}\frac{1}{\varepsilon} + a_{S,gq}^{(1)} + \varepsilon\, b_{S,gq}^{(1)}\right]\\ &+ \left(\frac{\hat{g}^2}{16\pi^2}\right)^2 S_\varepsilon^2 \left(\frac{-p^2}{\mu^2}\right)^\varepsilon \left[\left\{\frac{1}{2}\gamma_{S,gq}^{(0)}\,\left(\gamma_{S,qq}^{(0)} + \gamma_{S,gg}^{(0)}\right) - \beta_0\,\gamma_{S,gq}^{(0)}\right\}\frac{1}{\varepsilon^2}\right.\\ &+ \left\{\frac{1}{2}\gamma_{S,gq}^{(1)} - 2\beta_0\, a_{S,gq}^{(1)} + \gamma_{S,gq}^{(0)}\, a_{S,qq}^{(1)} + \gamma_{S,gg}^{(0)}\, a_{S,gq}^{(1)} - \hat{\alpha}\left(\frac{da_{s,gq}^{(1)}}{d\hat{\alpha}}\right) z_\alpha\right\}\frac{1}{\varepsilon}\\ &\left. + a_{S,gq}^{(2)}\right]_{\hat{\alpha}=1}, \end{aligned}\tag{2.30}$$

$$\begin{aligned}\hat{A}_{S,gg} &= 1 + \left(\frac{\hat{g}^2}{16\pi^2}\right) S_\varepsilon \left(\frac{-p^2}{\mu^2}\right)^{\varepsilon/2}\left[\gamma_{S,gg}^{(0)}\frac{1}{\varepsilon} + a_{S,gg}^{(1)} + \varepsilon\, b_{S,gg}^{(1)}\right]\\ &+ \left(\frac{\hat{g}^2}{16\pi^2}\right)^2 S_\varepsilon^2 \left(\frac{-p^2}{\mu^2}\right)^\varepsilon \left[\left\{\frac{1}{2}\,\left(\gamma_{S,gg}^{(0)}\right)^2 + \frac{1}{2}\,\gamma_{S,gq}^{(0)}\,\gamma_{S,qg}^{(0)} - \beta_0\,\gamma_{S,gg}^{(0)}\right\}\frac{1}{\varepsilon^2}\right.\\ &+ \left\{\frac{1}{2}\,\gamma_{S,gg}^{(1)} - 2\beta_0\, a_{S,gg}^{(1)} + \gamma_{S,gg}^{(0)}\, a_{S,gg}^{(1)} + \gamma_{S,gq}^{(0)}\, a_{S,qg}^{(1)} - \hat{\alpha}\left(\frac{da_{S,gg}^{(1)}}{d\hat{\alpha}}\right)\, z_\alpha\right\}\frac{1}{\varepsilon}\\ &\left. + a_{S,gg}^{(2)}\right]_{\hat{\alpha}=1}. \end{aligned}\tag{2.31}$$

Notice that in the above we have suppressed the Mellin index m. The expressions have been written in such a way that the anomalous dimensions take their values in the $\overline{\rm MS}$ scheme. Furthermore we have in lowest order the identity $\gamma_{S,qq}^{(0)} = \gamma_{NS,qq}^{(0)}$. The above form of the unrenormalized OME's $\hat{A}_{k,ij}$ follows from the property that the renormalized OME's $A_{k,ij}$ satisfy the Callan Symanzik equations (2.13), (2.14). These equations can be solved order by order in perturbation theory which provides us with the expressions presented at the end of this section. We only have to show that the latter follow from the renormalization of the OME's in (2.25)–(2.31).

The renormalization of the OME's proceeds as follows. First replace the bare coupling constant $\hat{g}$ by the renormalized one $g(\mu) = g$. Up to order $\hat{g}^4$ it is sufficient to substitute in the above OME's

$$\hat{g} = g\left(1 + \frac{g^2}{16\pi^2}\beta_0\, S_\varepsilon\, \frac{1}{\varepsilon}\right), \tag{2.32}$$

where β_0 is given by (2.16). Next one has to perform gauge constant renormalization. Notice that in the next section we will calculate the one-loop OME's in a general covariant gauge. The Feynman propagator in this gauge is given by

$$D_{\mu\nu}(k) = \frac{1}{k^2 + i\epsilon}\left(-g_{\mu\nu} + (1-\alpha)\frac{k_\mu k_\nu}{k^2 + i\epsilon}\right), \tag{2.33}$$

where α is the gauge constant. The two-loop OME's are computed in the Feynman gauge so that we have put in eqs. (2.25)–(2.31) $\hat{\alpha} = 1$.
Since the quarks and gluons are massless one has to put the external momenta p of the Feynman graphs off-shell. This implies that the OME's are no longer S-matrix elements and they become gauge (α) dependent. Therefore we also have to perform gauge constant renormalization which proceeds as follows. Replace the bare gauge constant $\hat{\alpha}$ by the renormalized one.

$$\hat{\alpha} = Z_\alpha \alpha. \tag{2.34}$$

In the covariant gauge one has the property $Z_\alpha = Z_A$ where Z_A is the gluon field renormalization constant. Hence Z_α is given by

$$Z_\alpha = 1 + \frac{g^2}{16\pi^2} z_\alpha \frac{1}{\varepsilon}, \tag{2.35}$$

where z_α is given in (2.19). After these two renormalizations the only ultraviolet divergences left in the OME's are removed by operator renormalization. Choosing the $\overline{\rm MS}$ scheme the operator renormalization constants are given by (see (2.21), (2.22))

$$\begin{aligned} Z_{NS,qq} &= 1 + \left(\frac{g^2}{16\pi^2}\right) S_\varepsilon\left[\frac{1}{\varepsilon}\gamma_{NS,qq}^{(0)}\right] \\ &\quad + \left(\frac{g^2}{16\pi^2}\right)^2 S_\varepsilon^2\left[\left\{\frac{1}{2}\left(\gamma_{NS,qq}^{(0)}\right)^2 + \beta_0\gamma_{NS,qq}^{(0)}\right\}\frac{1}{\varepsilon^2} + \frac{1}{2\varepsilon}\gamma_{NS,qq}^{(1)}\right], & (2.36) \\ Z_{S,qq} &= Z_{NS,qq} + Z_{PS,qq}, & (2.37) \end{aligned}$$

with

$$Z_{PS,qq} = \left(\frac{g^2}{16\pi^2}\right)^2 S_\varepsilon^2\left[\left\{\frac{1}{2}\gamma_{S,qg}^{(0)}\gamma_{S,gq}^{(0)}\right\}\frac{1}{\varepsilon^2} + \frac{1}{2\varepsilon}\gamma_{PS,qq}^{(1)}\right], \tag{2.38}$$

$$
\begin{aligned}
Z_{S,qg} &= \quad \left(\frac{g^2}{16\pi^2}\right) S_\varepsilon \left[\frac{1}{\varepsilon}\gamma_{S,qg}^{(0)}\right] \qquad (2.39)\\
&+ \left(\frac{g^2}{16\pi^2}\right)^2 S_\varepsilon^2 \left[\left\{\frac{1}{2}\gamma_{S,qg}^{(0)}\left(\gamma_{S,gg}^{(0)}+\gamma_{S,qq}^{(0)}\right)+\beta_0\,\gamma_{S,qg}^{(0)}\right\}\frac{1}{\varepsilon^2}+\frac{1}{2\varepsilon}\gamma_{S,qg}^{(1)}\right],\\
Z_{S,gq} &= \quad \left(\frac{g^2}{16\pi^2}\right) S_\varepsilon \left[\frac{1}{\varepsilon}\gamma_{S,gq}^{(0)}\right] \qquad (2.40)\\
&+ \left(\frac{g^2}{16\pi^2}\right)^2 S_\varepsilon^2 \left[\left\{\frac{1}{2}\gamma_{S,gq}^{(0)}\left(\gamma_{S,gg}^{(0)}+\gamma_{S,qq}^{(0)}\right)+\beta_0\,\gamma_{S,gq}^{(0)}\right\}\frac{1}{\varepsilon^2}+\frac{1}{2\varepsilon}\gamma_{S,gq}^{(1)}\right],\\
Z_{S,gg} &= 1+\left(\frac{g^2}{16\pi^2}\right) S_\varepsilon \left[\frac{1}{\varepsilon}\gamma_{S,gg}^{(0)}\right]\\
&+ \left(\frac{g^2}{16\pi^2}\right)^2 S_\varepsilon^2 \left[\left\{\frac{1}{2}\left(\gamma_{S,gg}^{(0)}\right)^2+\frac{1}{2}\,\gamma_{S,gq}^{(0)}\gamma_{S,qg}^{(0)}+\beta_0\,\gamma_{S,gg}^{(0)}\right\}\frac{1}{\varepsilon^2}\right.\\
&\left.+\frac{1}{2\varepsilon}\gamma_{S,gg}^{(1)}\right], \qquad (2.41)
\end{aligned}
$$

Notice that the anomalous dimensions $\gamma_{k,ij}^{(l)}$ (k =NS,S, $l=0,1$) are gauge independent so that $Z_{k,ij}$ have to be gauge independent too. The renormalized operator matrix elements are derived from

$$
A_{NS,qq}(p^2,\mu^2,g,\alpha) = Z_{NS,qq}^{-1}(g^2,\varepsilon)\hat{A}_{NS,qq}(p^2,\mu^2,\hat{g},\hat{\alpha})|_{\hat{g}\to Z_g g;\ \hat{\alpha}\to Z_\alpha\alpha} \qquad (2.42)
$$

and

$$
A_{S,ij}(p^2,\mu^2,g,\alpha) = (Z_S^{-1})_{il}(g^2,\varepsilon)\hat{A}_{S,lj}(p^2,\mu^2,\hat{g},\hat{\alpha})|_{g\to Z_g g;\ \hat{\alpha}\to Z_\alpha\alpha} \qquad (2.43)
$$

with the results

$$
\begin{aligned}
A_{NS,qq} &= 1+\frac{g^2}{16\pi^2}\left[\frac{1}{2}\gamma_{NS,qq}^{(0)}\ln\left(-\frac{p^2}{\mu^2}\right)+a_{NS,qq}^{(1)}\right]\\
&+\left(\frac{g^2}{16\pi^2}\right)^2\left[\left\{\frac{1}{8}\left(\gamma_{NS,qq}^{(0)}\right)^2-\frac{1}{4}\beta_0\,\gamma_{NS,qq}^{(0)}\right\}\ln^2\left(-\frac{p^2}{\mu^2}\right)\right.\\
&+\left\{\frac{1}{2}\gamma_{NS,qq}^{(1)}-\beta_0 a_{NS,qq}^{(1)}+\frac{1}{2}\gamma_{NS,qq}^{(0)}a_{NS,qq}^{(1)}-\frac{1}{2}\alpha\left(\frac{da_{NS,qq}^{(1)}}{d\alpha}\right)z_\alpha\right\}\ln\left(-\frac{p^2}{\mu^2}\right)\\
&\left. a_{NS,qq}^{(2)}+2\beta_0\,b_{NS,qq}^{(1)}-\gamma_{NS,qq}^{(0)}b_{NS,qq}^{(1)}+\alpha\left(\frac{d}{d\alpha}b_{NS,qq}^{(1)}\right)z_\alpha\right]_{\alpha=1}, \qquad (2.44)
\end{aligned}
$$

