Head-on collision of compact objects in general relativity: Comparison of post-Newtonian and perturbation approaches

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The gravitational-wave energy flux produced during the head-on infall and collision of two compact objects is calculated using two approaches: (i) a post-Newtonian method, carried to second post-Newtonian order beyond the quadrupole formula, valid for systems of arbitrary masses; and (ii) a black-hole perturbation method, valid for a test-body falling radially toward a black hole. In the test-body case, the methods are compared. The post-Newtonian method is shown to converge to the "exact" perturbation result more slowly than expected *a priori*. A surprisingly good approximation to the energy radiated during the infall phase, as calculated by perturbation theory, is found to be given by a Newtonian, or quadrupole, approximation combined with the exact test-body equations of motion in the Schwarzschild spacetime.

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I. INTRODUCTION AND SUMMARY

The head-on collision of two bodies in general relativity has long been a test-bed for various approaches to solving Einstein's equations in dynamical situations, despite its vanishingly small astrophysical probability. Because of the axial symmetry of the situation, the problem simplifies dramatically, yet can still retain features of potential astrophysical importance, such as strong curvature, fast motions, and the emission of gravitational radiation to infinity. The first attempt to solve this problem in a fully general relativistic manner was the 1971 calculation of perturbations of the Schwarzschild black-hole spacetime caused by a test particle in radial infall from infinity [1], using the recently developed Zerilli equation for black-hole perturbations [2]. The main result of this calculation was a formula for the total gravitational-wave energy emitted, $\Delta E = 0.0104\mu c^2(\mu/m)$, where μ is the mass of the test body, and m the mass of the black hole ($\mu \ll m$). Subsequently, these perturbation methods were extended to handle more general test-body orbits around black holes [3–5].

The head-on collision of two black holes has also been the starting point in attempts to solve general relativistic, dynamical problems by purely numerical techniques. The pioneering computation of the head-on-collision of two equal-mass black holes in the middle 1970s by Smarr and Eppley [6] stood for almost 15 years as the state of the art, until recent technical advances together with the use of supercomputers resulted in a substantial improvement in accuracy and reliability [7]. Generalizing these computations to the case of black holes in circular orbits is sufficiently difficult that it is viewed as a "grand challenge."

Another method for studying the interaction of compact objects and the emission of gravitational radiation is the post-Newtonian approximation. Although this method can be used for bodies of arbitrary masses in arbitrary orbits, it breaks down when the gravitational fields become strong and the velocities become comparable to the speed of light.

The ultimate goal toward which these calculations are pointed is the solution of the inspiral and coalescence of two compact objects (black holes or neutron stars), and the calculation to high accuracy of the emitted gravitational waveform. Such inspiralling systems are believed to be the most promising sources for detection of gravitational waves by kilometer-scale laser interferometric gravitational observatories (LIGO [8] in the US, VIRGO [9] in Europe). Furthermore, it has been shown that very accurate gravitational-waveform templates will be needed in order to infer characteristics of the sources from the data, using matched filtering techniques [10]. It is also widely believed that all three theoretical techniques will play a role in this effort: post-Newtonian techniques for describing much of the inspiral phase (until the system becomes too relativistic), numerical techniques for the late inspiral and coalescence, and test-body perturbation techniques for the special case of a small mass inspiralling into a very massive black hole. Consequently, it is important to explore carefully the regions of validity and the accuracy of these techniques, especially in regimes where they overlap.

In this paper we study the gravitational radiation emitted during the head-on collision of two compact objects. In particular, we compute the gravitational-wave luminosity as a function of their relative separation. We use two techniques: (i) the post-Newtonian approximation, carried to second order [order $(v/c)^4 \sim (Gm/rc^2)^2$] beyond the quadrupole approximation; and (ii) the test-body perturbation technique. Specializing the post-Newtonian method to the case of a test-body falling from infinity toward a massive body, we compare the two approaches. Our main findings are these:

(1) The post-Newtonian approximation begins to fail when the bodies are separated by about $10Gm/c^2$ (in harmonic coordinates), where m is the total mass of the system. By failure, we mean that post-Newtonian corrections to the energy flux become comparable to the leading-order, quadrupole term.

(2) The post-Newtonian approximation converges very slowly: at a separation of $100Gm/c^2$, the first and second post-Newtonian corrections to the luminosity are respectively 5 and 40 times their expected sizes of $Gm/rc^2 \simeq 10^{-2}$, and $(Gm/rc^2)^2 \simeq 10^{-4}$, relative to the leading-order term.

(3) The test-body approach shows that for separations larger than $4Gm/c^2$ (in Schwarzschild coordinates), the radiation emitted is dominated by bremsstrahlung radiation generated by the infalling particle. Thereafter, the radiation is dominated by black-hole quasi-normal ringing. The bremsstrahlung radiation contributes only approximately 3 percent of the total energy emitted.

(4) At separations greater than about $10Gm/c^2$, the perturbation results can actually be well approximated by a hybrid "quadrupole" formula for the luminosity, in which the exact Schwarzschild test-body equations of motion are used to evaluate the three time derivatives of the source quadrupole moment. The post-Newtonian values differ from the perturbation results by 20% at a separation of $45Gm/c^2$, 10% at $75Gm/c^2$, and 1% at $400Gm/c^2$, reflecting a very slow convergence. The slow convergence of the post-Newtonian approximation has also been noted in the case of circular test-body orbits [11–14].

The organization of this paper is as follows: Section II discusses the head-on collision of bodies of arbitrary masses to second post-Newtonian (2PN) order beyond the quadrupole approximation, including the effects of gravitational-wave tails. Section III discusses the perturbation approach, and presents the energy flux for test-body radial infall as a function of retarded time, showing both the bremsstrahlung radiation and the dominant quasi-normal ringing radiation. In section IV we compare the two approaches; to do so requires a careful mapping between the post-Newtonian retarded source variables, expressed in harmonic coordinates, and the retarded Schwarzschild coordinates of the perturbation approach. Section V discusses the accuracy of the post-Newtonian approach, and derives the hybrid "quadrupole" formula for the luminosity. In an Appendix we discuss a remarkable series of cancellations that occur in the post-Newtonian method for the radial infall from infinity.

II. HEAD-ON COLLISION OF COMPACT OBJECTS TO SECOND POST-NEWTONIAN ORDER

A. Gravitational wave generation to second post-Newtonian order

This first subsection summarizes from the literature the relevant equations for gravitational wave generation by a slowly moving system of two arbitrary, non spinning, masses [15].

We describe the motion of the two bodies in terms of their relative distance. In a harmonic coordinate system whose origin is at the center of mass, the relative position vector is $\boldsymbol{x} \equiv \boldsymbol{x}_1 - \boldsymbol{x}_2$, where $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are the positions of each body. If the masses are denoted m_1 and m_2, we define the total mass $m \equiv m_1 + m_2$, the reduced mass $\mu \equiv m_1 m_2/m$, $\eta \equiv \mu/m$, and $\delta m \equiv m_1 - m_2$. Also, $r \equiv |\boldsymbol{x}|$ and $\boldsymbol{v} \equiv \dot{\boldsymbol{x}}$, where a dot denotes differentiation with respect to time.

The luminosity (or energy flux) associated with the gravitational waves can be expressed in terms of the mass and current multipole moments of the radiation field [16]. We need only a few terms for an expression accurate through second post-Newtonian (2PN) order beyond the leading-order, quadrupole-formula expression:

$$\dot{E}(T) = \frac{1}{5}\left({}^{(3)}I\!\!\!/^{ij}\right)^2 + \frac{1}{189}\left({}^{(4)}I\!\!\!/^{ijk}\right)^2 + \frac{1}{9072}\left({}^{(5)}I\!\!\!/^{ijkl}\right)^2 + \cdots + \frac{16}{45}\left({}^{(3)}J\!\!\!/^{ij}\right)^2 + \frac{1}{84}\left({}^{(4)}J\!\!\!/^{ijk}\right)^2 + \cdots. \tag{2.1}$$

Here, $I\!\!\!/^{\{i_n\}}$ and $J\!\!\!/^{\{i_n\}}$ are respectively the symmetric-trace-free (STF) mass and current multipole moments of the radiation field, and the antescript denotes the number of derivatives taken with respect to time. The luminosity is expressed as a function of time T, which represents proper time as measured by static observers at large distances. The total energy radiated, from initial time T_i to final time T_f, is given by

$$\Delta E = \int_{T_i}^{T_f} \dot{E}(T)\, dT. \tag{2.2}$$

The radiative multipole moments in Eq. (2.1) can be related to the multipole moments of the source. These are expanded in powers of $v^2 \sim m/r$ to the order required for an expression for the luminosity accurate through 2PN order. They can be expressed as