$$
\begin{aligned}
A_{S,qq} &= A_{NS,qq}+A_{PS,qq},\\
A_{PS,qq} &= \left(\frac{g^2}{16\pi^2}\right)^2\left[\frac{1}{8}\gamma_{S,qg}^{(0)}\gamma_{S,gq}^{(0)}\ln^2\left(-\frac{p^2}{\mu^2}\right)\right.\\
&+\left\{\frac{1}{2}\gamma_{PS,qq}^{(1)}+\frac{1}{2}\gamma_{S,qg}^{(0)}a_{S,gq}^{(1)}\right\}\ln\left(-\frac{p^2}{\mu^2}\right)
\end{aligned}
$$

$$+a^{(2)}_{PS,qq} - \gamma^{(0)}_{S,qg} b^{(1)}_{S,gq}\Big] , \tag{2.45}$$

$$\begin{aligned}
A_{S,qg} &= \left(\frac{g^2}{16\pi^2}\right) \left[\frac{1}{2}\gamma^{(0)}_{S,qg} \ln\left(-\frac{p^2}{\mu^2}\right) + a^{(1)}_{S,qg}\right] \\
&+ \left(\frac{g^2}{16\pi^2}\right)^2 \Bigg[\left\{ \frac{1}{8}\gamma^{(0)}_{S,qg}\left(\gamma^{(0)}_{S,qq} + \gamma^{(0)}_{S,gg}\right) - \frac{1}{4}\beta_0\,\gamma^{(0)}_{S,qg} \right\} \ln^2\left(-\frac{p^2}{\mu^2}\right) \\
&\qquad + \left\{ \frac{1}{2}\gamma^{(1)}_{S,qg} - \beta_0 a^{(1)}_{S,qg} + \frac{1}{2}\gamma^{(0)}_{S,qg} a^{(1)}_{S,gg} + \frac{1}{2}\gamma^{(0)}_{S,qq} a^{(1)}_{S,qg} \right. \\
&\qquad \left. - \frac{1}{2}\alpha\left(\frac{d}{d\alpha}\,a^{(1)}_{S,qg}\right) z_\alpha \right\} \ln\left(-\frac{p^2}{\mu^2}\right) \\
&\qquad + a^{(2)}_{S,qg} + 2\beta_0 a^{(1)}_{S,qg} - \gamma^{(0)}_{S,qq} b^{(1)}_{S,qg} - \gamma^{(0)}_{S,qg} b^{(1)}_{S,gg} \\
&\qquad + \alpha\left(\frac{d}{d\alpha} b^{(1)}_{S,qg}\right) z_\alpha \Bigg]_{\alpha=1} ,
\end{aligned} \tag{2.46}$$

$$\begin{aligned}
A_{S,gq} &= \left(\frac{g^2}{16\pi^2}\right) \left[\frac{1}{2}\gamma^{(0)}_{S,gq} \ln\left(-\frac{p^2}{\mu^2}\right) + a^{(1)}_{S,gq}\right] \\
&+ \left(\frac{g^2}{16\pi^2}\right)^2 \Bigg[\left\{ \frac{1}{8}\gamma^{(0)}_{S,gq}\left(\gamma^{(0)}_{S,qq} + \gamma^{(0)}_{S,gg}\right) - \frac{1}{4}\beta_0\,\gamma^{(0)}_{S,gq} \right\} \ln^2\left(-\frac{p^2}{\mu^2}\right) \\
&\qquad + \left\{ \frac{1}{2}\gamma^{(1)}_{S,gq} - \beta_0 a^{(1)}_{S,gq} + \frac{1}{2}\gamma^{(0)}_{S,gq} a^{(1)}_{NS,qq} + \frac{1}{2}\gamma^{(0)}_{S,gg} a^{(1)}_{S,gq} \right. \\
&\qquad \left. - \frac{1}{2}\alpha\left(\frac{d}{d\alpha}\,a^{(1)}_{S,gq}\right) z_\alpha \right\} \ln\left(-\frac{p^2}{\mu^2}\right) \\
&\qquad + a^{(2)}_{S,gq} + 2\beta_0 a^{(1)}_{S,gq} - \gamma^{(0)}_{S,gq} b^{(1)}_{S,qq} - \gamma^{(0)}_{S,gg} b^{(1)}_{S,gq} \\
&\qquad + \alpha\left(\frac{d}{d\alpha} b^{(1)}_{S,gq}\right) z_\alpha \Bigg]_{\alpha=1} ,
\end{aligned} \tag{2.47}$$

$$\begin{aligned}
A_{S,gg} &= 1 + \left(\frac{g^2}{16\pi^2}\right) \left[\frac{1}{2}\gamma^{(0)}_{S,gg} \ln\left(-\frac{p^2}{\mu^2}\right) + a^{(1)}_{S,gg}\right] \\
&+ \left(\frac{g^2}{16\pi^2}\right)^2 \Bigg[\left\{ \frac{1}{8}\left(\gamma^{(0)}_{S,gg}\right)^2 + \gamma^{(0)}_{S,gq}\gamma^{(0)}_{S,qg} - \frac{1}{4}\beta_0\,\gamma^{(0)}_{S,gg} \right\} \ln^2\left(-\frac{p^2}{\mu^2}\right) \\
&\qquad + \left\{ \frac{1}{2}\gamma^{(1)}_{S,gg} - \beta_0 a^{(1)}_{S,gg} + \frac{1}{2}\gamma^{(0)}_{S,gg} a^{(1)}_{S,gg} + \frac{1}{2}\gamma^{(0)}_{S,gq} a^{(1)}_{S,qg} \right. \\
&\qquad \left. - \frac{1}{2}\alpha\left(\frac{d}{d\alpha}\,a^{(1)}_{S,gg}\right) z_\alpha \right\} \ln\left(-\frac{p^2}{\mu^2}\right)
\end{aligned}$$

$$+ a^{(2)}_{S,gg} + 2\beta_0 b^{(1)}_{S,gg} - \gamma^{(0)}_{S,gg} b^{(1)}_{S,gg} - \gamma^{(0)}_{S,gq} b^{(1)}_{S,qg}$$

$$\left. + \alpha \left(\frac{d}{d\alpha} b^{(1)}_{S,gg} \right) z_\alpha \right]_{\alpha=1} . \qquad (2.48)$$

The above renormalized OME's satisfy the Callan Symanzik equations in (2.13), (2.14) which proves that the ansatz for the unrenormalized OME's in (2.25)–(2.31) is correct. This is also corroborated by the expressions for the operator renormalization constants $Z_{k,ij}$ in (2.36)–(2.41) which after insertion in eqs. (2.23), (2.24) provides us with the anomalous dimensions in (2.20).

The above renormalization procedure was originally introduced by F.J. Dyson [18]. There exists an alternative possibility invented by Bogoliubov, Parasiuk, Hepp and Zimmermann (BPHZ) [19]. In the latter one renormalizes each Feynman graph independently using the counter-term method. These counter-terms appear in the effective lagrangian which is expressed into the renormalized (coupling- and gauge-) constants, fields and operators. The BPHZ-method has been used in the literature [14]–[17] to derive the anomalous dimensions of the spin averaged operators $O^{\mu_1 \cdots \mu_m}$ (i=NS,S) in (2.1). The advantage of this method is that the gauge dependent terms given by $\alpha \left(\frac{d}{d\alpha} a^{(1)}_{k,ij} \right) z_\alpha$ in (2.25)–(2.31) are automatically subtracted. We will come back to this method at the end of section 3. The reason for the algebraic exercise given above can be explained as follows. Since the lowest order coefficients $\gamma^{(0)}_{k,ij}$ and $a^{(1)}_{k,ij}$ can be very easily determined from the one-loop OME's one immediately can predict the double pole terms in the unrenormalized OME's (2.25)–(2.30). The coefficient of the single pole term can be also computed except for the second order anomalous dimensions $\gamma^{(1)}_{k,ij}$. By equating the predicted form of the two-loop OME's in (2.25)–(2.30) to the explicitly computed result in the next section one immediately can infer the results for $\gamma^{(1)}_{k,ij}$.

3 Calculation of the order α_s^2 contribution to the spin splitting functions

In this section we first give an outline of the procedure of the calculation of the OME's defined in (2.11). Then we present the analytical result for the OME's and extract from them the splitting functions (anomalous dimensions).

The calculation of the OME's proceeds as follows. Using the operator vertices in Appendix A and applying the standard Feynman rules we have computed the connected Green's functions, which are given by the graphs in figs. 1-6, up to two-loop order. The latter also involves the calculation of the diagrams which contain the self energies of the quark and the gluon in the external legs. These diagrams are not explicitly drawn in the figures but are included in our calculation. The computation of the one-loop graphs has been done in the general covariant gauge because one has to renormalize the gauge constant α even if one chooses the Feynman gauge $\alpha = 1$. The two-loop graphs have been calculated in the Feynman gauge which is sufficient to that order. The OME's are then obtained by multiplying the connected Green's function by the inverse of the external quark and gluon propagators. Since the external momenta are put off shell only ultraviolet divergences appear in the OME's which are regularized by using the method of n-dimensional regularization. This implies that we have to find a suitable prescription for the γ_5-matrix which appears in the quark operators $R_{k,q}$ for $k = NS$ (2.5) and $k = S$ (2.6). Here we will adopt the reading point method as explained in [20]. One can also adopt the method of 't Hooft and Veltman [21], which is equivalent to the one given by Breitenlohner and Maison [22] (see also [23]). The disadvantage of the last method is that the non-singlet axial vector operator $R^{(1)}_{NS,q}$ (2.5) gets renormalized in spite of the fact that it is conserved. This has to be undone by introducing an additional renormalization constant [23]. However for continuity this procedure has to be extended to higher spin non-singlet operators $R^m_{NS,q}$ (2.5) $(m > 1)$ otherwise the anomalous dimension of $R^m_{NS,q}$ will become unequal to the anomalous dimension of the spin averaged non-singlet operator $O^m_{NS,q}$ (see (2.1)). Notice that the same procedure has to be also carried out for some of the singlet operators $R^m_{S,q}$ (2.6). Using the reading point method [20] one can omit the additional renormalization constant. Anyhow we have checked that both methods lead to the same result.

As has been already mentioned in section 2 the Feynman rules for the operator vertices in Appendix A have been derived multiplying the operators $R^{\mu_1\cdots\mu_m}_{k,i}$ by the sources $J_{\mu_1\cdots\mu_m} = \Delta_{\mu_1}\cdots\Delta_{\mu_m}$ with $\Delta^2 = 0$. To simplify further we can choose $s = p$, where s is the spin vector in (2.4), without any loss of essential information. The operator matrix elements $\hat{A}^m_{k,ij}$ (2.25)–(2.31) are then given by

$$\hat{A}^m_{k,iq}(p^2,\mu^2,g,\varepsilon)(\Delta p)^m = \frac{1}{4}\,Tr\left\{\hat{G}^m_{k,iq}(p,\Delta,\mu^2,g,\varepsilon)\gamma_5\not p\right\}, \tag{3.49}$$

with $k = NS, S$ and $i = q, g$ and

$$\hat{A}^m_{S,ig}(p^2,\mu^2,g,\varepsilon)(\Delta p)^m = \frac{1}{2\,\Delta p}\,\varepsilon_{\mu\nu\lambda\sigma}\,\Delta^\lambda p^\sigma\,\hat{G}^{m\mu\nu}_{S,ig}(p,\Delta,\mu^2,g,\varepsilon). \tag{3.50}$$

Here $\hat{G}^m_{k,ij}$ stand for the unrenormalized Green's functions which are multiplied by the inverse of the external quark and gluon propagators.

We will now give a short outline of the calculation of $\hat{A}^m_{k,ij}$. Let us first start with the non-singlet OME $\hat{A}^m_{NS,qq}$. The Green's function $\hat{G}^m_{NS,qq}$, which is determined by the one-loop graphs in fig. 1a,b and by the two-loop graphs in fig. 2, consists out of Feynman integrals where the numerators are given by a string of γ-matrices. One of the γ - matrices represents the γ_5-matrix. The latter is then anticommuted with the other γ-matrices until it appears on the right side of the string next to the γ_5 in (3.49). Then we set $\gamma_5^2 = 1$ and simplify the trace by contracting over dummy Lorentz-indices. Finally we perform the trace in (3.49). In this way one obtains the identity

$$\hat{A}^m_{NS,qq}(p^2,\mu^2,\varepsilon) = \left(\hat{A}^m_{NS,qq}(p^2,\mu^2,\varepsilon)\right)_{\rm spin-averaged}, \tag{3.51}$$

without any additional renormalization constant. Notice that the calculation of $A^m_{NS,qq}$ (spin averaged OME) has been already done in the literature so that it will not be repeated here. Except for the non-singlet operator $\hat{A}^m_{NS,qq}$ the remaining spin OME's differ from their spin averaged analogues. Since we need the one-loop OME's as presented in fig. 1, for the renormalization of the two-loop OME's given by figs. 2-6 we have to calculate the former ones up to the non-pole term $a_{k,ij}$ defined in eqs. (2.25)–(2.31). The one-loop terms $b_{k,ij}$ which are proportional to $\varepsilon = n - 4$, do not play any role in the determination of the anomalous dimension and they will not be presented in this paper.

Starting with the one-loop contribution to $\hat{A}^m_{S,gq}$ (fig. 1c) we have to perform tensorial reduction of the tensor integrals appearing in $\hat{G}^m_{S,gq}$. These tensor integrals arise because the integration momentum q_μ appears in the numerators of the integrand. Examples of such one-loop integrals are given by eqs. (B.1)–(B.3). The result will be that $\hat{G}^m_{S,gq}$ gets terms of the form $\varepsilon^{\alpha\beta\lambda\sigma}\Delta_\sigma p_\lambda \gamma_\alpha \not{p} \gamma_\beta$, $\varepsilon^{\alpha\beta\lambda\sigma}\Delta_\sigma \gamma_\alpha \gamma_\lambda \gamma_\beta$, $\varepsilon^{\alpha\beta\lambda\sigma} p_\lambda \Delta_\sigma \gamma_\alpha \not{\Delta} \gamma_\beta$, where the Levi-Civita tensor $\varepsilon^{\alpha\beta\lambda\sigma}$ originates from the two-gluon operator vertex in (A.4). Performing the trace in (3.49) provides us with a second Levi-Civita tensor so that we have to contract over two and three dummy Lorentz-indices. The contraction has to be performed in 4 dimensions since the operator vertices have a unique meaning in 4 dimensions only. Next we discuss the calculation of the one-loop contribution to $\hat{A}^m_{S,qg}$ (3.50) (fig. 1d,e). To this OME we apply the reading point method [20] and put the γ_5 on the right hand side of the trace from the start. In this way we reproduce the Adler-Bell-Jackiw anomaly which can be traced back to the triangular fermion loop in fig. 1d. Notice that fig. 1e leads to a zero result because the external momentum p appears twice in the Levi-Civita tensor. The Green's function $\hat{G}^{m\mu\nu}_{S,qg}$ will then become proportional to $\varepsilon^{\mu\nu\lambda\sigma}\Delta_\lambda p_\sigma$. The latter will be contracted with the Levi-Civita tensor in (3.50) where the contraction is performed in 4 dimensions. The one-loop graphs contributing to $\hat{A}^m_{S,gg}$ are presented in figs. 1f, g. Because of the Levi-Civita tensor coming from the two-gluon operator vector in (A.4) $\hat{G}^{m,\mu\nu}_{S,gg}$ will, after tensorial reduction, become proportional to $\varepsilon^{\mu\nu\lambda\sigma}\Delta_\lambda p_\sigma$. Like in the case of $\hat{A}^m_{S,qg}$ the contraction with the Levi-Civita tensor in (3.50) has to take place in 4 dimensions.

Before we proceed with the two-loop graphs we want to emphasize that first the tensorial reduction has to be made before one can perform the contraction between the two Levi-Civita tensors. Both operations do not commute and lead to different results for the OME's. This holds for the one as well as two-loop calculation. If one contracts the Levi-Civita tensors in n dimensions both operations commute. However then the Lorentz indices of the operator vertices in Appendix A have to be generalized to n dimensions which is a non-unique procedure.

The calculation of the two-loop graphs in figs. 3-6 proceeds in an analogous way as in the one-loop case. However here there arise some extra complications. First of all we encounter the two-loop scalar Feynman integrals which have already been performed in [24] to calculate the spin averaged OME $\hat{A}^m_{S,gg}$. To check these integrals and the tensorial reduction algorithm we have recalculated all spin averaged anomalous dimensions (splitting functions) and we found complete agreement with the results published in the literature [14]–[17].

The second complication shows up in the tensorial reductions of the two-loop tensor Feynman integrals where the numerator now reveals the presence of two integration momenta q_1 and q_2. A more detailed explanation of how the tensor integrals are reduced into scalar integrals is presented in Appendix B. The third complication arises because of the appearance of a trace of six γ-matrices out of which two are contracted with the integration momenta q_1 and q_2. Such graphs (see e.g. fig. 3 and figs. 5.11) are calculated by the following procedure. First one performs tensorial reduction of the Feynman integrals as indicated in Appendix B. This will lead to an increase of the pairs of γ-matrices having the same Lorentz-index. Then one can eliminate these pairs using the standard rules for γ-algebra in n dimensions. This is possible without ever touching the γ_5 matrix because it is put at the right hand side of the string of γ-matrices. After this procedure one ends up with the expression $Tr(\not{a}\not{b}\not{c}\not{d}\gamma_5)$ which is uniquely defined (irrespective of the γ_5-scheme). The same holds for the other graphs which do not contain fermion loops. The final result is that all Green's functions $\hat{G}^m_{k,ij}$ get the same form as observed for the one-loop case. Four dimensional contraction of the two Levi-Civita tensors yields the OME's $\hat{A}^{(m)}_{k,ij}$ in (3.49), (3.50). Before finishing the technical part of this section we give a comment on the algebraic manipulation programs which are used to calculate $\hat{A}^m_{k,ij}$. The matrix elements (including the full tensorial reduction) were calculated using the package *FeynCalc* [25] which is written in *Mathematica* [26]. The two-loop scalar integrals were performed by using a program written in FORM [25] which was called in *FeynCalc*.

If one performs the inverse Mellin transform of the OME's the results for the one-loop calculation can be summarized as follows (see eqs. (2.21)–(2.27)). First we have the lowest order splitting functions which are already known in the literature [12, 13].

$$P^{(0)}_{NS,qq} = P^{(0)}_{S,qq} = C_F\left[8\left(\frac{1}{1-x}\right)_+ - 4 - 4x + 6\delta(1-x)\right], \tag{3.52}$$

$$P^{(0)}_{S,qg} = T_f\left[16x - 8\right], \tag{3.53}$$

$$P^{(0)}_{S,gq} = C_f\left[8 - 4x\right], \tag{3.54}$$

$$P^{(0)}_{S,gg} = C_A\left[8\left(\frac{1}{1-x}\right)_+ + 8 - 16x + \frac{22}{3}\delta(1-x)\right] - T_f\left[\frac{8}{3}\delta(1-x)\right], \tag{3.55}$$

where the colour factors in SU(N) are given by $C_F = (N^2-1)/(2N), C_A = N$ and $T_f = 1/2$ ($N = 3$ in QCD). The non-pole terms $a^{(1)}_{k,ij}$ appearing in expressions (2.25)–(2.31) read as follows

$$\begin{aligned} a^{(1)}_{NS,qq} = a^{(1)}_{S,qq} = C_F\Bigg[&-4\left(\frac{\ln(1-x)}{1-x}\right)_+ + 2(1+x)\ln(1-x) - 2\frac{1+x^2}{1-x}\ln x \\ &-4 + 2x + (1-\alpha)\left(2 - \left(\frac{1}{1-x}\right)_+\right) + \delta(1-x)(7 - 4\zeta(2)), \end{aligned} \tag{3.56}$$

$$a^{(1)}_{S,qg} = T_f\left[(4-8x)(\ln x + \ln(1-x)) - 4\right], \tag{3.57}$$

$$a^{(1)}_{S,gq} = C_F\left[(-4+2x)(\ln x + \ln(1-x)) + 2 - 4x\right], \tag{3.58}$$

$$\begin{aligned} a^{(1)}_{S,gg} = C_A\Bigg[&-4\left(\frac{\ln(1-x)}{1-x}\right)_+ + (-4+8x)\ln(1-x) + \left(-\frac{4}{1-x} - 4 + 8x\right)\ln x \\ &-(1-\alpha)\left(\frac{1}{1-x}\right)_+ + 2 + \delta(1-x)\left(\frac{67}{9} - 4\zeta(2) + (1-\alpha) - \frac{1}{4}(1-\alpha)^2\right)\Bigg] \\ &-T_f\left[\frac{20}{9}\delta(1-x)\right]. \end{aligned} \tag{3.59}$$

In the above expressions the distributions $\left(\frac{\ln^k(1-x)}{1-x}\right)_+$ are defined by

$$\int_0^1 dx\left(\frac{\ln^k(1-x)}{1-x}\right)_+ f(x) = \int_0^1 dx\left(\frac{\ln^k(1-x)}{1-x}\right)(f(x) - f(1)). \tag{3.60}$$

Notice that in the general covariant gauge only $a^{(1)}_{k,qq}$ (k=NS,S) and $a^{(1)}_{S,gg}$ depend on the gauge parameter α.