$$
\begin{aligned}
\mathcal{I}^{ij} &= \mu \Bigg\{ \left[1 + \frac{29}{42}\,(1-3\eta)\,v^2 - \frac{1}{7}\,(5-8\eta)\left(\frac{m}{r}\right) \right] x^i\,x^j - \frac{4}{7}\,(1-3\eta)\,r\,\dot r\,x^i\,v^j + \frac{11}{21}\,(1-3\eta)\,r^2\,v^i\,v^j \\
&\quad + x^i\,x^j \left[\frac{1}{504}\,(253 - 1835\,\eta + 3545\,\eta^2)\,v^4 + \frac{1}{756}\,(2021 - 5947\,\eta - 4883\,\eta^2)\,v^2 \left(\frac{m}{r}\right) \right. \\
&\quad \left. - \frac{1}{252}\,(355 + 1906\,\eta - 337\,\eta^2)\left(\frac{m}{r}\right)^2 - \frac{1}{756}\,(131 - 907\,\eta + 1273\,\eta^2)\,\dot r^2 \left(\frac{m}{r}\right) \right] \\
&\quad + r^2\,v^i\,v^j \left[\frac{1}{189}\,(742 - 335\,\eta - 985\,\eta^2)\left(\frac{m}{r}\right) + \frac{1}{126}\,(41 - 337\,\eta + 733\,\eta^2)\,v^2 + \frac{5}{63}\,(1 - 5\eta + 5\eta^2)\,\dot r^2 \right] \\
&\quad - r\,\dot r\,v^i\,x^j \left[\frac{1}{378}\,(1085 - 4057\,\eta - 1463\,\eta^2)\left(\frac{m}{r}\right) + \frac{1}{63}\,(26 - 202\,\eta + 418\,\eta^2)\,v^2 \right] \Bigg\}_{STF} + \mathcal{I}^{ij}_{Tail}, \qquad (2.3a) \\
\mathcal{I}^{ijk} &= -\mu\frac{\delta m}{m} \left\{ \left[1 + \frac{1}{6}\,(5 - 19\,\eta)\,v^2 - \frac{1}{6}\,(5 - 13\,\eta)\left(\frac{m}{r}\right) \right] x^i\,x^j\,x^k + (1-2\eta)(r^2\,v^i\,v^j\,x^k - r\,\dot r\,v^i\,x^j\,x^k) \right\}_{STF}, \qquad (2.3b) \\
\mathcal{I}^{ijkl} &= \mu\,(1-3\eta)\,(x^i\,x^j\,x^k\,x^l)_{STF}, \qquad (2.3c) \\
\mathcal{J}^{ij} &= -\mu\frac{\delta m}{m} \Bigg(\varepsilon^{iab} \Bigg\{ \left[1 + \frac{1}{2}\,(1-5\eta)\,v^2 + 2\,(1+\eta)\left(\frac{m}{r}\right) \right] x^j x^a v^b \\
&\quad + \frac{1}{28}\frac{d}{dt}\left[(1-2\eta)(3r^2 v^j - r\dot r x^j)\,x^a v^b \right] \Bigg\} \Bigg)_{STF}, \qquad (2.3d) \\
\mathcal{J}^{ijk} &= \mu\,(1-3\eta)\,(\varepsilon^{iab}\,x^a\,v^b\,x^j\,x^k\,x^l)_{STF}. \qquad (2.3e)
\end{aligned}
$$

The radiation measured at time T by an observer at a large distance R from the source is made up of two contributions. The first can be thought of as radiation propagating directly from the source; it depends on the state of the source at retarded time $u \simeq T - R$ (see below for a more precise expression). The second can be thought of as radiation scattered by the spacetime curvature as it propagates away from the source. This contribution, called the "tail," is emitted by the source at all times earlier than u. The tail term can also be decomposed into multipole moments. The dominant term, and the only one needed to evaluate $\dot E$ to 2PN order, is given by the following integral over retarded time along the source's trajectory $\mathcal{C}(u')$ prior to retarded time u [17]:

$$
\mathcal{I}^{ij}_{Tail}(u) = 2m \int_{\mathcal{C}(u')} {}^{(2)}\mathcal{I}^{ij}\,(u-u') \left[\ln\left(\frac{u'}{2\mathcal{S}}\right) + \frac{11}{12} \right] du'. \qquad (2.4)
$$

Here, u' parametrizes the trajectory up to the final point $u' = 0$. The constant $\mathcal{S}$ is an arbitrary scale parameter which arises from the joining of two coordinate systems (see Sec. IIC for a discussion); we will show that our results do not depend on the value of this parameter.

The last element needed to compute the energy flux are the equations of motion. For any two-body problem the relative acceleration $\boldsymbol{a} = \boldsymbol{a}_1 - \boldsymbol{a}_2$ in a harmonic coordinate system with origin at the center of mass, accurate to 2PN order, is given by [18]

$$
\boldsymbol{a} = -\left(\frac{m}{r^2}\right) \left[(1 + A_1 + A_2)\,\boldsymbol{n} + (B_1 + B_2)\,\boldsymbol{v} \right], \qquad (2.5)
$$

where $\boldsymbol{n} = \boldsymbol{x}/r$, and

$$
\begin{aligned}
A_1 &= -2\,(2+\eta)\left(\frac{m}{r}\right) + (1+3\eta)\,v^2 - \frac{3}{2}\,\eta\,\dot r^2, \qquad (2.6a) \\
A_2 &= \frac{3}{4}\,(12 + 29\,\eta)\left(\frac{m}{r}\right)^2 + \eta\,(3-4\eta)\,v^4 + \frac{15}{8}\,\eta\,(1-3\eta)\,\dot r^4 - \frac{3}{2}\,\eta\,(3-4\eta)\,v^2\,\dot r^2 \\
&\quad - \frac{1}{2}\,\eta\,(13-4\eta)\left(\frac{m}{r}\right) v^2 - (2 + 25\,\eta + 2\eta^2)\left(\frac{m}{r}\right)\dot r^2, \qquad (2.6b) \\
B_1 &= -2\,(2-\eta)\,\dot r, \qquad (2.6c) \\
B_2 &= \frac{1}{2}\,\dot r \left[(4 + 41\,\eta + 8\eta^2)\left(\frac{m}{r}\right) - \eta\,(15+4\eta)\,v^2 + 3\eta\,(3+2\eta)\,\dot r^2 \right]. \qquad (2.6d)
\end{aligned}
$$

We substitute Eq. (2.5) to the required order whenever time derivatives of the velocity appear in the various terms comprising Eq. (2.1).

Solving Eq. (2.5) yields the trajectory $\boldsymbol{x}(t)$. As in Newtonian mechanics, solving this equation is facilitated by making use of the first integrals: E the energy, and $\boldsymbol{J}$ the angular momentum. Both of these quantities are conserved to 2PN order. Below we will specialize to the case of a head-on collision, for which $\boldsymbol{J}=0$. We therefore only need an expression for the energy, which is given by [19]

$$\begin{aligned} E = \mu \Big\{ & \frac{1}{2}\, v^2 - \left(\frac{m}{r}\right) + \frac{3}{8}(1-3\eta)\, v^4 + \frac{1}{2}(3+\eta)\left(\frac{m}{r}\right)\, v^2 + \frac{1}{2}\eta\left(\frac{m}{r}\right)\dot{r}^2 + \frac{1}{2}\left(\frac{m}{r}\right)^2 + \frac{5}{16}(1-7\eta+13\,\eta^2)\, v^6 \\ & + \frac{1}{8}(21-23\,\eta-27\,\eta^2)\left(\frac{m}{r}\right)\, v^4 + \frac{1}{4}\eta\,(1-15\,\eta)\left(\frac{m}{r}\right)\, v^2\,\dot{r}^2 - \frac{3}{8}\eta\,(1-3\eta)\left(\frac{m}{r}\right)\dot{r}^4 \\ & + \frac{1}{8}(14-55\,\eta+4\eta^2)\left(\frac{m}{r}\right)^2 v^2 + \frac{1}{8}(4+69\,\eta+12\,\eta^2)\left(\frac{m}{r}\right)^2\dot{r}^2 - \frac{1}{4}(2+15\,\eta)\left(\frac{m}{r}\right)^3 \Big\}. \end{aligned} \tag{2.7}$$

B. Head-on collision

In this subsection we use the equations listed above to calculate the gravitational-wave luminosity produced during the head-on infall and collision of two bodies of arbitrary masses. We consider two different situations. In the first (A), the infall proceeds from rest at infinite initial separation; in the second (B), the infall proceeds from rest at finite initial separation.

There is only one direction of motion in this problem. We therefore have $\boldsymbol{x} = z\,\boldsymbol{n}$, $\boldsymbol{v} = \dot{z}\,\boldsymbol{n}$, $r=z$, and $v=\dot{r}=\dot{z}$. Using this, Eqs. (2.1) and (2.3) simplify considerably. Moreover, since the current moments are all proportional to $\boldsymbol{J}=0$, they all vanish. The mass multipole moments reduce to:

$$\begin{aligned} \mathcal{I}^{ij} = \mu z^2 \Big[& 1 + \frac{9}{14}\,(1-3\eta)\,\dot{z}^2 - \frac{1}{7}\,(5-8\eta)\left(\frac{m}{z}\right) - \frac{1}{252}\,(355+1096\,\eta-337\,\eta^2)\left(\frac{m}{z}\right)^2 \\ & + \frac{1}{126}\,(448+289\,\eta-1195\,\eta^2)\left(\frac{m}{z}\right)\dot{z}^2 + \frac{1}{168}\,(83-589\,\eta+1111\,\eta^2)\,\dot{z}^4 \Big]\,\bar{\boldsymbol{P}}^{(2)} + \mathcal{I}^{ij}_{Tail}, \end{aligned} \tag{2.8a}$$

$$\mathcal{I}^{ijk} = -\mu z^3\,\frac{\delta m}{m}\left[1 + \frac{1}{6}\,(5-19\eta)\,\dot{z}^2 - \frac{1}{6}\,(5-13\eta)\left(\frac{m}{z}\right)\right]\bar{\boldsymbol{P}}^{(3)}, \tag{2.8b}$$

$$\mathcal{I}^{ijkl} = \mu z^4\,(1-3\eta)\;\bar{\boldsymbol{P}}^{(4)}. \tag{2.8c}$$

Here, the $\bar{\boldsymbol{P}}^{(q)}$'s are STF products of a number q of unit vectors, as given by Pirani [20] and Thorne [16]:

$$\bar{\boldsymbol{P}}^{(2)} \equiv n^i n^j - \frac{1}{3}\,\delta^{ij} \equiv \boldsymbol{P}^{(2)} - \frac{1}{3}\,\boldsymbol{D}, \tag{2.9a}$$

$$\bar{\boldsymbol{P}}^{(3)} \equiv n^i n^j n^k - \frac{1}{5}\left(\delta^{ij} n^k + \delta^{ik} n^j + \delta^{jk} n^i\right) \equiv \boldsymbol{P}^{(3)} - \frac{3}{5}\left(\boldsymbol{D}\cdot\boldsymbol{P}^{(1)}\right), \tag{2.9b}$$

$$\begin{aligned} \bar{\boldsymbol{P}}^{(4)} \equiv\; & n^i n^j n^k n^l - \frac{1}{7}\left(\delta^{ij} n^k n^l + \delta^{ik} n^j n^l + \delta^{il} n^k n^j + \delta^{jk} n^i n^l + \delta^{jl} n^i n^k + \delta^{jk} n^i n^l\right) \\ & + \frac{1}{35}\left(\delta^{ij}\delta^{kl} + \delta^{ik}\delta^{jl} + \delta^{il}\delta^{kj}\right) \equiv \boldsymbol{P}^{(4)} - \frac{6}{7}\left(\boldsymbol{D}\cdot\boldsymbol{P}^{(2)}\right) + \frac{3}{35}\left(\boldsymbol{D}\cdot\boldsymbol{D}\right); \end{aligned} \tag{2.9c}$$

the $\boldsymbol{P}^{(q)}$'s are symmetric products of a number q of unit vectors, and $\boldsymbol{D}$ symbolically represents the Kronecker delta. The compact notation to the right-hand-side of Eqs. (2.9), with which we represent fully symmetrized products of symmetric tensors by enclosing them within parentheses [e.g., $(\boldsymbol{D}\cdot\boldsymbol{P}^{(1)})$ or $(\boldsymbol{D}\cdot\boldsymbol{D})$], corresponds to expressions (2.1) and (2.2) in Ref. [16]. Notice that $\mathcal{I}^{ij}_{Tail}$ in Eq. (2.8a) also contains a factor $\bar{\boldsymbol{P}}^{(2)}$; it arises via the term ${}^{(2)}\mathcal{I}^{ij}$ in the integrand of Eq. (2.4).