The two-loop contributions to the unrenormalized OME's are given by the inverse Mellin transforms (see eqs. (2.25)–(2.31))

$$\hat{A}_{PS,qq} = \frac{g^4}{(16\pi^2)^2} S_\varepsilon^2 \left(\frac{-p^2}{\mu^2}\right)^\varepsilon C_F T_f B_{Ff}^{qq}, \tag{3.61}$$

$$\hat{A}_{S,qg} = \frac{g^4}{(16\pi^2)^2} S_\varepsilon^2 \left(\frac{-p^2}{\mu^2}\right)^\varepsilon \left[T_f^2 B_{ff}^{qg} + C_F T_f B_{Ff}^{qg} + C_A T_f B_{Af}^{qg}\right], \tag{3.62}$$

$$\hat{A}_{S,gq} = \frac{g^4}{(16\pi^2)^2} S_\varepsilon^2 \left(\frac{-p^2}{\mu^2}\right)^\varepsilon \left[C_A C_F B_{AF}^{gq} + C_F T_f B_{Ff}^{gq} + C_F^2 B_{FF}^{gq}\right], \tag{3.63}$$

$$\hat{A}_{S,gg} = \frac{g^4}{(16\pi^2)^2} S_\varepsilon^2 \left(\frac{-p^2}{\mu^2}\right)^\varepsilon \left[C_A T_f B_{Af}^{gg} + C_F T_f B_{Ff}^{gg} + C_A^2 B_{AA}^{gg}\right], \tag{3.64}$$

$$\begin{aligned}
B_{Ff}^{qq} &= \frac{8}{\varepsilon^2}\left\{ 4\,(1+x)\ln x + 10\,(1-x) \right\} \\
&\quad + \frac{8}{\varepsilon}\left\{ 4\,(1+x)\mathrm{Li}_2(1-x) + 3\,(1+x)\ln^2 x + 4\,(1+x)\ln x\ln(1-x) \right. \\
&\qquad \left. + 10\,(1-x)\ln(1-x) + (7-5\,x)\ln x - 5\,(1-x) \right\}, \\
B_{ff}^{qg} &= \frac{64}{\varepsilon^2}\left\{\frac{1}{3}(1-2\,x)\right\} + \frac{32}{\varepsilon}\left\{-\frac{2}{9}(4-5\,x) + \frac{1}{3}(1-2\,x)\ln x\right\}, \\
B_{Ff}^{qg} &= \frac{4}{\varepsilon^2}\left\{ -8\,(1-2\,x)\ln(1-x) + 4\,(1-2\,x)\ln x + 6 \right\} \\
&\quad + \frac{4}{\varepsilon}\left\{ -12\,(1-2\,x)\,\mathrm{Li}_2(1-x) + 4\,(1-2\,x)\,\zeta(2) - 6\,(1-2\,x)\ln^2(1-x) \right. \\
&\qquad - 8\,(1-2\,x)\ln x\ln(1-x) + 3\,(1-2\,x)\ln^2 x + 2\,(4\,x+3)\ln(1-x) \\
&\qquad \left. -(8\,x+5)\ln x - 12 + 13\,x \right\}, \\
B_{Af}^{qg} &= \frac{16}{\varepsilon^2}\left\{-2\,(1-2\,x)\ln(1-x) + 4\,(1+x)\ln x + \frac{1}{3}(25-14\,x)\right\} \\
&\quad + \frac{8}{\varepsilon}\left\{ 12\,\mathrm{Li}_2(1-x) + 2(2\,x+1)\mathrm{Li}_2(-x) - 2\,(1-4\,x)\,\zeta(2) \right. \\
&\qquad - 3(1-2\,x)\ln^2(1-x) + 2\,(2\,x+1)\ln x\ln(1+x) + 8\,(1+x)\ln x\ln(1-x) \\
&\qquad \left. +(6\,x+5)\ln^2 x + \frac{2}{3}(11\,x+14)\ln x + \frac{1}{3}(73-62\,x)\ln(1-x) - \frac{1}{9}(44-x)\right\}, \\
B_{AF}^{gq} &= \frac{8}{\varepsilon^2}\left\{\frac{1}{3}(25\,x-14) + 2\,(2-x)\ln(1-x) - 2\,(x+4)\ln x\right\} \\
&\quad + \frac{4}{\varepsilon}\left\{ -12\,x\,\mathrm{Li}_2(1-x) - 2\,(x+2)\mathrm{Li}_2(-x) - 2\,(4-x)\,\zeta(2) - 2\,(x+2)\ln x\ln(1+x) \right. \\
&\qquad + 3\,(2-x)\ln^2(1-x) - (3\,x+10)\ln^2 x - 2\,(5x+2)\ln x\ln(1-x) \\
&\qquad \left. -\frac{1}{3}(50-73\,x)\ln(1-x) + \frac{1}{3}(17\,x+8)\ln x + \frac{1}{9}(109-119\,x)\right\},
\end{aligned}$$

$$
\begin{aligned}
B^{gq}_{Ff} &= \frac{32}{\varepsilon^2}\left\{\frac{1}{3}(x-2)\right\} + \frac{16}{\varepsilon}\left\{\frac{1}{9}(10-11x) + \frac{1}{3}(x-2)\ln(1-x) + \frac{2}{3}(x-2)\ln x\right\},\\
B^{gq}_{FF} &= \frac{4}{\varepsilon^2}\left\{3x - 4(x-2)\ln(1-x) + 2(x-2)\ln x\right\}\\
&+\frac{2}{\varepsilon}\left\{12(x-2)\,\mathrm{Li}_2(1-x) + 8(2-x)\zeta(2) + 6(2-x)\ln^2(1-x) + 3(x-2)\ln^2 x\right.\\
&\qquad \left.+4(x-2)\ln x\ln(1-x) - 2(4-7x)\ln(1-x) + 7x\ln x + 9 - 14x\right\},\\
B^{gg}_{Af} &= \frac{32}{\varepsilon^2}\left\{-\left(\frac{1}{1-x}\right)_+ + 2x - 1 - \frac{11}{9}\delta(1-x)\right\}\\
&+\frac{8}{\varepsilon}\left\{-\frac{8}{3}\left(\frac{\ln(1-x)}{1-x}\right)_+ + \left(\frac{16}{3}x - \frac{8}{3}\right)\ln(1-x) + 3\left(\frac{1}{1-x}\right)_+\right.\\
&\qquad \left.+\left(6x - \frac{8}{3(1-x)} - 2\right)\ln x - \frac{26}{3}x + \frac{20}{3} + \left(\frac{271}{27} - \frac{8}{3}\zeta(2)\right)\delta(1-x)\right\},\\
B^{gg}_{Ff} &= \frac{8}{\varepsilon^2}\left\{4(1+x)\ln x + 10(1-x)\right\}\\
&+\frac{4}{\varepsilon}\left\{8(1+x)\,\mathrm{Li}_2(1-x) + 8(1+x)\ln x\ln(1-x) + 6(1+x)\ln^2 x\right.\\
&\qquad \left.+20(1-x)\ln(1-x) + (6-14x)\ln x + 22(x-1) + \delta(1-x)\right\},\\
B^{gg}_{AA} &= \frac{4}{\varepsilon^2}\left\{16\left(\frac{\ln(1-x)}{1-x}\right)_+ + 16(1-2x)\ln(1-x) + 22\left(\frac{1}{1-x}\right)_+\right.\\
&\qquad \left.-8\left(3 + \frac{1}{1-x}\right)\ln x + 20x - 42 + \delta(1-x)\left(\frac{121}{9} - 8\zeta(2)\right)\right\}\\
&+\frac{2}{\varepsilon}\left\{-32(1+x)\mathrm{Li}_2(1-x) + \left(-16x - \frac{8}{1+x} - 8\right)\mathrm{Li}_2(-x)\right.\\
&\qquad +\left(-16x + 4\left(\frac{1}{1-x}\right)_+ - \frac{4}{1+x}\right)\zeta(2) + \left(-16x - \frac{8}{1+x} - 8\right)\ln x\ln(1+x)\\
&\qquad +24\left(\frac{\ln^2(1-x)}{1-x}\right)_+ + \frac{88}{3}\left(\frac{\ln(1-x)}{1-x}\right)_+ + (-48x + 24)\ln^2(1-x)\\
&\qquad +\left(-48x + \frac{8}{1-x} - 24\right)\ln x\ln(1-x) + \left(\frac{2}{1+x} - 32 - \frac{10}{1-x}\right)\ln^2 x\\
&\qquad +\left(\frac{208}{3}x - \frac{320}{3}\right)\ln(1-x) + \left(-14x + \frac{88}{3(1-x)} - 38\right)\ln x - 43\left(\frac{1}{1-x}\right)_+\\
&\qquad \left.+\frac{119}{3} - \frac{29}{3}x + \delta(1-x)\left(\frac{88}{3}\zeta(2) + 10\zeta(3) - \frac{1663}{27}\right)\right\}.
\end{aligned}
$$

Here the function $\mathrm{Li}_2(y)$ stands for the dilogarithm which can be found in [28]. After substitution of the one-loop order coefficients β_0 (2.15), z_α (2.19), the Mellin transforms of $P^{(0)}_{k,ij}$ $\left(=\gamma^{(0),m}_{k,ij}\right)$ (3.52)–(3.55) and $a^{(1)}_{k,ij}$ (3.56)–(3.59) into the algebraic expressions for $\hat{A}_{k,ij}$ in (2.25)–(2.31) one can equate the latter with the results obtained for $\hat{A}_{k,ij}$ as presented above in (3.61)–(3.64). From this one infers the two-loop contribution to the anomalous dimensions which are the unknown coefficients in eqs. (2.25)–(2.31). After performing the inverse Mellin transform we get the splitting functions. The non-singlet splitting function $P^{(1)}_{NS,qq}$ is the same as obtained in the spin-averaged case (see e.g. [16, 17]). In order to obtain the singlet splitting function $P^{(1)}_{S,qq}$ one has to add to $P^{(1)}_{NS,qq}$ the quantity $P^{(1)}_{PS,qq}$ given below.

$$P^{(1)}_{PS,qq} = C_F T_f \left[-16(1+x)\ln^2 x - 16(1-3x)\ln x + 16(1-x)\right]. \tag{3.65}$$

Furthermore we have the singlet splitting functions

$$\begin{aligned} P^{(1)}_{S,qg} = \; & 4\,C_A T_f\,[-8(1+2x)\mathrm{Li}_2(-x) - 8\zeta(2) - 8(1+2x)\ln x\ln(1+x) \\ & + 4(1-2x)\ln^2(1-x) - 4(1+2x)\ln^2 x \\ & -16\,(1-x)\ln(1-x) + 4\,(1+8x)\ln x - 44\,x + 48] \\ & +4\,C_F T_f \left[8\,(1-2x)\zeta(2) - 4\,(1-2x)\ln^2(1-x)\right. \\ & + 8\,(1-2x)\ln x\ln(1-x) - 2\,(1-2x)\ln^2 x \\ & +16\,(1-x)\ln(1-x) - 2(1-16x)\ln x + 4 + 6x], \end{aligned} \tag{3.66}$$

$$\begin{aligned} P^{(1)}_{S,gq} = \; & C_A C_F \left[16\,(2+x)\mathrm{Li}_2(-x) + 16\,x\,\zeta(2) + 8\,(2-x)\ln^2(1-x)\right. \\ & + 16\,(2+x)\ln x\ln(1+x) + 8\,(2+x)\ln^2 x \\ & + 16\,(x-2)\ln x\ln(1-x) + \left(\frac{80}{3} + \frac{8}{3}x\right)\ln(1-x) \\ & \left. +8\,(4-13\,x)\ln x + \frac{328}{9} + \frac{280}{9}x\right] \\ & +C_F^2 \left[8\,(x-2)\ln^2(1-x) - 4\,(x-2)\ln^2 x - 164 + 128\,x\right. \\ & -8\,(x+2)\ln(1-x) - 4\,(20+7\,x)\ln x] \\ & +C_F T_f \left[-\frac{32}{9}(4+x) + \frac{32}{3}(x-2)\ln(1-x)\right], \end{aligned} \tag{3.67}$$

$$\begin{aligned} P^{(1)}_{S,gg} = \; & C_A^2 \left[\left(64x + \frac{32}{1+x} + 32\right)\mathrm{Li}_2(-x) + \left(64x - 16\left(\frac{1}{1-x}\right)_+ + \frac{16}{1+x}\right)\zeta(2)\right. \\ & + \left(\frac{8}{1-x} - \frac{8}{1+x} + 32\right)\ln^2 x + \left(64x + \frac{32}{1+x} + 32\right)\ln x\ln(1+x) \\ & + \left(64x - \frac{32}{1-x} - 32\right)\ln x\ln(1-x) + \left(\frac{232}{3} - \frac{536}{3}x\right)\ln x \\ & \left. + \frac{536}{9}\left(\frac{1}{1-x}\right)_+ - \frac{388}{9}x - \frac{148}{9} + \delta(1-x)(24\zeta(3) + \frac{64}{3})\right] \\ & +C_A T_f \left[-\frac{160}{9}\left(\frac{1}{1-x}\right)_+ - \frac{32}{3}(1+x)\ln x - \frac{448}{9} + \frac{608}{9}x\right. \\ & \left. -\frac{32}{3}\delta(1-x)\right] \\ & +C_F T_f \left[-16(1+x)\ln^2 x + 16(x-5)\ln x - 80(1-x)\right. \\ & -8\delta(1-x)]. \end{aligned} \tag{3.68}$$

For practical purposes, and the discussion of the results obtained above in the next section, it is also useful to present the one- and two-loop anomalous dimensions which are related to the splitting functions via the Mellin transform

$$\gamma^m_{k,ij} = -\int_0^1 dx\, x^{m-1}\, P_{k,ij}(x). \tag{3.69}$$

The one-loop contribution to the anomalous dimensions become

$$\begin{aligned}
\gamma^{(0),m}_{NS,qq} &= \gamma^{(0),m}_{S,qq} = C_F\left[8S_1(m-1) + \frac{4}{m} + \frac{4}{m+1} - 6\right], & (3.70)\\
\gamma^{(0),m}_{S,qg} &= T_f\left[\frac{8}{m} - \frac{16}{m+1}\right], & (3.71)\\
\gamma^{(0),m}_{S,gq} &= C_F\left[\frac{4}{m+1} - \frac{8}{m}\right], & (3.72)\\
\gamma^{(0),m}_{S,gg} &= C_A\left[8S_1(m-1) - \frac{8}{m} + \frac{16}{m+1} - \frac{22}{3}\right] + \frac{8}{3}T_f. & (3.73)
\end{aligned}$$

The two-loop non-singlet anomalous dimension $\gamma^{(1),m}_{NS,qq}$ is the same as found for the spin averaged operator (see e.g. [15, 16]). To obtain the singlet anomalous dimension $\gamma^{(1),m}_{S,qq}$ one has to add $\gamma^{(1),m}_{NS,qq}$ the quantity $\gamma^{(1),m}_{PS,qq}$ which reads

$$\gamma^{(1),m}_{PS,qq} = C_F T_f\, 16\left[\frac{2}{(m+1)^3} + \frac{3}{(m+1)^2} + \frac{1}{(m+1)} + \frac{2}{m^3} - \frac{1}{m^2} - \frac{1}{m}\right]. \tag{3.74}$$

Furthermore we have the singlet anomalous dimensions

$$\begin{aligned}
\gamma^{(1),m}_{S,qg} = {}& 16\, C_A T_f\left[-\frac{S_1^2(m-1)}{m} + \frac{2\,S_1^2(m-1)}{m+1} - \frac{2\,S_1(m-1)}{m^2} + \frac{4\,S_1(m-1)}{(m+1)^2}\right.\\
&- \frac{S_2(m-1)}{m} + \frac{2\,S_2(m-1)}{m+1} - \frac{2\,\tilde{S}_2(m-1)}{m} + \frac{4\,\tilde{S}_2(m-1)}{m+1}\\
&\left.- \frac{4}{m} + \frac{3}{m+1} - \frac{3}{m^2} + \frac{8}{(m+1)^2} + \frac{2}{m^3} + \frac{12}{(m+1)^3}\right]\\
&+ 8\, C_F T_f\left[\frac{2\,S_1^2(m-1)}{m} - \frac{4\,S_1^2(m-1)}{m+1} - \frac{2\,S_2(m-1)}{m} + \frac{4\,S_2(m-1)}{m+1}\right.\\
&\left.- \frac{10}{m} + \frac{5}{m+1} + \frac{7}{m^2} + \frac{8}{(m+1)^2} - \frac{2}{m^3} + \frac{4}{(m+1)^3}\right],
\end{aligned} \tag{3.75}$$

$$\begin{aligned}
\gamma^{(1),m}_{S,gq} = {}& 8\, C_A C_F\left[-\frac{2\,S_1^2(m-1)}{m} + \frac{S_1^2(m-1)}{m+1} + \frac{16\,S_1(m-1)}{3m} - \frac{5S_1(m-1)}{3(m+1)}\right.\\
&+ \frac{2S_2(m-1)}{m} - \frac{S_2(m-1)}{m+1} + \frac{4\tilde{S}_2(m-1)}{m} - \frac{2\tilde{S}_2(m-1)}{m+1} - \frac{56}{9m}\\
&\left.- \frac{20}{9(m+1)} + \frac{28}{3m^2} - \frac{38}{3(m+1)^2} - \frac{4}{m^3} - \frac{6}{(m+1)^3}\right]\\
&+ 4\, C_F^2\left[\frac{4\,S_1^2(m-1)}{m} - \frac{2\,S_1^2(m-1)}{m+1} - \frac{8\,S_1(m-1)}{m} + \frac{2\,S_1(m-1)}{m+1}\right.
\end{aligned}$$

$$
\begin{aligned}
&+\frac{8\,S_1(m-1)}{m^2}-\frac{4\,S_1(m-1)}{(m+1)^2}+\frac{4\,S_2(m-1)}{m}-\frac{2\,S_2(m-2)}{m+1} \\
&+\frac{39}{m}-\frac{30}{m+1}-\frac{28}{m^2}-\frac{5}{(m+1)^2}+\frac{4}{m^3}-\frac{2}{(m+1)^3}\Bigg] \\
&+32\,C_F T_f\,\Bigg[-\frac{2\,S_1(m-1)}{3\,m}+\frac{S_1(m-1)}{3\,(m+1)}+\frac{7}{9\,m} \\
&-\frac{2}{9\,(m+1)}-\frac{2}{3\,m^2}+\frac{1}{3\,(m+1)^2}\Bigg],
\end{aligned}
\tag{3.76}
$$

$$
\begin{aligned}
\gamma_{S,gg}^{(1),m} &= 4\,C_A^2\Bigg[\frac{134}{9}\,S_1(m-1)+\frac{8\,S_1(m-1)}{m^2}-\frac{16\,S_1(m-1)}{(m+1)^2} \\
&+\frac{8\,S_2(m-1)}{m}-\frac{16\,S_2(m-1)}{m+1}+4\,S_3(m-1) \\
&-8\,S_{1,2}(m-1)-8\,S_{2,1}(m-1)+\frac{8\,\tilde{S}_2(m-1)}{m}-\frac{16\,\tilde{S}_2(m-1)}{m+1} \\
&+4\,\tilde{S}_3(m-1)-8\,\tilde{S}_{1,2}(m-1)-\frac{107}{9\,m}+\frac{241}{9\,(m+1)} \\
&+\frac{58}{3\,m^2}-\frac{86}{3\,(m+1)^2}-\frac{8}{m^3}-\frac{48}{(m+1)^3}-\frac{16}{3}\Bigg] \\
&+32\,C_A T_f\,\Bigg[\frac{-5\,S_1(m-1)}{9}+\frac{14}{9m}-\frac{19}{9\,(m+1)}-\frac{1}{3\,m^2}-\frac{1}{3\,(m+1)^2}+\frac{1}{3}\Bigg] \\
&+8\,C_F T_f\,\Bigg[-\frac{10}{m+1}+\frac{2}{(m+1)^2}+\frac{4}{(m+1)^3}+1+\frac{10}{m}-\frac{10}{m^2}+\frac{4}{m^3}\Bigg],
\end{aligned}
\tag{3.77}
$$

where we have introduced the following notations

$$
\begin{aligned}
S_k(m-1) &= \sum_{i=1}^{m-1}\frac{1}{i^k}, \\
\tilde{S}_k(m-1) &= \sum_{i=1}^{m-1}\frac{(-1)^i}{i^k}, \\
S_{k,l}(m-1) &= \sum_{i=1}^{m-1}\frac{1}{i^k}\,S_l(i), \\
\tilde{S}_{k,l}(m-1) &= \sum_{i=1}^{m-1}\frac{1}{i^k}\,\tilde{S}_l(i).
\end{aligned}
$$

To check our results for the two-loop splitting functions (anomalous dimensions) we have also used the BPHZ method [19] as mentioned at the end of section 2. Here we renormalized the OME's graph by graph and found finally the same results as listed in (3.65)–(3.67). As already mentioned in the beginning the above splitting functions and anomalous dimensions have been calculated in the $\overline{\mathrm{MS}}$ scheme. If one prefers another scheme the corresponding anomalous dimensions are related to the $\overline{\mathrm{MS}}$ ones in the following way

$$
\gamma_{NS,qq} = \bar{\gamma}_{NS,qq}+\beta(g)Z_{NS}\frac{d\,Z_{NS}^{-1}}{dg}, \tag{3.78}
$$

$$
\gamma_{S,ij} = Z_{il}\bar{\gamma}_{lm}\left(Z^{-1}\right)_{mj}+\beta(g)Z_{il}\frac{d\,(Z^{-1})_{lj}}{dg}, \tag{3.79}
$$

where $\bar{\gamma}_{k,qq}$ $(k = NS, S)$ denotes the anomalous dimension in the $\overline{\rm MS}$ scheme and Z_{NS}, Z_{ij} are finite operator renormalization constants. Up to order g^2 they can be expressed as follows

$$Z_{NS} = 1 + \frac{g^2}{16\pi^2} z_{qq}, \tag{3.80}$$

$$Z = \begin{pmatrix} 1 + \frac{g^2}{16\pi^2} z_{qq} & \frac{g^2}{16\pi^2} z_{qg} \\ \frac{g^2}{16\pi^2} z_{gq} & 1 + \frac{g^2}{16\pi^2} z_{gg} \end{pmatrix}. \tag{3.81}$$

Substitution of eqs. (3.80), (3.81) into eqs. (3.78), (3.79) yields

$$\gamma^{(1)}_{NS,qq} = \bar{\gamma}^{(1)}_{NS,qq} + 2\beta_0 z_{qq}, \tag{3.82}$$

$$\gamma^{(1)}_{S,qq} = \bar{\gamma}^{(1)}_{S,qq} + 2\beta_0 z_{qq} + z_{qg}\bar{\gamma}^{(0)}_{S,gq} - \bar{\gamma}^{(0)}_{S,qg} z_{gq}, \tag{3.83}$$

$$\gamma^{(1)}_{S,qg} = \bar{\gamma}^{(1)}_{S,qg} + 2\beta_0 Z_{qg} + z_{qg}\left(\bar{\gamma}^{(0)}_{S,gg} - \bar{\gamma}^{(0)}_{S,qq}\right) + \bar{\gamma}^{(0)}_{S,qg}\left(z_{qq} - z_{gg}\right), \tag{3.84}$$

$$\gamma^{(1)}_{S,gq} = \bar{\gamma}^{(1)}_{S,gq} + 2\beta_0 z_{gq} + z_{gq}\left(\bar{\gamma}^{(0)}_{S,qq} - \bar{\gamma}^{(0)}_{S,gg}\right) + \bar{\gamma}^{(0)}_{S,gq}\left(z_{gg} - z_{qq}\right), \tag{3.85}$$

$$\gamma^{(1)}_{S,gg} = \bar{\gamma}^{(1)}_{S,gg} + 2\beta_0 z_{gg} + z_{gq}\bar{\gamma}^{(0)}_{S,qg} - \bar{\gamma}^{(0)}_{S,gq} z_{qg}. \tag{3.86}$$

Before finishing this section we want to make a comment on the spin splitting functions and the anomalous dimensions calculated above. Two of them, i.e., $P^{(1)}_{PS,qq}$ $(\gamma^{(1)}_{PS,qq})$ and $P^{(1)}_{S,qg}$ $(\gamma^{(1)}_{S,qg})$ have been already calculated in the literature [9]. They were obtained via mass factorization of the partonic cross sectoin of the subprocesses $\gamma^* + q \to q + q + \bar{q}$ and $\gamma^* + g \to g + q + \bar{q}$ including the virtual corrections to $\gamma^* + g \to q + \bar{q}$. The result for $P^{(1)}_{PS,qq}$ (3.65) agrees with eq. (3.37) in [9]. However the expression for $P^{(1)}_{S,qg}$ in (3.66) differs from the one obtained in eq. (3.38) of [9] by a finite renormalization, i.e.,

$$P^{(1)}_{S,qg}([9]) - P^{(1)}_{S,qg}(3.66) = P^{(0)}_{S,qg} \otimes z_{qq}, \tag{3.87}$$

with

$$z_{qq} = -16\, C_F\,(1-x), \tag{3.88}$$

and $\otimes$ denotes the convolution symbol

$$(f \otimes g)(x) = \int_0^1 dx_1 \int_0^1 dx_2\, \delta(x - x_1 x_2)\, f(x_1)\, g(x_2). \tag{3.89}$$

This finite renormalization is due to a different γ_5-prescription used in [9].