As the movement is directed along the z axis, the $\boldsymbol{P}^{(q)}$'s have components along z only (e.g., $\boldsymbol{P}^{(2)} = \delta^{iz}\delta^{jz}$). The $\bar{\boldsymbol{P}}^{(q)}$'s are therefore diagonal tensors, constant in time, and they factor out of the time derivatives in Eq. (2.1). As a result, we need only calculate the squares of the $\bar{\boldsymbol{P}}^{(q)}$'s,

$$\bar{\boldsymbol{P}}^{(2)}\cdot\bar{\boldsymbol{P}}^{(2)} = \frac{2}{3}, \qquad \bar{\boldsymbol{P}}^{(3)}\cdot\bar{\boldsymbol{P}}^{(3)} = \frac{2}{5}, \qquad \bar{\boldsymbol{P}}^{(4)}\cdot\bar{\boldsymbol{P}}^{(4)} = \frac{8}{35}, \tag{2.10}$$

and then insert them into the squares of ${}^{(n+1)}\mathcal{I}^{i_1 \cdots i_n}$ in Eq. (2.1).

We must now evaluate the time derivatives of the mass multipole moments, substituting the expression for the acceleration $\ddot{z}$ when needed. For a head-on collision, Eq. (2.5) reduces to

$$\ddot{z} = -\frac{m}{z^2}\left[1 - 2\,(2+\eta)\left(\frac{m}{z}\right) - \left(3 - \frac{7}{2}\eta\right)\dot{z}^2 + \left(\frac{m}{z}\right)^2\left(9 + \frac{87}{4}\eta\right) - \left(\frac{m}{z}\right)\dot{z}^2\,\eta\,(11 - 4\eta) - \frac{21}{8}\,\dot{z}^4\,\eta\,(\eta+1)\right]. \quad (2.11)$$

It is also helpful to make use of our expression (2.7) for the conserved energy in order to express $\dot{z}$ as a function of m/z. For a radial infall, we have

$$E = \mu\left[\frac{1}{2}\,\dot{z}^2 - \left(\frac{m}{z}\right) + \frac{3}{8}\,(1-3\eta)\,\dot{z}^4 + \frac{1}{2}\,(3+2\eta)\left(\frac{m}{z}\right)\dot{z}^2 + \frac{1}{2}\left(\frac{m}{z}\right)^2 + \frac{5}{16}\,(1 - 7\eta + 13\,\eta^2)\,\dot{z}^6\right.$$
$$\left. + \frac{3}{8}\,(7 - 8\eta - 16\,\eta^2)\left(\frac{m}{z}\right)\dot{z}^4 + \frac{1}{4}\,(9 + 7\eta + 8\eta^2)\left(\frac{m}{z}\right)^2\dot{z}^2 - \frac{1}{4}\,(2 + 15\,\eta)\left(\frac{m}{z}\right)^3\right]. \quad (2.12)$$

Defining z_0 to be the initial separation (at which $\dot{z} = 0$), we can invert Eq. (2.12) for $\dot{z}(z, z_0)$. For case (A), in which the initial separation is infinite, $E(z) = E(\infty) = 0$, and we obtain

$$\dot{z} = -\left\{\frac{2m}{z}\left[1 - 5\Big(1 - \frac{\eta}{2}\Big)\Big(\frac{m}{z}\Big) + \Big(13 - \frac{81}{4}\eta + 5\,\eta^2\Big)\Big(\frac{m}{z}\Big)^2\right]\right\}^{1/2}. \quad (2.13)$$

For case (B), in which the initial separation is finite, we have

$$E(z) = E(z_0) = -\mu\left[\left(\frac{m}{z_0}\right) - \frac{1}{2}\left(\frac{m}{z_0}\right)^2 + \frac{1}{2}\left(1 + \frac{15}{2}\eta\right)\left(\frac{m}{z_0}\right)^3\right], \quad (2.14)$$

and we obtain

$$\dot{z} = -\left\{2\left(\frac{m}{z} - \frac{m}{z_0}\right)\left[1 - 5\left(\frac{m}{z}\right)\left(1 - \frac{\eta}{2}\right) + \left(\frac{m}{z_0}\right)\left(1 - \frac{9}{2}\,\eta\right) + \left(\frac{m}{z}\right)^2\left(13 - \frac{81}{4}\eta + 5\,\eta^2\right)\right.\right.$$
$$\left.\left. + \left(\frac{m}{z}\right)\left(\frac{m}{z_0}\right)\left(5 - \frac{173}{4}\eta + 13\,\eta^2\right) + \left(\frac{m}{z_0}\right)^2\left(1 - \frac{5}{4}\eta + 8\,\eta^2\right)\right]\right\}^{1/2}. \quad (2.15)$$

We can then insert Eqs. (2.11), (2.13), and Eq. (2.15) into the time derivatives of the mass moments, Eqs. (2.8), and calculate the gravitational-wave luminosity in terms of m/z and m/z_0.

We discuss in the Appendix an interesting cancellation of terms which occurs in the case of an infall from infinity.

C. Tail terms

We now turn to the evaluation of the tail contribution to the gravitational-wave energy flux. To this end, we must first consider the tail correction to the mass quadrupole moment, as given by Eq. (2.4).

To leading order, we have $\mathcal{I}^{zz} \propto z^2$. Because $\dot{z} \approx (m/z)^{1/2}$, each time derivative has the effect of multiplying this quantity by $m^{1/2} z^{-3/2}$. Integration with respect to retarded time then has the effect of multiplying the resulting expression by by $z/\dot{z} \sim z^{3/2} m^{-1/2}$. As a result, we have that $\mathcal{I}^{zz}_{Tail} \propto (m/z)^{3/2}\,\mathcal{I}^{zz}$, which shows that the tail correction is of 1.5PN order relative to the direct quadrupole term. Post-Newtonian corrections to $\mathcal{I}^{zz}_{Tail}$ will then be of 2.5PN order relative to the same term, and can be neglected in an expansion accurate through second post-Newtonian order. We therefore have that in the calculation of $\mathcal{I}^{ij}_{Tail}$, we may put $\mathcal{I}^{ij} \equiv Q^{zz}\,\bar{\boldsymbol{P}}^{(2)} = \mu z^2\,\bar{\boldsymbol{P}}^{(2)}$, and take $\mathcal{C}(u)$ to represent the source's Newtonian trajectory.

For case (A) — infall from infinity — the tail term contributes to the luminosity via

$${}^{(3)}\mathcal{I}^{ij}_{Tail} = 2m\bar{\boldsymbol{P}}^{(2)}\int_0^{\infty} {}^{(5)}Q^{zz}\,(u - u')\left[\ln\left(\frac{u'}{2\mathcal{S}}\right) + \frac{11}{12}\right]du', \quad (2.16)$$

where the relation between separation z and time t on the Newtonian trajectory is given by

$$t = -\frac{4m}{3}\left(\frac{z}{2m}\right)^{3/2}. \tag{2.17}$$

The integral in Eq. (2.16) can be evaluated in closed form, and the result is

$${}^{(3)}\!I^{ij}_{Tail} = -2\eta\left(\frac{m}{z}\right)^4\left[\frac{71}{6} + \frac{5\pi}{\sqrt{3}} + 15\ln\left(\frac{m}{z}\right) + 10\ln\left(\sqrt{\frac{2}{3}}\frac{\mathcal{S}}{m}\right)\right]\bar{\boldsymbol{P}}^{(2)}. \tag{2.18}$$

For case (B) — infall from a finite distance z_0 — the tail term is

$${}^{(3)}\!I^{ij}_{Tail} = 2m\bar{\boldsymbol{P}}^{(2)}\int_0^{u(z_0)} {}^{(5)}Q^{zz}\left(u-u'\right)\left[\ln\left(\frac{u'}{2\mathcal{S}}\right) + \frac{11}{12}\right]du'. \tag{2.19}$$

The Newtonian trajectory is here given by

$$t = \frac{m}{\sqrt{2}}\left(\frac{m}{z_0}\right)^{-3/2} g(x), \tag{2.20}$$

where $g(x) = \sqrt{x}\sqrt{1-x} - \arcsin\sqrt{x}$, with $x = z/z_0 < 1$. Evaluating the integral, we obtain

$${}^{(3)}\!I^{ij}_{Tail} = 4\eta\left(\frac{m}{z_0}\right)^4\left\{\left(\frac{5-4x}{x^4}\right)\left[\frac{11}{12} - \frac{3}{2}\ln\left(\frac{m}{z_0}\right) - \ln\left(2\sqrt{2}\frac{\mathcal{S}}{m}\right)\right] + Int(x)\right\}\bar{\boldsymbol{P}}^{(2)}, \tag{2.21}$$

where $\mathrm{Int}(x)$ is given by

$$Int(x) = 4\int_x^1\left(\frac{5-3y}{y^5}\right)\ln\left[g(x)-g(y)\right]\,dy. \tag{2.22}$$

In this form, $\mathrm{Int}(x)$ has a very slow numerical convergence, because the argument of the logarithm vanishes at the lower limit. To evaluate it, we integrate by parts and change variables, so that

$$\begin{aligned} Int(x) = &-\frac{4\,(5-3x)}{x^5}\sqrt{\frac{1-x}{x}}\,g(x) \\ &+ 4\int_y^{\pi/2}\frac{(24\sin^4 w - 77\sin^2 w + 55)}{\sin^{12} w}\Big\{\left[h(y)-h(w)\right]\ln\left[h(y)-h(w)\right] + h(w)\Big\}\,dw, \end{aligned} \tag{2.23}$$

where $y = \arcsin\sqrt{x}$, and $h(w) = \sin w\cos w - w$. This expression is much better suited for numerical calculation.

We now show that the gravitational-wave luminosity is independent of the arbitrary parameter $\mathcal{S}$ appearing in Eqs. (2.18) and (2.21). For simplicity, we will only consider case (A) — infall from infinity; the argument for case (B) is entirely analogous.

In the approach of Blanchet and Damour [17], the energy flux at infinity is obtained by first solving the generation of gravitational waves in the near zone containing the source, using harmonic coordinates and expanding about flat spacetime, and then matching this solution outside the source to a solution of the vacuum equations in the far zone, expressed in radiative coordinates. In the near-zone solution, the waves propagate along "flat" null cones given by

$$t - r = c, \tag{2.24}$$

while in the far-zone solution, they propagate along the true null cones given by

$$t - r - 2m\ln r = c', \tag{2.25}$$

where c and c' are constants. The two solutions are matched at an intermediate distance $\mathcal{S} \sim \lambda \sim r/v$, where λ is the gravitational wavelength. This matching is the origin of the tail term, Eq. (2.18), together with its apparent $\mathcal{S}$-dependence.

When evaluating the energy flux, we must trace an observation made at time T by an observer at distance R back along the null cone to the corresponding time t_{CM} (evaluated at $r = 0$, the center of mass) which determines the state of the source. The null cone that reaches the observer (T, R) is characterized by $c' = T - R - 2m\ln R$; the match at $\mathcal{S}$ gives $c' = c - 2m\ln\mathcal{S}$, and the null cone at $r = 0$ is characterized by $t_{CM} = c$. The result is

$$t_{CM} = T - R - 2m\ln\left(\frac{R}{\mathcal{S}}\right). \tag{2.26}$$

We find it convenient to rescale $\mathcal{S}$ by m, and to define retarded time $u = T - R - 2m\ln(R/m)$. Then

$$t_{CM} = u - 2m\ln\left(\frac{m}{\mathcal{S}}\right). \tag{2.27}$$

This exhibits the explicit $\mathcal{S}$-dependence in the connection between $z(t)$ and the observer. We now show that this $\mathcal{S}$-dependence cancels the $\mathcal{S}$-dependence in the tail term.

The trajectory at Newtonian order becomes

$$\begin{aligned}\frac{2m}{z} &= \left(\frac{4m}{3|t_{CM}|}\right)^{2/3} = \left\{\frac{4m}{3\left[|u| + 2m\ln(m/\mathcal{S})\right]}\right\}^{2/3} \simeq \left(\frac{4m}{3|u|}\right)^{2/3}\left[1 - \frac{4m}{3|u|}\ln\left(\frac{m}{\mathcal{S}}\right)\right] \\ &\simeq \frac{2m}{z'}\left[1 - \left(\frac{2m}{z'}\right)^{3/2}\ln\left(\frac{m}{\mathcal{S}}\right)\right],\end{aligned} \tag{2.28}$$

where $z' = z(u)$; we have used the fact that $t_{CM} < 0$, and have performed the expansions assuming $m/|u| \ll 1$. The Newtonian contribution to ${}^{(3)}\mathcal{I}^{ij}$ is just the corresponding derivative of the quadrupole moment $\mathcal{I}^{ij}_N = \mu z^2 \bar{\boldsymbol{P}}^{(2)}$,

$${}^{(3)}\mathcal{I}^{ij}_N = \frac{\eta}{2}\left(\frac{2m}{z}\right)^{5/2}\bar{\boldsymbol{P}}^{(2)}, \tag{2.29}$$

where the Newtonian expressions for $\dot{z}$ and $\ddot{z}$ were substituted to yield a result in terms of z only. Applying the shift (2.28) to this contribution, we obtain

$$\begin{aligned}{}^{(3)}\mathcal{I}^{ij}_N &= \frac{\eta}{2}\left(\frac{2m}{z}\right)^{5/2}\bar{\boldsymbol{P}}^{(2)} \simeq \frac{\eta}{2}\left(\frac{2m}{z'}\right)^{5/2}\left[1 - \left(\frac{2m}{z'}\right)^{3/2}\ln\left(\frac{m}{\mathcal{S}}\right)\right]^{5/2}\bar{\boldsymbol{P}}^{(2)} \\ &\simeq {}^{(3)}(\mathcal{I}^{ij}_N)' - 20\,\eta\left(\frac{m}{z}\right)^4\ln\left(\frac{m}{\mathcal{S}}\right)\bar{\boldsymbol{P}}^{(2)}.\end{aligned} \tag{2.30}$$

The additional term in Eq. (2.30) cancels the dependence on $\ln(m/\mathcal{S})$ in the tail term, Eq. (2.18).

Thus the gravitational-wave luminosity is truly independent of the matching scale $\mathcal{S}$. In practice therefore, we may assume that we have identified the orbital variable z at the correct retarded time, and eliminate the $\mathcal{S}$-dependence in the tail term by setting $\mathcal{S} = m$.

D. Luminosity and energy loss

We now have all the elements needed to calculate the gravitational-wave luminosity for both types of collision, (A) and (B). We use the acceleration (2.11) and the velocities (2.13) and (2.15) to obtain the time derivatives of the multipole moments (2.8), including the tail terms, Eq. (2.18) and (2.21). We then square each term, inserting the squares of the STF tensors, Eq. (2.10). Finally, we compute $\dot{E}$.

1. Infall from infinity

For case (A) — infall from infinity — the luminosity is given by

$$\begin{aligned}\dot{E} = \frac{16}{15}\eta^2\left(\frac{m}{z}\right)^5 \Bigg\{ &1 - \frac{1}{7}\left(\frac{m}{z}\right)\left(43 - \frac{111}{2}\eta\right) \\ &- \sqrt{2}\left(\frac{m}{z}\right)^{3/2}\left[\frac{71}{6} + \frac{5\pi}{\sqrt{3}} + 15\ln\left(\frac{m}{z}\right) + 5\ln\left(\frac{2}{3}\right)\right] - \frac{1}{3}\left(\frac{m}{z}\right)^2\left(\frac{1127}{9} + \frac{803}{12}\eta - 112\,\eta^2\right)\Bigg\}.\end{aligned} \tag{2.31}$$

In Eq. (2.31), within the curly brackets, we label the terms of order $O(1)$, $O(m/z)$, $O[(m/z)^{3/2}]$, and $O[(m/z)^2]$, as [N], [1PN], [1.5PN], and [2PN], respectively.

In Fig. 1 we plot $\dot{E}/\eta^2$ as a function of z for different values of $\eta = \mu/m$, from $\eta = 0$ (test-body limit) to $\eta = 0.25$ (equal-mass case). We see that $\dot{E}$ increases monotonically with increasing η, and that the curves converge at large z; for such large separations, $\dot{E}/\eta^2 \simeq (16/15)(m/z)^5$, independent of η. The plot also shows that $\dot{E}$ increases with decreasing z, until it reaches a maximum and then starts decreasing. This behavior signals the breakdown of the post-Newtonian expansion, as we now discuss.

In Fig. 2a (for $\eta = 0$) and 2b (for $\eta = 0.25$) we separate the luminosity (2.31) into the various contributions corresponding to the [1PN], [1.5PN], and [2PN] terms, and plot their ratio to the leading-order, Newtonian [N] expression. Notice that both the [1PN] and the [2PN] contributions are negative, while the [1.5PN] contribution is positive, because of its logarithmic term. In both figures, for large z, the post-Newtonian terms are all smaller than the Newtonian expression. However, they are larger than might be expected *a priori*: at $z = 100m$, the [PN], [1.5PN], and [2PN] contributions would be expected to be of order 10^{-2}, 10^{-3}, and 10^{-4}, respectively. Instead, we find that they actually are of order $4-6\times10^{-2}$, 7×10^{-2}, and 4×10^{-3}, respectively. This reflects the presence of large numerical coefficients in Eq. (2.31). As z decreases, the post-Newtonian terms increase until they become comparable to each other and to the Newtonian expression; this occurs within the interval $7 \lesssim z/m \leq 10$. For $z \lesssim 7m$, the negative post-Newtonian corrections overcome the Newtonian term, and the luminosity formally changes sign. Evidently, the post-Newtonian approximation is no longer reliable in this region. Finally, comparing Fig. 2a with Fig. 2b, we see that the term which varies the most with η is the [1PN] contribution; the [2PN] term varies only weakly with η, and the [1.5PN] term is independent of η, as Eq. (2.31) directly shows.

The total energy radiated, up to time T_f, is given by Eq. (2.2),

$$\Delta E(z_f) = \int_{-\infty}^{T_f} \dot{E}(T)\, dT = -\int_{z_f}^{\infty} \dot{E}(z)\, \frac{dz}{\dot{z}}. \tag{2.32}$$

The luminosity is given by (2.31) and $\dot{z}$ by Eq. (2.13). It is not possible to evaluate this integral in closed form. Instead, we approximate $1/\dot{z}$ by its expansion in powers of m/z, up to the second order, and then integrate. We obtain

$$\begin{aligned}\Delta E(z_f) = {}& \frac{16\sqrt{2}}{105}\,\eta^2 m \left(\frac{m}{z_f}\right)^{7/2} \left\{1 - \frac{17}{6}\left(\frac{m}{z_f}\right)\left(1 - \frac{11}{6}\eta\right)\right. \\ & \left. - \frac{7}{5\sqrt{2}}\left(\frac{m}{z_f}\right)^{3/2}\left[\frac{53}{6} + \frac{5\pi}{\sqrt{3}} + 15\ln\left(\frac{m}{z_f}\right) - \ln\left(\frac{2}{3}\right)\right] - \frac{1}{66}\left(\frac{m}{z_f}\right)^2\left[\frac{81985}{36} - \frac{749}{3}\eta - \frac{18323}{16}\eta^2\right]\right\}. \end{aligned} \tag{2.33}$$

In Fig. 3 we plot $\Delta E/\eta^2 m$ as a function of z_f, for different values of η. For large z_f, the energy radiated behaves as its Newtonian expression, $\Delta E/\eta^2 m \simeq (16\sqrt{2}/105)(m/z_f)^{7/2}$, independent of η. For small z_f, $\Delta E/\eta^2 m$ increases with increasing η. In Fig. 3, the smallest value of z_f at which ΔE was calculated corresponds to where the gravitational-wave luminosity formally vanishes.

2. Infall from a finite distance

For case (B) — infall from a finite distance z_0 — the luminosity is given by

$$\begin{aligned}\dot{E} = {}& \frac{16}{15}\eta^2\left(\frac{m}{z}\right)^5 \left(1 - x - \frac{1}{7}\left(\frac{m}{z}\right)\left[\left(43 - \frac{111}{2}\eta\right) - x\,(116 - 131\,\eta) + x^2\left(71 - \frac{135}{2}\eta\right)\right]\right. \\ & + \sqrt{2}\left(\frac{m}{z} - \frac{m}{z_0}\right)^{1/2}\left\{\left(\frac{m}{z_0}\right)\left(x^3 + 4 - \frac{5}{x}\right)\left[3\ln\left(\frac{2m}{z_0}\right) - \frac{11}{6}\right] + 2x^3\left(\frac{m}{z_0}\right) Int(x)\right\} \\ & - \frac{1}{3}\left(\frac{m}{z}\right)^2\left[\left(\frac{1127}{9} + \frac{803}{12}\eta - 112\,\eta^2\right) + \frac{1}{7}x\left(\frac{4471}{9} - \frac{15481}{3}\eta + 2864\,\eta^2\right)\right. \\ & \left.\left. - \frac{1}{7}x^2\left(1870 - \frac{38521}{6}\eta + \frac{8800}{3}\eta^2\right) + x^3\left(83 - \frac{1183}{4}\eta + \frac{872}{7}\eta^2\right)\right]\right), \end{aligned} \tag{2.34}$$

where $x = z/z_0$ and $\mathrm{Int}(x)$ is given by Eq. (2.22). This luminosity has characteristics similar to that calculated previously. The main difference is that here, contrary to case (A), the luminosity does not vanish at the beginning of the infall. This, we shall now explain, is due to the fact that although the velocity vanishes at $z = z_0$, the acceleration does not; that $\dot{E}$ does not vanish at $z = z_0$ merely reflects the time symmetry of the trajectory at $t = t(z_0)$. At Newtonian order the general expression for the energy flux, $\dot{E} = (8\eta^2/15)(m/r)^4(12\,v^2 - 11\,\dot{r})$, necessarily vanishes at a moment of stationarity ($v = \dot{r} = 0$). This is because it arises from an odd number (3) of time derivatives of a mass multipole moment. At higher order, however, contributions arising from even numbers of time derivatives appear, e.g., ${}^{(4)}\!I^{ijk}$ in Eq. (2.1). These contain terms involving the acceleration, which does not vanish at z_0. As a consequence, the luminosity also will not vanish.

As in case (A), we see here also that the luminosity changes sign at some small value of z, when the negative higher-order terms become comparable to the positive Newtonian term. This signals the breakdown of the post-Newtonian method.

The total energy radiated, from the moment of time symmetry to the final separation z_f, is given by

$$\Delta E(z_f) = -\int_{z_f}^{z_0} \dot{E}(z)\,\frac{dz}{\dot{z}}, \tag{2.35}$$

where $\dot{E}(z)$ is given by Eq. (2.34) and $\dot{z}$ by Eq. (2.15). We have integrated this equation numerically for selected values of z_0; the smallest value of z_f is chosen to be the one for which the luminosity formally vanishes.

In Figs. 4a (for $\eta = 0$) and 4b (for $\eta = 0.25$) we plot $\Delta E(z_f)/\eta^2 m$ as a function of the final separation z_f, for different values of the initial separation z_0, including the limit $z_0 = \infty$ considered previously. We see that as z_f decreases, the curves all approach each other. This results from the fact that most of the energy is generated near the end of the infall.

III. HEAD-ON COLLISION OF A SMALL MASS AND A MASSIVE BLACK HOLE

In this section we calculate the gravitational-wave luminosity produced during the radial infall, proceeding from rest at infinity, of a particle with small mass into a massive, nonrotating black hole. The smaller mass will now be denoted μ, and that of the black hole m. We assume $\mu \ll m$; in this limit μ and m are equivalent to the reduced mass and total mass of the previous section.

This restriction on the mass ratio implies that the problem considered here is a limiting case of the one considered in the previous section, in which no restriction was put on the masses and the nature of the colliding objects. On the other hand, the problem considered here can also be seen to be an extension of the one considered in the previous section: contrary to Sec. II, we shall here put no restriction on the velocity of the infalling mass, and correspondingly, put no restriction on the strength of the gravitational field at the particle's location. While the results of the previous section were accurate through second post-Newtonian order, the results presented here are accurate to all orders in v/c.

The stress-energy tensor associated with the infalling particle creates a small perturbation in the gravitational field of the nonrotating black hole (whose metric is given by the Schwarzschild solution). Part of this perturbation represents the Coulomb field of the infalling particle; the remaining part represents radiative degrees of freedom, and these propagate away from the source as gravitational waves.

We use the Teukolsky perturbation formalism [21] to calculate $\dot{E}(T)$, the gravitational-wave luminosity as measured near future null infinity; the luminosity is expressed as a function of time T (to be defined precisely below). In Sec. IV this result will be compared to that obtained in Sec. II using post-Newtonian theory.

The total amount $\Delta E = \int \dot{E}(T)dT$ of gravitational-wave energy produced during the radial infall of a particle into a Schwarzschild black hole was first calculated by Davis, Ruffini, Press, and Price [1]. This calculation was subsequently generalized to nonradial motion by Detweiler and Szedenits [3]. Infall into a Kerr black hole was considered by Sasaki and Nakamura [4] (radial motion) and Kojima and Nakamura [5] (nonradial motion).

The Teukolsky perturbation formalism is described in detail in Ref. [22]; we shall make frequent use of equations contained in this paper. In the Teukolsky formalism, the gravitational perturbations are described by a single, complex-valued function Ψ_4, which represents a particular component of the perturbed Weyl tensor. The second-order differential equation governing Ψ_4 can be completely separated by decomposing Ψ_4 into Fourier modes and spherical-harmonic components. The radial function $R_{LM}(\omega; r)$, where ω is the angular frequency and L, M the spherical-harmonic indices, satisfies an inhomogeneous, second-order, ordinary differential equation, whose source term is constructed from the particle's stress-energy tensor. This equation can readily be solved by means of a Green's function.

As shown in Sec. II of Ref. [22], the gravitational waveforms, h_+ and $h_\times$, as measured by a fictitious detector situated near future null infinity, can be obtained from the asymptotic behavior $R_{LM}(\omega; r \to \infty)$ of the radial function. With a slight change in notation, Eq. (2.14) of Ref. [22] becomes

$$h(T, R, \theta, \phi) = \frac{2\mu}{R} \sum_{LM} Z_{LM}(u) \, {}_{-2}Y_{LM}(\theta, \phi). \tag{3.1}$$

Here, $h \equiv h_+ - ih_\times$, T is proper time at the detector's location, R is the distance from the source to the gravitational-wave detector, and

$$u = T - R^* = T - R - 2m \ln\left(\frac{R}{2m} - 1\right) \tag{3.2}$$

is retarded time; T and R are Schwarzschild coordinates, which are distinct from the harmonic coordinates used in the preceding section. The functions $Z_{LM}(u)$ represent the multipole moments of the gravitational-wave field; they are expressed as the Fourier integrals

$$Z_{LM}(u) = \int \tilde{Z}_{LM}(\omega) \, e^{-i\omega u} \, d\omega, \tag{3.3}$$

where $\tilde{Z}_{LM}(\omega)$ will be given below. Finally, ${}_{-2}Y_{LM}(\theta, \phi)$ are the spherical harmonics of spin-weight -2 [23].

In terms of h given above, the gravitational-wave luminosity can be expressed as

$$\dot{E}(T) = \frac{R^2}{16\pi} \int \frac{\partial \bar{h}}{\partial T} \frac{\partial h}{\partial T} \, d\Omega, \tag{3.4}$$

where an overbar denotes complex conjugation, and the integration is over solid angles. Substituting Eq. (3.1) into (3.4), and using the orthonormality of the spin-weighted spherical harmonics, we obtain

$$\dot{E}(T) = \frac{\mu^2}{4\pi} \sum_{LM} \left| \dot{Z}_{LM}(u) \right|^2, \tag{3.5}$$

where

$$\dot{Z}_{LM}(u) = \int (-i\omega) \, \tilde{Z}_{LM}(\omega) \, e^{-i\omega u} \, d\omega. \tag{3.6}$$

It is straightforward to follow the prescription described in Ref. [22] and to calculate an expression for $\tilde{Z}_{LM}(\omega)$ for the problem under consideration. The first step is to describe the motion of the particle, whose world line is taken to be a marginally bound, radial geodesic of the Schwarzschild spacetime. We take $\tilde{E} \equiv u_t = 1$, where u^α is the particle's four-velocity, and $\theta = 0$ along the world line. The geodesic equations are

$$\frac{dr}{d\tau} = -\left(\frac{2m}{r}\right)^{1/2}, \qquad \frac{dt}{d\tau} = \frac{1}{f}, \tag{3.7}$$

where τ denotes proper time and $f = 1 - 2m/r$. These imply the following relation along the world line, where $x = (r/2m)^{1/2}$:

$$t(r) = -4mg(x), \tag{3.8a}$$

$$g(x) = \frac{1}{3} x^3 + x + \frac{1}{2} \ln \frac{x-1}{x+1}. \tag{3.8b}$$

Proceeding along the lines described in Sec. II of Ref. [22], we obtain the following expression for $\tilde{Z}_{LM}(\omega)$:

$$\tilde{Z}_{LM}(\omega) = \tilde{Z}_L(\omega) \, \delta_{M0}. \tag{3.9}$$

The fact that only the modes with $M = 0$ contribute to the radiation reflects the axial symmetry of the problem. We also obtain

$$\tilde{Z}_L(\omega) = -\frac{i}{2m\omega}\,\frac{\sqrt{(L-1)L(L+1)(L+2)}}{(L-1)L(L+1)(L+2)-12im\omega}\,\sqrt{\frac{2L+1}{4\pi}}\,\frac{1}{A_L^{in}(\omega)}\int_{2m}^{\infty}\frac{\sqrt{2m/r}}{\left(1+\sqrt{2m/r}\right)^2}e^{-i\omega t(r)}\,\Gamma_L(\omega)X_L(\omega;r)\,dr. \tag{3.10}$$

The quantities $A_L^{in}(\omega)$, $\Gamma_L(\omega)$, and $X_L(\omega;r)$ have not yet been introduced. We shall explain their meaning in the following paragraph.