The above splitting functions, which are calculated in the $\overline{\rm MS}$ scheme, have to be combined with the quark and gluon coefficient functions (2.10) computed in the same scheme in order to perform a complete next-to-leading order analysis. The quark coefficient function can be found in [9, 11] and it equals to

$$\begin{aligned} \tilde{E}_q(x, Q^2, \mu^2) = {} & \delta(1-x) + \frac{g^2}{16\pi^2} C_F \left[\left\{4\left(\frac{1}{1-x}\right)_+ - 2 - 2\,x\right.\right. \\ & \left. + 3\,\delta(1-x)\right\}\ln\frac{Q^2}{\mu^2} + 4\left(\frac{\ln(1-x)}{1-x}\right)_+ - 2\,(1+x)\ln(1-x) \\ & \left. -2\,\frac{1+x^2}{1-x}\ln x - 3\left(\frac{1}{1-x}\right)_+ + 4 + 2\,x - \delta(1-x)(9 + 4\zeta(2))\right] \end{aligned} \tag{3.90}$$

The gluon coefficient function (see e.g. [9, 29]) gets the form

$$\tilde{E}_g(x, Q^2, \mu^2) = \frac{g^2}{16\pi^2} T_f \Big[(8\,x - 4) \ln \frac{Q^2}{\mu^2} + (8\,x - 4) \ln(1 - x) \\ -(8\,x - 4) \ln x + 12 - 16\,x \Big] . \tag{3.91}$$

The Mellin transforms of $\tilde{E}_k$ $(k = q, g)$ become

$$\tilde{E}_q^m(Q^2, \mu^2) = 1 + \frac{g^2}{16\pi^2} C_F \Bigg[\left(3 - \frac{2}{m} - \frac{2}{m+1} - 4 S_1(m-1) \right) \ln \frac{Q^2}{\mu^2} \\ + \left(\frac{2}{m} + \frac{2}{m+1} + 3 \right) S_1(m-1) + 4\, S_{1,1}(m-1) \\ -4\, S_2(m-1) + \frac{6}{m} - 9 \Bigg] , \tag{3.92}$$

$$\tilde{E}_g^m(Q^2, \mu^2) = \frac{g^2}{16\pi^2} T_f \Bigg[\left(\frac{8}{m+1} - \frac{4}{m} \right) \ln \frac{Q^2}{\mu^2} + \left(\frac{4}{m} - \frac{8}{m+1} \right) S_1(m-1) \\ + \frac{4}{m} - \frac{8}{m+1} \Bigg] . \tag{3.93}$$

Notice that the first moment $\tilde{E}_q^1 = 1 - 3(g^2/(16\pi^2))C_F$ agrees with eq. 6 of [10]. Furthermore we have $\tilde{E}_g^1 = 0$ (see [11]). Both properties are characteristic of our choice of the γ_5-prescription and the fact that the anomalous dimensions are calculated in the $\overline{\mathrm{MS}}$ scheme.

4 Properties of the spin anomalous dimensions

In this section we will discuss some of the properties of the splitting functions and anomalous dimensions which have been calculated in the last section. Let us start with the first moments of the spin anomalous dimensions in the $\overline{\mathrm{MS}}$ scheme.

$$\gamma_{NS,qq}^{(0),1} = 0 \qquad \gamma_{NS,qq}^{(1),1} = 0, \tag{4.94}$$

$$\gamma_{S,qq}^{(0),1} = 0 \qquad \gamma_{S,qq}^{(1),1} = 24\, C_F\, T_f, \tag{4.95}$$

$$\gamma_{S,qg}^{(0),1} = 0 \qquad \gamma_{S,qg}^{(1),1} = 0, \tag{4.96}$$

$$\gamma_{S,gq}^{(0),1} = -6\, C_F \qquad \gamma_{S,gq}^{(1),1} = -6\, C_F^2 - \frac{142}{3}\, C_A\, C_F + \frac{8}{3}\, C_F\, T_f, \tag{4.97}$$

$$\gamma_{S,gg}^{(0),1} = -2\,\beta_0 = -\left(\frac{22}{3}\, C_A - \frac{8}{3}\, T_f \right) , \tag{4.98}$$

$$\gamma_{S,gg}^{(1),1} = -2\,\beta_1 = -\frac{68}{3}\, C_A^2 + 8\, C_F\, T_f + \frac{40}{3}\, C_A\, T_f, \tag{4.99}$$

where β_0 and β_1 are the first and second order coefficients in the perturbation series of the β-function (2.15).

In the above we have assumed that there is one flavour only in the fermion loops of the OME graphs. If there are more light flavours the T_f in the above expressions have to be multiplied by the number of light flavours indicated by n_f (see (2.16), (2.17)). The vanishing of the first moment of the non-singlet anomalous dimension follows from the conservation of the axial vector current $R^{\mu}_{NS,qq}(x)$. The value of the singlet anomalous dimension $\gamma^{(1),1}_{S,qq}$ was already calculated in [29]. It is due to the anomaly of the singlet axial vector current $R^{\mu}_{S,q}(x)$ which contributes via the triangular fermion loops to $\gamma^{(1),1}_{S,qq}$ in second order perturbation theory. The vanishing of $\gamma^{(1),1}_{S,gq}$ was shown on general grounds in [30], see also [31]. From the last reference we also infer (see eq. (22) in [31]) that $\gamma^{(1),1}_{S,gg} = -2\beta_1$, provided the anomalous dimension is calculated in the $\overline{\rm MS}$ scheme. Finally we want to investigate an interesting relation which is conjectured for an $N = 1$ supersymmetric Yang-Mills field theory. It can be derived from QCD by putting the colour factors $C_F = C_A = N$ and $T_f = N/2$ [32]. The relation reads as follows. First define

$$\delta\gamma = \gamma_{S,qq} + \gamma_{S,gq} - \gamma_{S,qg} - \gamma_{S,gg}. \tag{4.100}$$

For an $N = 1$ supersymmetric Yang-Mills field theory one has

$$\delta\gamma = 0, \tag{4.101}$$

provided $\delta\gamma$ is calculated in a renormalization scheme which preserves the supersymmetric Ward identities. In many cases one has shown that at least up to two loops n-dimensional reduction is a regularization method which respects the supersymmetric Ward identities. Therefore a renormalization scheme where the pole terms plus the additional constants γ_E (Euler constant) and $\ln 4\pi$ are subtracted ($\overline{\rm MS}$ scheme) will respect these Ward identities too. In lowest order, where there is no difference between n-dimensional regularization and n-dimensional reduction, the above relation holds for the spin as well as spin averaged anomalous dimensions. If one assumes that the two-loop anomalous dimensions calculated in the two regularization schemes (n-dimensional reduction and n-dimensional regularization) are related to each other via a finite renormalization one can derive the following relation [33]

$$\delta\,\gamma^{(1),m}_{\rm RED} - \delta\,\gamma^{(1),m}_{\rm REG} = \left(2\,\beta_0 - \gamma^{(0),m}_{S,qg} - \gamma^{(0),m}_{S,gq}\right)\left(\delta\,a^{(1),m}_{\rm RED} - \delta\,a^{(1),m}_{\rm REG}\right), \tag{4.102}$$

with

$$\delta\,a^{(1)} = a^{(1)}_{S,qq} + a^{(1)}_{S,gq} - a^{(1)}_{S,qg} - a^{(1)}_{S,gg}, \tag{4.103}$$

where the terms $a^{(1)}_{S,ij,\rm REG}$ and $a^{(1)}_{S,ij,\rm RED}$ are the non-pole parts of the OME's in (2.25)–(2.31) which are calculated using n-dimensional regularization and n-dimensional reduction respectively. Equation (4.102) can be easily derived from eqs. (3.82)–(3.86) by putting $\delta\gamma^{(0)} = 0$ and $z_{ij} = a_{S,ij,\rm REG} - a_{S,ij,\rm RED}$. If one makes the additional assumption $\delta\bar{\gamma}^{(1)} = \delta\bar{\gamma}^{(1)}_{\rm RED} = 0$ (4.101) relation (4.102) turns out to be valid for the two-loop spin averaged anomalous dimensions which is checked in [33].

In the case of the spin anomalous dimensions we obtain the following results

$$a^{(1)}_{S,qq,\rm REG} - a^{(1)}_{S,qq,\rm RED} = N\left[-2 + 2\,x + \delta(1-x)\right], \tag{4.104}$$

$$a^{(1)}_{S,qg,\rm REG} - a^{(1)}_{S,qg,\rm RED} = 0, \tag{4.105}$$

$$a^{(1)}_{S,gq,\rm REG} - a^{(1)}_{S,gq,\rm RED} = 0, \tag{4.106}$$

$$a^{(1)}_{S,gg,\rm REG} - a^{(1)}_{S,gg,\rm RED} = N\left[\frac{1}{3}\,\delta(1-x)\right]. \tag{4.107}$$

If we assume that $\delta\,\gamma^{(1)}_{\rm RED} = 0$ then (from (4.102))

$$\delta\,\gamma^{(1),m}_{\rm REG} = 4 - \frac{8}{m^2} + \frac{8}{(m+1)^2} - \frac{28}{3}\frac{1}{m} + \frac{44}{3}\frac{1}{m+1}, \tag{4.108}$$

which is in disagreement with the result of our calculation derived from eqs. (3.74) (3.77) which is equal to

$$\delta\,\gamma^{(1),m}_{\rm REG} = 4 + \frac{548}{3}\frac{1}{m} - \frac{532}{3}\frac{1}{m+1} - \frac{104}{m^2} - \frac{88}{(m+1)^2}. \tag{4.109}$$

We tried to explain the difference between the prediction in (4.108), which is based on supersymmetry, and the result (4.109) obtained by our calculations. Therefore we investigated the findings in [33] for the spin averaged anomalous dimensions and found a surprising result. Since all external legs of the OME's are put off shell one can split $\hat{G}^m_{k,iq}$ (3.49) and $\hat{G}^{m,\mu,\nu}_{k,ig}$ (3.50) into a so-called physical and unphysical part. In the case of $\hat{G}^m_{k,iq}$ the former part is proportional to $\not\Delta$ whereas the latter part is multiplied by $\not p$. This property holds for the spin as well as spin averaged operators. In the spin case $\hat{G}^{m,\mu\nu}_{S,ig}$ has a physical part only which is proportional to $\varepsilon^{\mu\nu\alpha\beta}\Delta_\alpha p_\beta$. However for the spin averaged case one also encounters unphysical parts in the OME's. Here the physical part is the coefficient of the tensor $g^{\mu\nu} - (\Delta_\mu p_\nu + p_\mu \Delta_\nu)/(\Delta p) + \Delta_\mu\Delta_\nu p^2/(\Delta p)^2$. Limiting ourselves to the physical parts of the non-pole terms $a^{(1)}_{S,ij}$ we find the following results. The spin averaged OME's satisfy the relation (Feynman gauge)

$$\delta a^{(1)}_{\rm RED} = 0, \tag{4.110}$$

whereas the spin OME's lead to

$$\delta a^{(1)}_{\rm RED} = N\left[-4 + 4\,x\right]. \tag{4.111}$$

Property (4.110) was not mentioned in [33]. However it might explain why $\delta\gamma^{(1)}_{\rm RED} = 0$ for the spin averaged case since the physical part of the unrenormalized one-loop OME's already satisfy the supersymmetric relation. Hence we have the suspicion that because of (4.111), $\delta\gamma^{(1)}_{\rm RED} \neq 0$ for the spin anomalous dimensions which explains the discrepancy between (4.108) and (4.109). Notice that (4.111) is obtained in the $\overline{\rm MS}$ scheme. Therefore we have made an oversubtraction so that $\delta a^{(1)}_{\rm RED} = 0$. If we now assume that after this subtraction $\delta\gamma^{(1)}_{\rm RED} = 0$ and recalculate $\delta a^{(1)}_{\rm REG}$ (Feynman gauge) by keeping the physical part only we obtain [2]

$$\delta a^{(1)}_{\rm REG} = N\left[-6 + 6\,x + \frac{2}{3}\,\delta(1-x)\right], \tag{4.112}$$

and from (4.102)

$$\delta\gamma^{(1),m}_{\rm REG} = 4 - \frac{24}{m^2} + \frac{24}{(m+1)^2} - \frac{100}{3}\frac{1}{m} + \frac{116}{3}\frac{1}{m+1}. \tag{4.113}$$

By comparing (4.109) with (4.113) we observe again a discrepancy between our calculation and the prediction obtained from the supersymmetric relation except for $m = 1$ where we

[2] Notice that the gauge dependent terms cancel in $\delta a^{(1),m}_{\rm RED} - \delta a^{(1),m}_{\rm REG}$.

have agreement. The reason for the violation of the supersymmetric relation (4.101) in the case of the spin anomalous dimensions is not known to us. It cannot be attributed to an error in the scalar Feynman integrals because they were also used to recalculate the spin averaged anomalous dimensions which we found to be in agreement with the results quoted in the literature. It might be due to our γ_5-prescription. However different γ_5-prescriptions are related via finite renormalizations and the discrepancy between (4.109) and (4.108) or (4.113) cannot be explained by such an effect. Finally we want to emphasize that to our knowledge the formal proof of the supersymmetric relation $\delta\gamma = 0$ is still lacking in the literature.

Appendix A: The operator vertices

In this appendix we present the twist-2 operator vertices. All momenta are flowing into the operator vertex.

A.1 Quark-(gluon) operator vertices

The quark-antiquark vertex is equal to

$$O(p) \quad = \quad -\not{\Delta}\gamma_5(\Delta p)^{m-1}, \tag{A.1}$$

where p denotes the momentum of the incoming quark line.
The quark-quark-gluon vertex is given by

$$O_a^{\mu}(p,q) \quad = \quad -g\,T_a\,\Delta^{\mu}\not{\Delta}\gamma_5\sum_{i=0}^{m-2}(\Delta p)^{m-i-2}(-\Delta q)^i, \tag{A.2}$$

where p and q are the momenta of the incoming quark and antiquark respectively.
The quark-quark-gluon-gluon vertex equals

$$\begin{aligned} O_{ab}^{\mu\nu}(p,q,r,s) \quad = \quad & g^2\,\Delta^{\mu}\,\Delta^{\nu}\not{\Delta}\gamma_5 \\ & \times\left[T_aT_b\left((-1)^m\sum_{j=0}^{m-3}\sum_{i=0}^{j}(-1)^j(\Delta p)^i(\Delta q)^{m-j-3}((\Delta p)+(\Delta s))^{j-i}\right)\right. \\ & \left.-T_bT_a\left(\sum_{j=0}^{m-3}\sum_{i=0}^{j}(-1)^j(\Delta p)^{m-j-3}(\Delta q)^j((\Delta q)+(\Delta s))^{j-i}\right)\right]. \end{aligned} \tag{A.3}$$

A.2 Gluon-operator vertices

The 2-gluon vertex is given by

$$O_{ab}^{\mu\nu}(p) = i\varepsilon^{\mu\nu\Delta p}(1-(-1)^m)(\Delta p)^{m-1}\delta_{ab}. \tag{A.4}$$

The 3-gluon vertex is equal to

$$O_{abc}^{\mu\nu\rho}(p,q,r) \quad = \quad g(1-(-1)^m)\,f_{abc}\,O^{\mu\nu\rho}(p,q,r), \tag{A.5}$$

with

$$
\begin{aligned}
O^{\mu\nu\rho}(p,q,r) \;=\; & -(\varepsilon^{\mu\rho p\Delta}\Delta^{\nu}-\varepsilon^{\mu\nu p\Delta}\Delta^{\rho})(\Delta p)^{m-2}-(\varepsilon^{\nu\rho\Delta q}\Delta^{\mu}+\varepsilon^{\mu\nu\Delta q}\Delta^{\rho})(\Delta q)^{m-2} \\
& -(\varepsilon^{\nu\rho\Delta r}\Delta^{\mu}-\varepsilon^{\mu\rho\Delta r}\Delta^{\nu})(\Delta r)^{m-2} \\
& +\sum_{i=0}^{m-3}(-\Delta p)^{i}(\Delta q)^{m-i-3}\Delta^{\rho}(\varepsilon^{\nu p\Delta q}\Delta^{\mu}+\varepsilon^{\mu\nu\Delta q}(\Delta p)) \\
& -\sum_{i=0}^{m-3}(-\Delta r)^{i}(\Delta p)^{m-i-3}\Delta^{\nu}(\varepsilon^{\mu p\Delta r}\Delta^{\rho}+\varepsilon^{\mu\rho\Delta p}(\Delta r)) \\
& -\sum_{i=0}^{m-3}(-\Delta r)^{i}(\Delta q)^{m-i-3}\Delta^{\mu}(\varepsilon^{\nu\Delta qr}\Delta^{\rho}-\varepsilon^{\nu\rho\Delta q}(\Delta r)).
\end{aligned}
\tag{A.6}
$$

The 4-gluon vertex equals

$$
\begin{aligned}
O^{\mu\nu\rho\sigma}_{abcd}(p,q,r,s) \;=\; & i\,g^{2}(1-(-1)^{m})\,[f_{abe}f_{cde}O^{\mu\nu\rho\sigma}(p,q,r,s) \\
& +f_{ace}f_{bde}O^{\mu\rho\nu\sigma}(p,r,q,s)-f_{ade}f_{bce}O^{\rho\nu\mu\sigma}(r,q,p,s)]\,,
\end{aligned}
\tag{A.7}
$$

where

$$
\begin{aligned}
& O^{\mu\nu\rho\sigma}(p,q,r,s) \;=\; (\varepsilon^{\Delta\nu\rho\sigma}\Delta^{\mu}-\varepsilon^{\Delta\mu\rho\sigma}\Delta^{\nu})((\Delta r)+(\Delta s))^{m-2} \\
& -\Delta^{\rho}(\varepsilon^{\nu\sigma\Delta s}\Delta^{\mu}-\varepsilon^{\mu\sigma\Delta s}\Delta^{\nu})\sum_{i=0}^{m-3}((\Delta r)+(\Delta s))^{i}(\Delta s)^{m-i-3} \\
& +\Delta^{\sigma}(\varepsilon^{\rho\nu r\Delta}\Delta^{\mu}-\varepsilon^{\rho\mu r\Delta}\Delta^{\nu})\sum_{i=0}^{m-3}((\Delta p)+(\Delta q))^{m-i-3}(-(\Delta r))^{i} \\
& +\Delta^{\nu}\sum_{i=0}^{m-3}((\Delta r)+(\Delta s))^{m-i-3}(-(\Delta p))^{i}(\varepsilon^{\mu\sigma\Delta p}\Delta^{\rho}-\varepsilon^{\mu\rho\Delta p}\Delta^{\sigma}) \\
& +\Delta^{\mu}\sum_{i=0}^{m-3}((\Delta r)+(\Delta s))^{m-i-3}(-(\Delta q))^{i}(\varepsilon^{\nu\sigma\Delta q}\Delta^{\rho}-\varepsilon^{\nu\rho\Delta q}\Delta^{\sigma}) \\
& +\Delta^{\nu}\Delta^{\rho}\sum_{j=0}^{m-4}\sum_{i=0}^{j}(\Delta p)^{m-j-4}((\Delta p)+(\Delta q))^{j-i}(-(\Delta s))^{i}(\varepsilon^{\Delta\sigma ps}\Delta^{\mu}+\varepsilon^{\mu\sigma\Delta s}(\Delta p)) \\
& -\Delta^{\mu}\Delta^{\rho}\sum_{j=0}^{m-4}\sum_{i=0}^{j}(\Delta q)^{m-j-4}((\Delta p)+(\Delta q))^{j-i}(-(\Delta s))^{i}(\varepsilon^{\Delta\sigma qs}\Delta^{\nu}+\varepsilon^{\nu\sigma\Delta s}(\Delta q)) \\
& -\Delta^{\nu}\Delta^{\sigma}\sum_{j=0}^{m-4}\sum_{i=0}^{j}(\Delta p)^{m-j-4}(-(\Delta r))^{i}((\Delta p)+(\Delta q))^{j-i}(\varepsilon^{\Delta\mu pr}\Delta^{\rho}+\varepsilon^{\mu\rho\Delta p}(\Delta r)) \\
& +\Delta^{\mu}\Delta^{\sigma}\sum_{j=0}^{m-4}\sum_{i=0}^{j}(\Delta q)^{m-j-4}(-(\Delta r))^{i}((\Delta p)+(\Delta q))^{j-i}(\varepsilon^{\Delta\nu qr}\Delta^{\rho}+\varepsilon^{\nu\rho\Delta q}(\Delta r)).
\end{aligned}
\tag{A.8}
$$

Appendix B: The tensorial reduction

In this appendix we present a more detailed explanation of the tensorial reduction of the tensor Feynman integrals into scalar integrals.