The function $X_L(\omega;r)$ is a solution to the Regge-Wheeler equation [24],

$$\left\{\frac{d^2}{dr^{*2}}+\omega^2+f\left[\frac{L(L+1)}{r^2}-\frac{6m}{r^3}\right]\right\}X_L(\omega;r)=0, \tag{3.11}$$

where $d/dr^* = f d/dr$. It is chosen so as to have the asymptotic behavior

$$X_L(\omega;r\to 2m)\sim e^{-i\omega r^*} \tag{3.12}$$

near the black-hole horizon. Correspondingly,

$$X_L(\omega;r\to\infty)\sim A_L^{in}(\omega)e^{-i\omega r^*}+O\big(e^{i\omega r^*}\big); \tag{3.13}$$

this equation defines the constant $A_L^{in}(\omega)$ appearing in Eq. (3.10). Our expression for $\tilde{Z}_L(\omega)$ also involves the first-order differential operator

$$\begin{aligned}\Gamma_L(\omega) = {} & 2(1-3m/r+i\omega r)rf\frac{d}{dr}+f\left[L(L+1)-6m/r\right]\\ & +2i\omega r(1-3m/r+i\omega r).\end{aligned} \tag{3.14}$$

The strategy to calculate the gravitational-wave luminosity is the following. For given values of L and ω, the Regge-Wheeler equation (3.11) is integrated numerically, starting near $r=2m$ and using Eq. (3.12) to specify the initial conditions. The Regge-Wheeler function and its first derivative are evaluated at values of r lying in the interval $(2m,\infty)$. The constant $A_L^{in}(\omega)$ is computed using Eq. (3.13). The integral to the right of Eq. (3.10) is then carried out, numerically, for the given values of L and ω. These steps are repeated for many relevant values of the frequency, and the integral to the right of Eq. (3.6) is evaluated for many values of u. Finally, the sum over L in Eq. (3.5) is carried out, and $\dot{E}(u)$ is obtained for the selected values of u.

The integral to the right of Eq. (3.10) is actually divergent, and must be regularized before carrying out the program outlined in the preceding paragraph. To do this we follow Detweiler and Szedenits [3], who showed that the divergent behavior can be removed by performing an integration by parts, and then discarding a (formally infinite) boundary term. The ultimate justification for this somewhat dangerous procedure comes from the eventual agreement with the previous analysis of Davis *et al.* [1], which is based on a manifestly finite perturbation formalism.

We denote the integral to the right of Eq. (3.10) by $I_L(\omega)$. It is easy to see that this integral can be split into convergent and divergent parts, so that $I = I_{conv} + I_{div}$ (indices are now suppressed for simplicity), with

$$I_{div} = \int_{2m}^{\infty} a(r)\,e^{i\omega t(r)}\,\mathcal{L}X(r)\,dr, \tag{3.15}$$

where $\mathcal{L} = f d/dr + i\omega$, and

$$a(r) = \frac{2r\sqrt{r/2m}}{\left(\sqrt{r/2m}+1\right)^2}\,(1-3m/r+i\omega r)\,; \tag{3.16}$$

the remainder of the integral gives I_{conv}.

To regularize I_{div} we rewrite Eq. (3.15) as

$$\begin{aligned}I_{div} = {} & \int_{2m}^{\infty}\left[a\,e^{i\omega t}\,\mathcal{L}X+\frac{d}{dr}\Big(b\,e^{i\omega t}\,\mathcal{L}X\Big)\right]dr\\ & -b\,e^{i\omega t}\,\mathcal{L}X\Big|_{2m}^{\infty},\end{aligned} \tag{3.17}$$

and seek a function $b(r)$ such that the new integral converges. The boundary term at infinity will then be seen to be infinite, and will be discarded; the boundary term at the horizon will be seen to vanish. It is easy to check that if $b(r)$ satisfies the differential equation

$$\frac{db}{dr} + i\omega\left(\frac{dt}{dr} + \frac{1}{f}\right)b + a = 0, \tag{3.18}$$

then the new integral will converge. A particular solution to Eq. (3.18) is $b(r) = -4im\omega^{-1}(1+x)^{-1}(1+x+2im\omega x^3)$, where $x = (r/2m)^{1/2}$. Substituting this into Eq. (3.17), discarding the boundary terms, and combining with I_{conv}, we find that the integral to the right of Eq. (3.10) can be written in the regularized form

$$\begin{aligned} I_L(\omega) = &\int_{2m}^{\infty} \frac{1 + im\omega\,(r/2m)^{3/2}}{im\omega\,(r/2m)^2}\, e^{i\omega t(r)} \\ &\times [L(L+1) - 6m/r]\, X_L(\omega; r)\, dr. \end{aligned} \tag{3.19}$$

It is evident that this integral is convergent at $r = \infty$.

The numerical code written for the purpose of calculating $\dot{E}(u)$ was built upon FORTRAN subroutines given in *Numerical Recipes* [25]. The Regge-Wheeler equation is integrated to high accuracy using the Bulirsh-Stoer method, which is also used to evaluate the integral (3.19). [This integral is truncated at a radius r_0 large compared with $1/\omega$; the remainder is calculated analytically using the asymptotic form for the Regge-Wheeler function, as given in Eq. (3.13).] These steps are repeated for many values of ω; we have calculated more than 600 points in the relevant interval $0 \leq m\omega \leq 0.9$ [1]. The Fourier transform of Eq. (3.6) is carried out by first selecting a value for u, and then evaluating the integral using a Romberg integrator. An interpolator is used to compute the integrand at those values of ω which the integrator selects; these are in general distinct from the values selected when previously calculating $\tilde{Z}_L(\omega)$. These steps are repeated for many values of u, lying in the relevant interval $-80m < u < 40m$. Finally, the sum over L is carried out. Due to the rapid convergence of the multipole expansion, only the modes with $L = 2$ and $L = 3$ need be included in the sum.

A plot of $\dot{E}(u)$, the gravitational-wave luminosity as a function of retarded time, is displayed in Fig. 5. It shows two main features: bremsstrahlung radiation and black-hole quasi-normal ringing. The bremsstrahlung radiation dominates the luminosity at retarded times $u < -10m$. This radiation is generated by the particle's infalling motion at early times (large distances from the black hole), and propagates directly from the particle to the detector (scattering by the spacetime curvature is insignificant). The black-hole ringing dominates the luminosity at retarded times $u > -10m$. This radiation is generated by the dynamics of the gravitational field in the vicinity of the black-hole horizon; it is scattered many times by the spacetime curvature, and is essentially disconnected from the particle's motion. It is clear that quasi-normal ringing dominates (by approximately two orders of magnitude) the energetics of the problem.

By integrating $\dot{E}(u)$ over retarded time we were able to reproduce the standard result for the total energy radiated,

$$\Delta E = (0.0092 + 0.0011 + \cdots)\mu^2/m, \tag{3.20}$$

first calculated by Davis *et al.* [1]. In Eq. (3.20), the first term comes from the $L = 2$ mode, while the second corresponds to $L = 3$. The total coefficient, as computed in Ref. [1] and reproduced here, is equal to 0.0104.

IV. COMPARISON BETWEEN POST-NEWTONIAN AND PERTURBATION-THEORY RESULTS

The purpose of this section is to compare the results obtained in Sec. II using post-Newtonian theory to those obtained in Sec. III using black-hole perturbation theory. More precisely, we shall consider the $\eta \to 0$ limit (where $\eta = \mu/m$) of Eq. (2.31) and compare it to the numerical results displayed in Fig. 5. While the numerical results are valid to all orders in the post-Newtonian expansion, the analytic expression is accurate only through second post-Newtonian order.

In the limit of small mass ratios, Eq. (2.31) takes the form

$$\dot{E}(T_h) = \frac{16}{15}\eta^2\left(\frac{m}{r_h}\right)^5\left\{1 - \frac{43}{7}\left(\frac{m}{r_h}\right) - \sqrt{2}\left[\frac{71}{6} + \frac{5\pi}{\sqrt{3}} + 15\ln\left(\frac{m}{r_h}\right) + 5\ln\left(\frac{2}{3}\right)\right]\left(\frac{m}{r_h}\right)^{3/2} - \frac{1127}{27}\left(\frac{m}{r_h}\right)^2\right\}. \tag{4.1}$$

For convenience, we have made slight changes to the notation used in Sec. II: we have replaced z by r_h, where the subscript indicates that the coordinates are harmonic. The gravitational-wave luminosity is expressed as a function of time T_h, which operationally has the same meaning as in Sec. III: it corresponds to proper time as measured by a static gravitational-wave detector situated near future null infinity.

Equation (4.1) is incomplete without the relationship between T_h (observer harmonic time at large distances), Which appears to the left, and r_h, which appears to the right. This relationship is discussed in Sec. IIC: If $r_h(t_h)$ describes the particle's world line in harmonic coordinates, then the quantity r_h appearing to the right of Eq. (4.1) is $r_h(u)$, where u is the retarded time

$$u = T_h - R_h - 2m \ln\left(\frac{R_h}{m}\right). \tag{4.2}$$

Here, $R_h \gg m$ is the distance from the detector to the system's center of mass (which here is identical to the fixed position of the large mass m). Notice that Eq. (4.2) takes the same form as in Eq. (3.2), in the limit of large R/m, apart from two slight differences. The first is that Eq. (3.2) involves Schwarzschild coordinates, T_S and R_S, instead of harmonic coordinates. The relationship between these coordinates is

$$T_S = T_h + \text{const.}, \qquad R_S = R_h + m. \tag{4.3}$$

The second is that in Eq. (4.2) R_h is divided by m, while in Eq. (3.2) R_S is divided by $2m$. This difference originates in the freedom in selecting the value of the parameter $\mathcal{S}$, as was discussed in Sec. IIC. This corresponds to the freedom in choosing the origin of the retarded time u. In particular, the origin of time might be chosen differently in the post-Newtonian and perturbation-theory calculations.

Because of this ambiguity in the origin of u, it would be unwise to compare directly Eq. (4.1) with Fig. 5. A much better way of carrying out the comparison is to re-express the luminosity function in terms of a fully unambiguous parameter. We shall choose to parametrize the world line with the Schwarzschild coordinate r_S, and express the luminosity in terms of this parameter. It might appear that this re-expression has already been effected (apart from the transformation $r_S = r_h + m$) in Eq. (3.2). We shall see, however, that the right-hand side of Eq. (3.2) is *not* the desired expression.

Let us begin with the perturbation-theory results for $\dot{E}(u)$, and let us adopt a mapping between u, the time coordinate running along future null infinity, and r_S, the adopted parameter along the particle's world line. The simplest and most natural mapping is depicted in Fig. 6: we connect the event labeled by r_S on the world line and the event labeled by u on future null infinity with a radial, outgoing null geodesic. This null geodesic is taken to propagate in the direction directly opposite to the direction in which the particle moves. Radial, outgoing null geodesics are such that $t_S - r_S - 2m\ln(r_S/2m - 1)$ is constant along them. The relation $u(r_S)$ therefore follows directly from Eq. (3.2), with r_S substituted for R, and $t_S(r_S)$, given by Eq. (3.8), substituted for T. The result is

$$u(r_S) = -4m\left[\frac{1}{3}x^3 + \frac{1}{2}x^2 + x + \ln(x-1)\right], \tag{4.4}$$

where $x = (r_S/2m)^{1/2}$. Using Eq. (4.4), the luminosity function can be plotted as a function of r_S, the parameter along the world line. This plot is displayed as a solid curve in Fig. 7.

The figure shows that the numerical curve tends to become pointwise unreliable as the radius increases beyond $r_S = 20m$: the computed curve undergoes small oscillations about a mean curve which presumably represents the true luminosity function. These oscillations are due to numerical error. More precisely, they are a consequence of the fact that the integration over angular frequencies, cf. Eq. (3.6), must necessarily be cut off at some finite upper bound ω_0. Reducing this upper bound produces larger oscillations about approximately the same mean curve.

The mapping between world line and future null infinity adopted in Eq. (4.1) is different from the one constructed in the preceding paragraph. This is understood from Eq. (4.2), which involves the distance R_h to the system's center of mass, instead of the distance to the particle itself. Therefore our plot for $\dot{E}(r_S)$ cannot be compared directly with Eq. (4.1). To carry out a meaningful comparison, we must first map r_A, the position of the particle at the retarded time of the center of mass, into the true-retarded position r_B, as illustrated in Fig. 8. (The coordinates used here are the harmonic coordinates, but we suppress our use of the subscript h.) We now turn to this task.

Figure 8 illustrates how to construct the mapping from r_A to r_B. As mentioned previously, the coordinates used for this purpose are the harmonic coordinates. Let $t = p(r)$ represent the particle's world line, and let $t = q(r) + u$ represent a light ray propagating in the opposite direction and meeting future null infinity at retarded time u. The center of mass is assumed to have the fixed position $r = 0$. The radius r_A is determined by inverting the equation

$$p(r_A) = q(0) + u. \tag{4.5}$$

The radius r_B, on the other hand, is obtained by solving

$$p(r_B) = q(r_B) + u. \tag{4.6}$$

By eliminating u from these equations, we obtain the desired (implicit) relationship between r_A and r_B:

$$p(r_A) = p(r_B) - q(r_B) + q(0). \tag{4.7}$$

Our goal now is to transform Eq. (4.7) into something more concrete.

We need to derive appropriate expressions for $p(r)$ and $q(r)$. The situation for $q(r)$ is quite simple. An expression for it can be derived from Eqs. (2.24)–(2.27). With $\mathcal{S}$ consistently set equal to m, we obtain simply

$$q(r) = r. \tag{4.8}$$

An expression for $p(r)$ can be derived by integrating dr/dt, which was written down in Sec. IIB. In the limit $\eta \to 0$, Eq. (2.13) becomes

$$\frac{dr}{dt} = -y^{1/2}\left(1 - \frac{5}{4}y + \frac{27}{32}y^2\right), \tag{4.9}$$

where $y = 2m/r$; Eq. (4.9) is valid through second post-Newtonian order. Integrating Eq. (4.9) would give an expression for $p(r)$ also accurate to second post-Newtonian order. This would imply that the mapping is carried out with the same degree of accuracy as the calculation of $\dot{E}$, as given in Eq. (4.1). However, we have here the opportunity of using an expression for dr/dt that is more accurate than Eq. (4.9). This comes about because in the $\eta \to 0$ limit, the equations of motion are given exactly by the Schwarzschild expressions (3.7). Once written in terms of the harmonic coordinates, these equations imply

$$\frac{dr}{dt} = -y^{1/2}\left(1 - \tfrac{1}{2}y\right)\left(1 + \tfrac{1}{2}y\right)^{-3/2}. \tag{4.10}$$

Expanding this in powers of y [26], we obtain

$$\begin{aligned}\frac{dr}{dt} = -y^{1/2}\Big(&1 - \frac{5}{4}y + \frac{27}{32}y^2 - \frac{65}{128}y^3 \\ &+ \frac{595}{2048}y^4 - \frac{1323}{8192}y^5 + \cdots\Big),\end{aligned} \tag{4.11}$$

which generalizes Eq. (4.9). For the purpose of carrying out the mapping we will deal with Eq. (4.11), whose integration gives $p(r)$:

$$\begin{aligned}p(r) = -\frac{4m}{3}\,y^{-3/2}\Big(&1 + \frac{15}{4}y - \frac{69}{32}y^2 - \frac{45}{128}y^3 \\ &- \frac{1089}{10240}y^4 - \frac{2169}{57344}y^5 + \cdots\Big).\end{aligned} \tag{4.12}$$

The mapping between r_A and r_B is given by Eqs. (4.7) and (4.8): $p(r_A) = p(r_B) - r_B$. This equation could be inverted numerically to yield $r_A(r_B)$. However, the following approximate, analytic inversion proves to be sufficiently accurate for our purposes. First, we define an auxiliary variable $Z \equiv [-3p(r_A)/4m]^{-2/3}$. Next, Eq. (4.12) is used to express Z as a power series in $y_A \equiv 2m/r_A$; this series is then inverted to yield y_A as a power series in Z:

$$y_A = Z\Big(1 + \frac{5}{2}Z + \frac{13}{4}Z^2 - \frac{65}{24}Z^3 - \frac{5821}{240}Z^4 + \cdots\Big). \tag{4.13}$$

Finally, we use our mapping equation to write $Z = [-3p(r_B)/4m + 3r_B/4m]^{-2/3}$; substitution of Eq. (4.12) then gives Z in terms of $y_B \equiv 2m/r_B$:

$$\begin{aligned}Z = y_B\Big(&1 + \frac{3}{2}{y_B}^{1/2} + \frac{15}{4}y_B - \frac{69}{32}{y_B}^2 - \frac{45}{128}{y_B}^3 \\ &- \frac{1089}{10240}{y_B}^4 - \frac{2169}{57344}{y_B}^5 + \cdots\Big)^{-2/3}.\end{aligned} \tag{4.14}$$

We have obtained the desired inversion: a value of r_B is selected, Z is calculated using Eq. (4.14), and r_A is then obtained from Eq. (4.13). It is this radius which must be substituted into Eq. (4.1) in order to obtain the gravitational-wave luminosity. This can then be plotted against the Schwarzschild radius $r_S = r_B + m$, which gives the dotted curve in Fig. 7.

The accuracy of the post-Newtonian expression can be ascertained from the figure. We find that the post-Newtonian value differs from the numerical one by a factor of approximately 2.5 at $r_S = 10m$, 1.5 at $r_S = 20m$, and 1.3 at $r_S = 30m$. We see that the post-Newtonian values tend to converge at large distances to the exact values, but that the rate of convergence is slow. We will explore this issue of the convergence of the post-Newtonian values more fully in the following section.

V. ACCURACY OF THE POST-NEWTONIAN EXPANSION

We have seen in the preceding section that the post-Newtonian expression for the gravitational-wave luminosity, as given by Eq. (4.1), only approximately reproduces the exact, numerical curve, as represented in Fig. 7. And indeed, the degree of accuracy is far less than could be expected: An expression valid to second post-Newtonian order could be expected to have a fractional accuracy of order $(m/r)^3$; for $r = 10m$ this is of order 10^{-3}. Instead, comparison with the numerical results shows that the fractional accuracy is actually of order 0.5.

In the previous section, the comparison between the post-Newtonian and perturbation-theory results could be carried out for $r_S < 30m$ only. The purpose of this section is to push the comparison to larger values of the radius, in spite of the fact that the numerical results are not reliable beyond $r_S = 30m$.

To this end, we consider the following calculation, based on the hybrid formalism proposed by Kidder, Will, and Wiseman [19]. We calculate the gravitational-wave luminosity associated with the radial infall of a particle with small mass into a much more massive, spherically symmetric object (not necessarily a black hole). The masses are denoted μ and m respectively, and we assume $\mu \ll m$. We do so by using the multipole expansion of Eq. (2.1), and by employing the *exact*, Schwarzschild equations of motion given by Eq. (4.10). Taking advantage of the rapid convergence of the multipole expansion, only the mass quadrupole term will be considered; tail effects will also be ignored.

The calculation proceeds as follows. In a Cartesian coordinate system based on the harmonic coordinates, the system's trace-free quadrupole moment can be expressed as $\mathcal{I}^{ij} = \mu r^2 \bar{\boldsymbol{P}}^{(2)}$, where $\bar{\boldsymbol{P}}^{(2)} = \mathrm{diag}(-1/3, -1/3, 2/3)$. Taking three time derivatives, we obtain

$$ {}^{(3)}\mathcal{I}^{ij} = 2\mu \left(r \frac{d^3 r}{dt^3} + 3 \frac{dr}{dt} \frac{d^2 r}{dt^2} \right) \bar{\boldsymbol{P}}^{(2)}. \tag{5.1} $$

Use of Eq. (4.10) and substitution into the first term of Eq. (2.1) finally yields

$$ \dot{E} = \frac{16}{15} \eta^2 \left(\frac{m}{r} \right)^5 \mathcal{R}^2, \tag{5.2} $$

where

$$ \begin{aligned} \mathcal{R} = {} & \left(1 - \frac{m}{r}\right) \left(1 + \frac{m}{r}\right)^{-13/2} \\ & \times \left[1 + 5 \left(\frac{m}{r}\right) - 29 \left(\frac{m}{r}\right)^2 + 15 \left(\frac{m}{r}\right)^3 \right]. \end{aligned} \tag{5.3} $$

Equation (5.2) with $\mathcal{R} = 1$ gives the standard quadrupole-formula result; the factor $\mathcal{R}^2$ is the relativistic correction. Notice that $\mathcal{R}$ tends toward unity at large r, and that $\mathcal{R} = 0$ at $r = m$ (the event horizon) and $r = 3m$.

Equation (5.2) is to be interpreted as the fully relativistic analogue of Eq. (4.1). This expression can be compared with the exact, numerical curve of Fig. 7, provided that the mapping illustrated in Fig. 8 is carefully carried out. The result is displayed as the dotted curve in Fig. 9. We see that Eq. (5.2) reproduces the numerical results to very high accuracy for $r_S > 7m$. In particular, this expression does much better than its post-Newtonian analogue. This agreement should not be considered to be of deep physical significance; it is essentially a fluke. However, it will allow us to use Eq. (5.2) as a tool for determining the large-distance accuracy of the post-Newtonian results, as we now explain.

Although the numerical curve cannot, because of numerical error, be reliably constructed for radii larger than approximately $30m$, it is most plausible that the agreement with Eq. (5.2) would persist at larger values of r. It therefore appears appropriate to investigate the large-distance convergence property of the post-Newtonian luminosity, with respect to the exact curve, by comparing Eq. (4.1) directly with Eq. (5.2). This comparison, which has the major advantage of not involving the mapping illustrated in Fig. 8, reveals that the relative difference between the post-Newtonian values and the exact ones is 20% at $r_h \simeq 45m$, 10% at $r_h \simeq 75m$, and 1% at $r_h \simeq 400m$. These results establish the slow convergence of the post-Newtonian expansion.

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APPENDIX: CANCELLATION OF POST-NEWTONIAN TERMS

The purpose of this Appendix is to point out a remarkable cancellation of terms occurring in the calculation of the gravitational-wave luminosity in the case of a head-on collision proceeding from rest at infinity.

Substituting Eq. (2.13) into Eqs. (2.8) we find that the mass multipole moments can be expressed as expansions in powers of m/z only. To second post-Newtonian order, the generic form is

$$\mathcal{I}^{zz} = \mu z^2 \left[A_{2,0} + A_{2,1} \left(\frac{m}{z} \right) + A_{2,2} \left(\frac{m}{z} \right)^2 \right], \tag{A1a}$$

$$\mathcal{I}^{zzz} = \mu z^3 \left[A_{3,0} + A_{3,1} \left(\frac{m}{z} \right) \right], \tag{A1b}$$

$$\mathcal{I}^{zzzz} = \mu z^4 A_{4,0}. \tag{A1c}$$

Assuming that such an expansion holds at least to order $(m/z)^n$, we write the mass n-pole moment in the form

$$\mathcal{I}^{zz\cdots} = \mu z^n \left\{ A_{n,0} + A_{n,1} \left(\frac{m}{z} \right) + \cdots + A_{n,n} \left(\frac{m}{z} \right)^n + O\left[\left(\frac{m}{z} \right)^{n+1} \right] \right\}. \tag{A2}$$

We have represented the first few moments in Fig. 10, in which a row corresponds to a given multipole order, and a column to a given power of m/z in the expansion. The thick dashed line separates the coefficients needed in order to express the luminosity accurately through 2PN order; these appear to the left of the boundary.

First, we observe that in Eq. (A2), the term of order $O[(m/z)^n]$ is just the constant $A_{n,n}\, \mu\, m^n$. The contribution from this term to the energy flux therefore vanishes when time derivatives are taken. We designate this cancellation in Fig. 10 by the darkly shaded diagonal beginning at $A_{2,2}$.

Second, because $\dot{z} \propto z^{-1/2}$ to leading order, the k^{th} time derivative of a term of the form $A_{l,n}\, z^{l-n}$, with $l > n$, goes like $z^{(l-n-3k/2)}$. Consequently, if $l - n = 0_{mod\ 3}$, this term becomes a constant after $k_0 = 2(l-n)/3$ derivatives, and will vanish after an additional derivative is taken. But since each l-pole moment is differentiated $l+1$ times to calculate the energy flux, this term will vanish whenever $k_0 \leq l - 1$, or $l + 2n \geq 3$. Such terms are indicated in Fig. 10 by the lightly shaded diagonals.

This does not imply, however, that contributions from the PN order corresponding to a shaded region do not appear at all in the time-derivatives of the corresponding STF moment. If the term in question is a higher-order correction to the moment (e.g., $A_{2,2}$ or $A_{4,1}$), then contributions to ${}^{(3)}\mathcal{I}^{ij}$ of the same order could be generated by correction terms in the equations of motion, applied to derivatives of lower-order terms. Only if the term is a leading-order term (e.g., $A_{3,0}$ or $A_{6,0}$) is the contribution at that order identically zero. It is interesting to note that because of this effect, the coefficient $A_{2,2}$ is not explicitly needed in calculating the energy flux to 2PN order. This coefficient has only recently been calculated [27].

In the case of infall from a finite distance, the argument used above fails. To leading order, the velocity is now

$$\dot{z} \propto \left(\frac{m}{z} - \frac{m}{z_0} \right)^{1/2}, \tag{A3}$$

and the 2PN expansion for I^{ij} contains powers of both (m/z) and (m/z_0). For example, terms of the form m^2/zz_0 and $(m/z_0)^2$ appear in the 2PN expansion for the quadrupole moment; when multiplied by μz^2, these terms are not killed by time derivatives.

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FIG. 1. Gravitational-wave luminosity, in units of η^2, as a function of separation z, for case (A): infall from infinite initial separation. We show the luminosity curves for $\eta = \{0, 0.15, 0.25\}$.

FIG. 2. A plot of the absolute value of the 1PN, 1.5PN, and 2PN contributions to the luminosity divided by the Newtonian expression. The 1PN and 2PN contributions are negative; the 1.5PN contribution is positive. Fig. 2a shows the test-body limit ($\eta = 0$), and Fig. 2b shows the equal-mass case ($\eta = 0.25$).

FIG. 3. Total energy radiated during an infall from infinite initial separation to final separation z_f. We show curves for $\eta = \{0, 0.15, 0.25\}$.

FIG. 4. Total energy radiated during an infall from initial separation z_0 to final separation z_f, including the limit $z_0 = \infty$. We show curves for $z_0/m = \{15, 30, 50, \infty\}$. Fig. 4a shows the test-body limit ($\eta = 0$); Fig. 4b shows the equal-mass case ($\eta = 0.25$).

FIG. 5. A plot of $\dot{E}/\eta^2$ (on a logarithmic scale) as a function of retarded time u/m, as calculated using perturbation theory. The luminosity represents bremsstrahlung radiation for $u/m < -10$ and black-hole quasi-normal ringing for $u/m > -10$.

FIG. 6. Conformal diagram representing the Schwarzschild spacetime. Shown are: past future null infinity (lower diagonal with positive slope), future null infinity (higher diagonal with negative slope), the future horizon (higher diagonal with positive slope), the past horizon (lower diagonal with negative slope), and the singularity (broken horizontal). We also represent the world line of the infalling particle, parametrized by the Schwarzschild coordinate r_S, and the radial, outgoing null geodesic reaching future null infinity at retarded time u.

FIG. 7. A plot of $\dot{E}/\eta^2$ (on a logarithmic scale) as a function of r_S/m, the parameter along the particle's world line. The solid curve represents the exact, numerical results obtained using perturbation theory. The dotted curve represents the post-Newtonian approximation given by Eq. (4.1).

FIG. 8. The mapping between r_A, the center-of-mass-retarded position, and r_B, the true-retarded position. The curve $t = p(r)$ represents the world line of the particle. The curve $t = q(r) + u$ represents an outgoing null ray propagating in the opposite direction, reaching future null infinity at retarded time u. The center of mass is located at the center $r = 0$.

FIG. 9. A plot of $\dot{E}/\eta^2$ (on a logarithmic scale) as a function of r_S/m, the parameter along the particle's world line. The solid curve represents the exact, numerical results obtained using perturbation theory. The dotted curve represents the quadrupole-relativistic approximation given by Eq. (5.2).

FIG. 10. Cancellations of terms in the case of infall from infinity. Each row represents a multipole order, with a common factor $\mu\, z^i$, while each column labels the contribution of order $O(m/z)^j$ to the multipole moment in a post-Newtonian expansion. Tail terms are ignored. A consistent expansion of the luminosity up to a certain order n involves the coefficients included in the upper-left corner of the table (the requirement for an expansion of the luminosity accurate to 2PN order is marked by the heavy dashed line). The coefficients which have currently been calculated explicitly are indicated in bold face.