According to the reading point method [20] we can put the γ^5-matrix at the right end of the traces. Then one can perform all straightforward simplifications of the γ-matrix algebra inside the traces. Furthermore we leave the γ^5-matrix untouched except that at the end we take $Tr\left(\gamma^\alpha\gamma^\beta\gamma^\sigma\gamma^\delta\gamma^5\right) = -4\,i\,\epsilon^{\alpha\beta\sigma\delta}$ In the case of the one-loop integrals the tensorial reduction can be very easily achieved via the standard Feynman parameter techniques. Since we do not need tensor integrals beyond rank two it is sufficient to list the following integrals

$$
\begin{aligned}
I_{ij} &= \int \frac{d^n q}{(2\pi)^n} \frac{(\Delta q)^m}{[q^2]^i\,[(q-p)^2]^j} \\
&= i\,S_n \frac{(-p^2)^{n/2}}{(p^2)^{i+j}} (\Delta p)^m \frac{\Gamma\left(i+j-\frac{n}{2}\right)}{\Gamma(i)\Gamma(j)} \int_0^1 dx\; x^m\; x^{n/2-1-i}(1-x)^{n/2-1-j}, \qquad \text{(B.1)} \\
I_{ij}^{\mu} &= \int \frac{d^n q}{(2\pi)^n} \frac{q^\mu\,(\Delta q)^m}{[q^2]^i\,[(q-p)^2]^j} \\
&= i\,S_n \frac{(-p^2)^{n/2}}{(p^2)^{i+j}} (\Delta p)^m \frac{1}{\Gamma(i)\Gamma(j)} \int_0^1 dx\; x^m\; x^{n/2-1-i}(1-x)^{n/2-1-j} \\
&\times \Bigg[\Gamma(i+j-\frac{n}{2}) x p^\mu + \frac{1}{2}\left\{\left(i+j-\frac{n}{2}\right)(1-2\,x)\right. \\
&\left. +\Gamma(i+j-1-\frac{n}{2})((1-x)(1-j)-x(1-i))\right\}\frac{\Delta^\mu p^2}{\Delta p}\Bigg], \qquad \text{(B.2)} \\
I_{ij}^{\mu\nu} &= \int \frac{d^n q}{(2\pi)^n} \frac{q^\mu\,q^\nu\,(\Delta q)^m}{[q^2]^i\,[(q-p)^2]^j} \\
&= i\,S_n \frac{(-p^2)^{n/2}}{(p^2)^{i+j}} (\Delta p)^m \frac{1}{\Gamma(i)\Gamma(j)} \int_0^1 dx\; x^m\; x^{n/2-1-i}(1-x)^{n/2-1-j} \\
&\times \Bigg[\Gamma(i+j-\frac{n}{2}) x^2 p^\mu p^\nu + \frac{1}{2}\Gamma\left(i+j-1-\frac{n}{2}\right) x(1-x) g^{\mu\nu} p^2 + \\
&+ \frac{\Delta^\mu p^\nu + p^\mu \Delta^\nu}{2\,\Delta p} p^2 \left\{\Gamma\left(i+j-\frac{n}{2}\right)(x-2\,x^2)\right. \\
&\left. +\Gamma\left(i+j-1-\frac{n}{2}\right)(-x(1-x)\,j + x^2(i-1))\right. \\
&+ \frac{\Delta^\mu\Delta^\nu}{4\,(\Delta p)^2}(p^2)^2 \left\{\Gamma\left(i+j-\frac{n}{2}\right)(1-4\,x+4\,x^2)\right. \\
&+2\,\Gamma(i+j-1-\frac{n}{2})\Big((1-x)^2(1-j) + x^2(1-i) + x(1-x)(i+j-1)\Big) \\
&+\Gamma\left(i+j-2-\frac{n}{2}\right)\Big\{(1-x)^2(j-1)(j-2) + x^2(i-1)(i-2) \\
&-2\,x(1-x)(i-1)(j-1)\}\Big], \qquad \text{(B.3)}
\end{aligned}
$$

where S_n is the spherical factor $S_n = \pi^{\frac{n}{2}}/(2\pi)^n$.
The tensorial reduction of the two-loop tensor Feynman integrals is much more complicated and has been performed by using the program *FeynCalc* [31]. The numerators of the two-loop Feynman integrals have the following structure.

$$
A_i = f_i^{\sigma_1\cdots\sigma_k}\;\tilde{f}_{i\sigma_1\cdots\sigma_k}(q_1,q_2), \qquad \text{(B.4)}
$$

where q_1 and q_2 denote the integration momenta. Explicit forms of $f_i^{\sigma_1\cdots\sigma_k}$ and $\tilde{f}_{i,\sigma_1\cdots\sigma_k}(q_1,q_2)$

are ($\varepsilon^{\alpha\beta\sigma\delta} p_\alpha \Delta_\beta = \varepsilon^{p\Delta\sigma\delta}$, etc.)

$$
\begin{aligned}
f_1(q_1,q_2) &= \varepsilon^{q_1 q_2 \Delta p}, & \tilde{f}_1(q_1,q_2) &= \varepsilon^{q_1 q_2 \Delta p}, && \text{(B.5)}\\
f_2^{\alpha}(q_1,q_2) &= \varepsilon^{q_1 p \Delta \alpha}, & \tilde{f}_{2,\alpha}(q_1,q_2) &= \varepsilon^{q_2 p \Delta \alpha}, && \text{(B.6)}\\
f_3^{\alpha\beta}(q_1,q_2) &= \varepsilon^{q_1 q_2 \alpha\beta}, & \tilde{f}_{3,\alpha\beta}(q_1,q_2) &= \mathrm{Tr}(\not{q}_1 \not{q}_2 \not{\Delta} \not{p}\, \gamma_\alpha \gamma_\beta \gamma_5). && \text{(B.7)}
\end{aligned}
$$

The tensor integrals can be represented as

$$
\begin{aligned}
I^{\alpha;\alpha\beta;\alpha\beta\sigma;\alpha\beta\sigma\delta}(\Delta,p) &= \int \frac{d^n q_1}{(2\pi)^n} \int \frac{d^n q_2}{(2\pi)^n}\, K(q_1,q_2,\Delta,p) \\
&\quad \times \left\{ q_i^\alpha; q_i^\alpha q_j^\beta; q_i^\alpha q_j^\beta q_k^\sigma; q_i^\alpha q_j^\beta q_k^\sigma q_l^\delta \right\}, \qquad \text{(B.8)}
\end{aligned}
$$

where $i = 1,2$ and $j = 1,2$. Further we have the definition

$$
K(q_1,q_2,\Delta,p) = \frac{(\Delta q_1)^a (\Delta q_2)^b (\Delta(p-q_1))^c (\Delta(p-q_2))^d (\Delta(q_1-q_2))^e}{(q_1^2)^f (q_2^2)^g ((q_1-p)^2)^h ((q_2-p)^2)^i ((q_1-q_2)^2)^j}. \qquad \text{(B.9)}
$$

Notice that the integers $a - g$ can take positive as well as negative integer values. By virtue of Lorentz covariance the integral $I(\Delta,p)$ can now be written as

$$
\begin{aligned}
I^{\alpha;\alpha\beta;\alpha\beta\sigma;\alpha\beta\sigma\delta}(\Delta,p) &= \sum_s \left\{ T_s^\alpha, T_s^{\alpha\beta}, T_s^{\alpha\beta\sigma}, T_s^{\alpha\beta\sigma\delta} \right\} \\
&\quad \times \int \frac{d^n q_1}{(2\pi)^n} \int \frac{d^n q_1}{(2\pi)^n} \sum_r f_r(p^2,n) K_r(q_1,q_2,\Delta,p), \qquad \text{(B.10)}
\end{aligned}
$$

with

$$
\begin{aligned}
T_s^\alpha &= \{p^\alpha; \Delta^\alpha\}, && \text{(B.11)}\\
T_s^{\alpha\beta} &= \left\{ g^{\alpha\beta}; p^\alpha p^\beta; \Delta^\alpha \Delta^\beta \right\}, && \text{(B.12)}\\
T_s^{\alpha\beta\sigma} &= \left\{ g^{\alpha\beta} p^\sigma; \cdots; \Delta^\alpha \Delta^\beta \Delta^\sigma \right\}, && \text{(B.13)}\\
T_s^{\alpha\beta\sigma\delta} &= \left\{ g^{\alpha\beta} g^{\sigma\delta}; \cdots; \Delta^\alpha \Delta^\beta \Delta^\sigma \Delta^\delta \right\}, && \text{(B.14)}
\end{aligned}
$$

where the K_r are of the same type as the K in (B.8) (but with different indices a-j) and the $f_r(p^2,n)$ are simple polynomial-like functions determined by the tensorial reduction.
In this way all Lorentz indices are transformed away from the integration momenta to the external momentum p and the lightlike vector Δ. The advantage of the tensorial reduction method is revealed when one evaluates e.g. the expression $\tilde{f}_{3,\alpha\beta}$ (B.7). This gets simplified to $\mathrm{Tr}(\not{\Delta}\not{p}\,\gamma_\alpha\gamma_\beta\gamma_5) = -4\, i\, \varepsilon^{\Delta p \alpha\beta}$. Hence one can avoid any γ_5-prescription dependence arising from the non-unique way of calculating a trace of six γ-matrices plus the γ_5-matrix in n dimensions. The explicit reduction formalae, which are too lengthy to be presented here, are obtained by using projection methods. They are incorporated in the program *FeynCalc 3.0* [25]. The scalar integrals which appear on the right hand side of (B.10) are calculated in [10] using the algebraic manipulation program FORM [27]. The two-loop integrals including the tensorial reduction have been checked by recalculating the spin averaged splitting functions which have been computed in the past (see [14]-[17]) and we found full agreement.

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Figure captions.

Fig. 1 One-loop graphs contributing to the spin OME's; (a), (b): $A^{(1)}_{NS,qq}$, $A^{(1)}_{S,qq}$; (c): $A^{(1)}_{S,gq}$; (d), (e): $A^{(1)}_{S,qg}$; (f), (g): $A^{(1)}_{S,gg}$. Graphs with external self-energies and with triangular fermion-loops where the arrows are reversed have been included in the calculation but are not shown in the figure. Graphs which are not symmetric with respect to the vertical line through the operator vertex have to be counted twice.

Fig. 2 Two-loop graphs contributing to the spin non-singlet OME $A^{(2)}_{NS,qq}$. Graphs with external self-energies have been included in the calculation but are not drawn in the figure. Graphs which are not symmetric with respect to the vertical line through the operator vertex have to be counted twice.

Fig. 3 Two-loop graphs contributing to the spin pure-singlet OME $A^{(2)}_{PS,qq}$. Graphs with triangular fermion loops where the arrows are reversed have been included in the calculation but are not shown in the figure. Graphs which are not symmetric with respect to the vertical line through the operator vertex have to be counted twice.

Fig. 4 Two-loop graphs contributing to the spin singlet OME $A^{(2)}_{S,qg}$. Graphs with triangular fermion loops where the arrows are reversed and diagrams containing external self energies have been included but are not shown in the figure. Graphs which are not symmetric with respect to the vertical line through the operator vertex have to be counted twice.

Fig. 5 Two-loop graphs contributing to the spin singlet OME $A^{(2)}_{S,gq}$. Graphs with external self-energies have been included in the calculation but are not drawn in the figure. Graphs which are not symmetric with respect to the vertical line through the operator vertex have to be counted twice.

Fig. 6 Two-loop graphs contributing to the spin singlet OME $A^{(2)}_{S,gg}$. Graphs with external self-energies and diagrams with ghost and triangular fermion loops where the arrows are reversed have been included in the calculation but are not drawn in the figure. Graphs which are not symmetric with respect to the vertical line through the operator vertex have to be counted twice.

Figure 1:

Figure 2:

Figure 3:

Figure 4:

Figure 5:

Figure 6: