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The (2+1)-Dimensional Black Hole

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Abstract

I review the classical and quantum properties of the (2+1)-dimensional black hole of Bañados, Teitelboim, and Zanelli. This solution of the Einstein field equations in three spacetime dimensions shares many of the characteristics of the Kerr black hole: it has an event horizon, an inner horizon, and an ergosphere; it occurs as an endpoint of gravitational collapse; it exhibits mass inflation; and it has a nonvanishing Hawking temperature and interesting thermodynamic properties. At the same time, its structure is simple enough to allow a number of exact computations, particularly in the quantum realm, that are impractical in 3+1 dimensions.

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Since the seminal work of Deser, Jackiw, and 't Hooft [1–3] and Witten [4, 5], general relativity in three spacetime dimensions has become an increasingly popular model in which to explore the foundations of classical and quantum gravity [6]. But although (2+1)-dimensional gravity has been widely recognized as a useful laboratory for studying conceptual issues—the nature of observables, for example, and the "problem of time"—it has been widely believed that the model is too physically unrealistic to give much insight into real gravitating systems in 3+1 dimensions. In particular, general relativity in 2+1 dimensions has no Newtonian limit [7] and no propagating degrees of freedom.

It therefore came as a considerable surprise when Bañados, Teitelboim, and Zanelli (BTZ) showed in 1992 that (2+1)-dimensional gravity has a black hole solution [8]. The BTZ black hole differs from the Schwarzschild and Kerr solutions in some important respects: it is asymptotically anti-de Sitter rather than asymptotically flat, and has no curvature singularity at the origin. Nonetheless, it is clearly a black hole: it has an event horizon and (in the rotating case) an inner horizon, it appears as the final state of collapsing matter, and it has thermodynamic properties much like those of a (3+1)-dimensional black hole.

The purpose of this article is to briefly review the past three years' work on the BTZ black hole. The first four sections deal with classical properties, while the last four discuss quantum mechanics, thermodynamics, and possible generalizations. For the most part, I will skip complicated derivations, referring the reader instead to the literature. For a recent review with a somewhat complementary choice of topics, see Ref. 9.

The structure of the paper is as follows. In section 1, I introduce the BTZ solution in standard Schwarzschild-like coordinates and in Eddington-Finkelstein and Kruskal coordinates, and summarize its basic physical characteristics. Section 2 deals with the global geometry of the (2+1)-dimensional black hole, and outlines its description in the Chern-Simons formulation of (2+1)-dimensional general relativity. Section 3 describes the formation of BTZ black holes from collapsing matter, and reports on some recent work on critical phenomena, while section 4 summarizes the physics of black hole interiors, focusing on the phenomenon of mass inflation.

I next turn to the quantum mechanical properties of the BTZ solution. Section 5 addresses the problem of quantum field theory in a (classical) black hole background. Sections 6 and 7 discuss the thermodynamic and statistical mechanical properties of the quantized black hole, including attempts to explain black hole entropy in terms of the "microscopic" physics of quantum gravitational states. Finally, section 8 briefly reviews a number of generalizations of the BTZ solution, including electrically charged black holes, dilatonic black holes, black holes in string theory, black holes in topologically massive gravity, and black holes formed from "topological" matter.

1 The BTZ Black Hole

The BTZ black hole in "Schwarzschild" coordinates is described by the metric

$$ds^2 = -(N^\perp)^2 dt^2 + f^{-2}dr^2 + r^2\left(d\phi + N^\phi dt\right)^2 \tag{1.1}$$

with lapse and shift functions and radial metric

$$N^\perp = f = \left(-M + \frac{r^2}{\ell^2} + \frac{J^2}{4r^2}\right)^{1/2}, \quad N^\phi = -\frac{J}{2r^2} \qquad (|J| \le M\ell). \tag{1.2}$$

It is straightforward to check that this metric satisfies the ordinary vacuum field equations of (2+1)-dimensional general relativity,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{1}{\ell^2}g_{\mu\nu} \tag{1.3}$$

with a cosmological constant $\Lambda = -1/\ell^2$. The metric (1.1) is stationary and axially symmetric, with Killing vectors ∂_t and ∂_ϕ, and generically has no other symmetries.

As the notation suggests, the parameters M and J, which determine the asymptotic behavior of the solution, are the standard ADM mass and angular momentum. To see this, one can write the Einstein action in ADM form,*

$$I = \frac{1}{2\pi}\int_0^T dt \int d^2x \left[\pi^{ij}\dot{g}_{ij} - N^\perp\mathcal{H} - N^i\mathcal{H}_i\right] + B, \tag{1.4}$$

where the boundary term B is required to cancel surface integrals in the variation of I and ensure that the action has genuine extrema [10]. If one now considers variations of the spatial metric that preserve the asymptotic form of the BTZ solution, one finds that [11]

$$\delta B = T\left[-\delta M + N^\phi \delta J\right]. \tag{1.5}$$

As in 3+1 dimensions, the conserved charges can be read off from δB: J is the angular momentum as measured at infinity, while M is the mass associated with asymptotic translations in the "Killing time" t.

This analysis can be formalized by noting that M and J are the Noether charges, as defined by Lee and Wald [12], associated with asymptotic time translations and rotations (see [13] for a computation in the first-order formalism). It may also be checked that M and J are the conserved charges associated with the asymptotic Killing vectors ∂_t and ∂_ϕ as defined by Abbott and Deser [14] for asymptotically anti-de Sitter spacetimes, and that they can be obtained from the anti-de Sitter version of the stress-energy pseudotensor considered by Bak et al. [15]. Alternatively, M and J may be expressed in terms of the quasilocal energy and angular momentum of Brown

*Unless otherwise stated, I use the units of Ref. 8, in which $8G = 1$.

and York [16]: if one places the BTZ black hole in a circular box of radius R and treats the boundary terms at R carefully, one obtains a mass and angular momentum measured at the boundary that approach M and J as $R \to \infty$ [17, 9].

The metric (1.1) is singular when $r = r_\pm$, where

$$r_\pm^2 = \frac{M\ell^2}{2}\left\{1 \pm \left[1 - \left(\frac{J}{M\ell}\right)^2\right]^{1/2}\right\}, \tag{1.6}$$

i.e.,

$$M = \frac{r_+^2 + r_-^2}{\ell^2}, \quad J = \frac{2r_+r_-}{\ell} . \tag{1.7}$$

As we shall see below, these are merely coordinate singularities, closely analogous to the singularity at $r = 2m$ of the ordinary Schwarzschild metric. The time-time component g_{00} of the metric vanishes at $r = r_{erg}$, where

$$r_{erg} = M^{1/2}\ell = \left(r_+^2 + r_-^2\right)^{1/2}. \tag{1.8}$$

As in the Kerr solution in 3+1 dimensions, $r < r_{erg}$ determines an ergosphere: timelike curves in this region necessarily have $d\phi/d\tau > 0$ (when $J > 0$), so all observers are dragged along by the rotation of the black hole. Note that the $r_\pm$ become complex if $|J| > M\ell$, and the horizons disappear, leaving a metric that has a naked conical singularity at $r = 0$. The $M = -1$, $J = 0$ metric may be recognized as that of ordinary anti-de Sitter space; it is separated by a mass gap from the $M = 0$, $J = 0$ "massless black hole," whose geometry is discussed in Refs. 11 and 18.

That the BTZ metric is a genuine black hole can be seen most easily by transforming to Eddington-Finkelstein-like coordinates [19],

$$dv = dt + \frac{dr}{(N^\perp)^2}, \quad d\tilde{\phi} = d\phi - \frac{N^\phi}{(N^\perp)^2}dr, \tag{1.9}$$

in which the metric becomes

$$ds^2 = -(N^\perp)^2 dv^2 + 2dvdr + r^2\left(d\tilde{\phi} + N^\phi dv\right)^2 . \tag{1.10}$$

It is now easy to see that the horizon $r = r_+$, where $N^\perp$ vanishes, is a null surface, generated by geodesics

$$r(\lambda) = r_+, \quad \frac{d\tilde{\phi}}{d\lambda} + N^\phi(r_+)\frac{dv}{d\lambda} = 0. \tag{1.11}$$

Moreover, this surface is evidently a marginally trapped surface: at $r = r_+$, any null geodesic satisfies

$$\frac{dv}{d\lambda}\frac{dr}{d\lambda} = -\frac{r_+^2}{2}\left(\frac{d\tilde{\phi}}{d\lambda} + N^\phi(r_+)\frac{dv}{d\lambda}\right)^2 \leq 0, \tag{1.12}$$

so r decreases or (for the geodesics (1.11)) remains constant as v increases.

Like the outer horizon of the Kerr metric, the surface $r = r_+$ is also a Killing horizon. The Killing vector normal to this surface is

$$\chi = \partial_v - N^\phi(r_+)\partial_{\tilde\phi}, \tag{1.13}$$

from which the surface gravity κ, defined by [20]

$$\kappa^2 = -\frac{1}{2}\nabla^a\chi^b\nabla_a\chi_b, \tag{1.14}$$

may be computed to be

$$\kappa = \frac{r_+^2 - r_-^2}{\ell^2 r_+}. \tag{1.15}$$

For a more complete description of the BTZ solution, we can transform instead to Kruskal-like coordinates [11]. To do so, let us define new null coordinates

$$u = \rho(r)e^{-at}, \qquad v = \rho(r)e^{at}, \qquad \text{with} \quad \frac{d\rho}{dr} = \frac{a\rho}{(N^\perp)^2}. \tag{1.16}$$

As in the case of the Kerr metric, we need two patches, $r_- < r < \infty$ and $0 < r < r_+$, to cover the BTZ spacetime. In each patch, the metric (1.1) takes the form

$$ds^2 = \Omega^2 du dv + r^2(d\tilde\phi + N^\phi dt)^2 \tag{1.17}$$

where

$$\begin{aligned}
\Omega_+^2 &= \frac{(r^2 - r_-^2)(r + r_+)^2}{a_+^2 r^2 \ell^2}\left(\frac{r - r_-}{r + r_-}\right)^{r_-/r_+}, \\
&\tilde\phi_+ = \phi + N^\phi(r_+)t\,, \qquad a_+ = \frac{r_+^2 - r_-^2}{\ell^2 r_+} \qquad\qquad (r_- < r < \infty)
\end{aligned} \tag{1.18}$$

$$\begin{aligned}
\Omega_-^2 &= \frac{(r_+^2 - r^2)(r + r_-)^2}{a_-^2 r^2 \ell^2}\left(\frac{r_+ - r}{r_+ + r}\right)^{r_+/r_-}, \\
&\tilde\phi_- = \phi + N^\phi(r_-)t\,, \qquad a_- = \frac{r_-^2 - r_+^2}{\ell^2 r_-} \qquad\qquad (0 < r < r_+)
\end{aligned} \tag{1.19}$$

with r and t viewed as implicit functions of u and v. (For explicit coordinate transformations, see Ref. 11.†)

As in the case of the Kerr black hole, an infinite number of such Kruskal patches may be joined together to form a maximal solution, whose Penrose diagram is shown

†Note that the coordinates U and V in [11] are, unconventionally, not null; my u and v are the sum and difference of the coordinates of this reference.

in figure 1a. This diagram differs from that of the Kerr metric at $r = \infty$, reflecting the fact that the BTZ black hole is asymptotically anti-de Sitter rather than asymptotically flat, but the overall structure is similar. In particular, it is evident that $r = r_+$ is an event horizon, while the inner horizon $r = r_-$ is a Cauchy horizon for region I. When $J = 0$, the Penrose diagram collapses to that of figure 1b, which is similar in structure—except for its asymptotic behavior—to the diagram for the ordinary Schwarzschild solution, while for the extreme case, $J = \pm M\ell$, the Penrose diagram is that of figure 1c.

Although the classical behavior of the BTZ black hole has not been investigated as thoroughly as, for example, the Schwarzschild solution, a fair amount is known. In particular, the behavior of geodesics has been studied by Cruz et al. [21] and Farina et al. [22], and the propagation of strings in a BTZ background has been analyzed by Larsen and Sanchez [23]. The static BTZ black hole has also been studied in the York time slicing, in which surfaces of constant mean (extrinsic) curvature $\mathrm{Tr}K = T$ are used as constant time surfaces; the resulting metric is equivalent to (1.1), via a complicated coordinate transformation involving elliptic integrals [24]. There have also been some recent attempts to find exact multi-black hole solutions. Clément has reported the existence of solutions representing multiple freely falling black holes, but the metrics also contain conical singularities [25]. Coussaert and Henneaux claim that all static multi-black hole solutions contain such singularities [26], but it is not known whether exact nonstatic nonsingular solutions can be found.

2 Global Geometry

In the last section, I emphasized similarities between the BTZ solution and ordinary (3+1)-dimensional black holes. But there are important differences as well, rooted in the simplicity of (2+1)-dimensional gravity. In three spacetime dimensions, the full curvature tensor is completely determined by the Ricci tensor,

$$R_{\mu\nu\rho\sigma} = g_{\mu\rho}R_{\nu\sigma} + g_{\nu\sigma}R_{\mu\rho} - g_{\nu\rho}R_{\mu\sigma} - g_{\mu\sigma}R_{\nu\rho} - \frac{1}{2}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})R. \tag{2.1}$$

Hence any solution of the vacuum Einstein field equations with a cosmological constant Λ,

$$R_{\mu\nu} = 2\Lambda g_{\mu\nu}, \tag{2.2}$$

has constant curvature. In particular, the BTZ metric (1.1) has constant negative curvature: any point in the black hole spacetime has a neighborhood isometric to anti-de Sitter space, and the whole spacetime is expressible as a collection of such neighborhoods appropriately patched together. Since the maximal simply connected spacetime of constant negative curvature is the universal covering space $\widetilde{\mathrm{adS}}$ of anti-de Sitter space, we might hope to represent the BTZ black hole as a quotient space of $\widetilde{\text{adS}}$ by some group of isometries.* Such a quotient construction provides a powerful mathematical tool, permitting, for example, the exact computation of Greens functions in a black hole background.

Geometrically, three-dimensional anti-de Sitter space (adS) may be obtained from flat $\mathbb{R}^{2,2}$, with coordinates (X_1, X_2, T_1, T_2) and metric

$$dS^2 = dX_1^2 + dX_2^2 - dT_1^2 - dT_2^2, \tag{2.3}$$

by restricting to the submanifold

$$X_1^2 - T_1^2 + X_2^2 - T_2^2 = -\ell^2 \tag{2.4}$$

with the induced metric. In this formulation, the isometry group is evidently $\mathrm{SO}(2,2)$. Equivalently, we can combine (X_1, X_2, T_1, T_2) into a 2×2 matrix,

$$\mathbf{X} = \frac{1}{\ell}\begin{pmatrix} T_1 + X_1 & T_2 + X_2 \\ -T_2 + X_2 & T_1 - X_1 \end{pmatrix}, \qquad \det|\mathbf{X}| = 1, \tag{2.5}$$

i.e., $\mathbf{X} \in \mathrm{SL}(2,\mathbb{R})$. Isometries may now be represented as elements of the group $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})/\mathbb{Z}_2 \approx \mathrm{SO}(2,2)$: the two copies of $\mathrm{SL}(2,\mathbb{R})$ act by left and right multiplication, $\mathbf{X} \to \rho_L \mathbf{X} \rho_R$, with $(\rho_L, \rho_R) \sim (-\rho_L, -\rho_R)$.

The relevant region of the universal covering space of anti-de Sitter space (see Ref. 11) may be covered by an infinite set of coordinate patches of three types, corresponding to the regions of the Penrose diagram of figure 1a:

I. $(r \geq r_+)$

$$\begin{aligned} X_1 &= \ell\sqrt{\alpha}\sinh\left(\frac{r_+}{\ell}\phi - \frac{r_-}{\ell^2}t\right), \quad X_2 = \ell\sqrt{\alpha - 1}\cosh\left(\frac{r_+}{\ell^2}t - \frac{r_-}{\ell}\phi\right) \\ T_1 &= \ell\sqrt{\alpha}\cosh\left(\frac{r_+}{\ell}\phi - \frac{r_-}{\ell^2}t\right), \quad T_2 = \ell\sqrt{\alpha - 1}\sinh\left(\frac{r_+}{\ell^2}t - \frac{r_-}{\ell}\phi\right) \end{aligned}$$

II. $(r_- \leq r \leq r_+)$

$$\begin{aligned} X_1 &= \ell\sqrt{\alpha}\sinh\left(\frac{r_+}{\ell}\phi - \frac{r_-}{\ell^2}t\right), \quad X_2 = -\ell\sqrt{1-\alpha}\sinh\left(\frac{r_+}{\ell^2}t - \frac{r_-}{\ell}\phi\right) \\ T_1 &= \ell\sqrt{\alpha}\cosh\left(\frac{r_+}{\ell}\phi - \frac{r_-}{\ell^2}t\right), \quad T_2 = -\ell\sqrt{1-\alpha}\cosh\left(\frac{r_+}{\ell^2}t - \frac{r_-}{\ell}\phi\right) \end{aligned}$$

III. $(0 \leq r \leq r_-)$

$$\begin{aligned} X_1 &= \ell\sqrt{-\alpha}\cosh\left(\frac{r_+}{\ell}\phi - \frac{r_-}{\ell^2}t\right), \quad X_2 = -\ell\sqrt{1-\alpha}\sinh\left(\frac{r_+}{\ell^2}t - \frac{r_-}{\ell}\phi\right) \\ T_1 &= \ell\sqrt{-\alpha}\sinh\left(\frac{r_+}{\ell}\phi - \frac{r_-}{\ell^2}t\right), \quad T_2 = -\ell\sqrt{1-\alpha}\cosh\left(\frac{r_+}{\ell^2}t - \frac{r_-}{\ell}\phi\right) \end{aligned} \tag{2.6}$$

*For certain spacetime topologies, the general existence of such a quotient space construction has been proven by Mess [27].

where

$$\alpha(r) = \left(\frac{r^2 - r_-^2}{r_+^2 - r_-^2}\right), \quad \phi \in (-\infty, \infty), \ t \in (-\infty, \infty). \tag{2.7}$$

It is straightforward to show that the standard adS metric dS^2 then transforms to the BTZ metric (1.1) in each patch. The "angle" ϕ in equation (2.6) has infinite range, however; to make it into a true angular variable, we must identify ϕ with $\phi + 2\pi$. This identification is an isometry of anti-de Sitter space—it is a boost in the X_1-T_1 and the X_2-T_2 planes—and corresponds to an element (ρ_L, ρ_R) of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})/\mathbb{Z}_2$ with

$$\rho_L = \begin{pmatrix} e^{\pi(r_+ - r_-)/\ell} & 0 \\ 0 & e^{-\pi(r_+ - r_-)/\ell} \end{pmatrix}, \qquad \rho_R = \begin{pmatrix} e^{\pi(r_+ + r_-)/\ell} & 0 \\ 0 & e^{-\pi(r_+ + r_-)/\ell} \end{pmatrix}. \tag{2.8}$$

The BTZ black hole may thus be viewed as a quotient space $\widetilde{\mathrm{adS}}/\langle(\rho_L, \rho_R)\rangle$, where $\langle(\rho_L, \rho_R)\rangle$ denotes the group generated by (ρ_L, ρ_R). This is an extraordinary result: anti-de Sitter space is an extremely simple, virtually structureless manifold, but appropriate identifications nevertheless convert it into a spacetime very much like the (3+1)-dimensional Kerr black hole.

A slightly different representation for the region $r \geq r_+$ will be useful for investigating the first-order formulation of the BTZ black hole. This "upper half-space" metric may be obtained from (1.1) by the coordinate transformation

$$\begin{aligned} x &= \left(\frac{r^2 - r_+^2}{r^2 - r_-^2}\right)^{1/2} \cosh\left(\frac{r_+}{\ell^2}t - \frac{r_-}{\ell}\phi\right) \exp\left\{\frac{r_+}{\ell}\phi - \frac{r_-}{\ell^2}t\right\} \\ y &= \left(\frac{r^2 - r_+^2}{r^2 - r_-^2}\right)^{1/2} \sinh\left(\frac{r_+}{\ell^2}t - \frac{r_-}{\ell}\phi\right) \exp\left\{\frac{r_+}{\ell}\phi - \frac{r_-}{\ell^2}t\right\} \\ z &= \left(\frac{r_+^2 - r_-^2}{r^2 - r_-^2}\right)^{1/2} \exp\left\{\frac{r_+}{\ell}\phi - \frac{r_-}{\ell^2}t\right\}, \end{aligned} \tag{2.9}$$

for which the BTZ metric becomes

$$ds^2 = \frac{\ell^2}{z^2}(dx^2 - dy^2 + dz^2) \qquad (z > 0). \tag{2.10}$$

Again, periodicity in the Schwarzschild angular coordinate ϕ requires that we identify points under the action $\phi \to \phi + 2\pi$, that is,

$$(x, y, z) \sim \tag{2.11}$$

$$\left(e^{2\pi r_+/\ell}(x \cosh\frac{2\pi r_-}{\ell} - y \sinh\frac{2\pi r_-}{\ell}),\ e^{2\pi r_+/\ell}(y \cosh\frac{2\pi r_-}{\ell} - x \sinh\frac{2\pi r_-}{\ell}),\ e^{2\pi r_+/\ell} z\right).$$

These identifications correspond once more to the isometry (2.8) of $\widetilde{\mathrm{adS}}$.

The physical significance of the quotient space representation of the BTZ black hole may be clarified by turning to the first-order formulation of general relativity. As discovered by Achúcarro and Townsend [28] and developed by Witten [4, 5], (2+1)-dimensional gravity can be rewritten as a Chern-Simons gauge theory. The fundamental variables in the first-order formalism are a triad $e^a = e_\mu{}^a dx^\mu$ (where $g_{\mu\nu} = \eta_{ab} e_\mu{}^a e_\nu{}^b$) and a spin connection $\omega^a = \frac{1}{2}\epsilon^{abc}\omega_{\mu bc}dx^\mu$, which can be combined to give a pair of SO(2, 1) or SL(2, $\mathbb{R}$) connection one-forms

$$A^{(\pm)a} = \omega^a \pm \frac{1}{\ell} e^a. \tag{2.12}$$

It is not hard to show that the Einstein-Hilbert action written in terms of these "gauge fields" becomes

$$I_{\text{grav}} = I_{\text{CS}}[A^{(+)}] - I_{\text{CS}}[A^{(-)}], \tag{2.13}$$

where

$$I_{\text{CS}} = \frac{k}{4\pi}\int_M \text{Tr}\left\{A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right\}, \tag{2.14}$$

is the Chern-Simons action. (Here k is a constant proportional to ℓ/G; its exact value depends on normalizations, and will not be important until section 7.) The Chern-Simons field equations require that the connections $A^{(\pm)}$ be flat, and it may be checked that this is equivalent to the constant curvature condition (2.1).

Now, any connection is completely determined by its holonomies, that is, by the Wilson loops

$$H[\gamma] = P\exp\left\{\int_\gamma A\right\} \tag{2.15}$$

around closed curves γ, where P denotes path ordering. In 3+1 dimensions, such holonomies are the fundamental variables in the "loop representation" [29], and in 2+1 dimensions they play an equally important role.† For a flat connection, the holonomies depend only on the homotopy class of γ, and may be thought of as non-Abelian Aharonov-Bohm phases. The $H[\gamma]$ are not quite gauge invariant, but transform by overall conjugation, $H[\gamma]\rightarrow g\cdot H[\gamma]\cdot g^{-1}$; their traces give a gauge-invariant and diffeomorphism-invariant characterization of the geometry.

For the metric (2.9)–(2.10) with the identifications (2.11), the closed curve

$$\gamma : \phi(s) = 2\pi s, \quad s\in[0,1] \tag{2.16}$$

is homotopically nontrivial, and it may be shown that up to overall conjugation, the holonomies of the connections $A^{(\pm)}$ are precisely the elements ρ_L and ρ_R given by equation (2.8). The holonomies thus unite the gauge theoretic and the geometric descriptions of the black hole—the same group elements that determine the Chern-Simons connection also give the set of identifications that fix the geometry. (See also

†For a general discussion of the geometrical significance of these holonomies in 2+1 dimensions, see Ref. 30.

Vaz and Witten [31] for a discussion of the geometric interpretation of the $H[\gamma]$.) An equivalent expression for the holonomies in the "Schwarzschild" coordinates (1.1) was first discovered by Cangemi et al. [32], who pointed out that the mass and angular momentum (1.7) have a natural interpretation in terms of the two quadratic Casimir operators of $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})$. This connection between mass, angular momentum, and holonomies has also been investigated by Izquierdo and Townsend [33], who discuss the possibility that bounds on the holonomies might give a version of the positive mass theorem for nonsingular (2+1)-dimensional asymptotically anti-de Sitter spacetimes.

The quotient space construction of the BTZ black hole has a number of uses. For example, we can now investigate the nature of the singularity at $r=0$ more easily. Note first that there is no curvature singularity: from (2.1), the curvature is constant everywhere. Moreover, it would seem from (2.6) that the metric could be extended in region III past $r=0$ to negative values of r^2. This is indeed possible, but it is shown in Ref. 11 that the resulting manifold contains closed timelike curves: the Killing vector describing the identifications $\langle(\rho_L,\rho_R)\rangle$ of $\widetilde{\mathrm{adS}}$ becomes timelike in these regions, so curves such as (2.16) become timelike. The surface $r=0$ is thus a singularity in the causal structure.‡

The quotient space construction is also useful for identifying Killing spinors. Such spinors are solutions of the equation

$$\nabla_\lambda\psi = \frac{\epsilon}{2\ell}\gamma_\lambda\psi, \tag{2.17}$$

where $\epsilon=\pm1$ [26]; the choice of sign of ϵ may be viewed as a choice of which of the two factors of $\mathrm{SL}(2,\mathbb{R})$ in the isometry group to gauge. The existence of Killing spinors is an indication of supersymmetry: if one views the BTZ black hole as a solution of (1,1)-adS supergravity with vanishing gravitino fields, a Killing spinor represents a remaining supersymmetry transformation that leaves this metric and gravitino configuration invariant.

The Killing spinors of anti-de Sitter space are easy to find—there are four, two for each sign of ϵ—and their existence in the BTZ spacetime is simply a question of whether the adS solutions are preserved by the identifications $\langle(\rho_L,\rho_R)\rangle$. In general, they are not: as Coussaert and Henneaux have shown, Killing spinors exist only for extreme black holes, those for which $J=\pm M\ell$ [26]. When $M\neq0$, an extreme black hole has a single supersymmetry; the $M=0$ black hole has two, each periodic in the angular coordinate ϕ, and may be viewed as the ground state of the Ramond sector of (1,1)-adS supergravity. Steif has recently shown that these Killing spinors have an even more direct geometric interpretation [18]: anti-de Sitter space may be embedded in the group manifold of the supergroup $\mathrm{OSp}(1|2;\mathbb{R})$, whose isometry group is $\mathrm{OSp}(1|2;\mathbb{R})\times\mathrm{OSp}(1|2;\mathbb{R})/\mathbb{Z}_2$, and the Killing spinors are simply the generators

‡For $J=0$, the spacetime fails to be Hausdorff at $r=0$, and the singularity resembles that of Taub-NUT space, but this is not the case if $J\neq0$; see appendix B of Ref. 11.

of odd elements of this group that commute with the even elements (ρ_L, ρ_R). Supersymmetric solutions of (2,0)-adS supergravity have also been investigated [33]; a number of supersymmetric configurations exist, but they typically involve naked conical singularities.

3 Black Holes and Gravitational Collapse

The physical importance of the (3+1)-dimensional black hole comes from its role as the final state of gravitational collapse. Since (2+1)-dimensional gravity has no Newtonian limit, one might fear that no such interpretation exists for the BTZ black hole. In fact, however, it was shown shortly after the discovery of the BTZ solution that this black hole arises naturally from collapsing matter [34].

Consider a (2+1)-dimensional spacetime containing a spherical cloud of dust surrounded by empty space. For the exterior, we take the metric to be of the BTZ form (1.1)–(1.2), with $J = 0$ for simplicity; for the interior, we may choose comoving coordinates, in which the geometry is given by a Robertson-Walker metric,

$$ds^2 = -dt^2 + a^2(t)\left(\frac{d\tilde{r}^2}{1 - k\tilde{r}^2} + \tilde{r}^2 d\phi^2\right). \tag{3.1}$$

The stress-energy tensor for pressureless dust is

$$T_{\mu\nu} = \rho(t) u_\mu u_\nu, \tag{3.2}$$

with $u_\mu = (1, 0, 0)$ in comoving coordinates; as usual, conservation implies that $\rho a^2 = \rho_0 {a_0}^2$. It is then easy to show that the field equations in the interior are solved by

$$\begin{aligned} a(t) &= a_0 \cos\frac{t}{\ell} + \ell \dot{a}_0 \sin\frac{t}{\ell}, \\ {\dot{a}_0}^2 &= 8\pi G \rho_0 {a_0}^2 - k - \frac{{a_0}^2}{\ell^2}. \end{aligned} \tag{3.3}$$

Note that for arbitrary initial values, $a(t)$ always reaches zero in a finite proper time.

We must now join the interior and exterior solutions, using the standard matching conditions that the spatial metric g_{ij} and the extrinsic curvature K_{ij} be continuous at the boundary. As Mann and Ross show [34], this requires that

$$M = 8\pi G \rho_0 {a_0}^2 {r_0}^2 - 1, \tag{3.4}$$

where $\tilde{r} = r_0$ is the position of the surface of the collapsing dust in the interior (comoving) coordinate, equivalent to the exterior radial coordinate $r = r_0 a(t)$. The collapse closely resembles the Oppenheimer-Snyder solution in 3+1 dimensions. In particular, the surface of the collapsing dust crosses the horizon in a finite amount of comoving time, but light emitted from the surface is infinitely red-shifted at the horizon, and the collapse appears to take infinitely long to a static exterior observer.

The mass M of the final black hole depends on three parameters, ρ_0, r_0, and a_0, or equivalently ρ_0, r_0, and v_0, where $v_0 = \dot{r}|_{t=0} = r_0\dot{a}_0$ is the initial velocity. If these parameters are such that $M < 0$ in eqn. (3.4), the final state is not a black hole, but rather a naked conical singularity in an asymptotically anti-de Sitter spacetime. We have thus found a sort of "phase transition" in the space of initial values, quite similar to the transition discovered numerically by Choptuik in 3+1 dimensions [35]. This transition has been investigated in detail by Peleg and Steif for the related case of a collapsing thin shell of dust [36]. In that case, there are four possible final states: open conical adS space, the BTZ black hole interior, the BTZ black hole exterior, and closed conical adS space. A phase diagram is shown in figure 2. Peleg and Steif show that the transitions between black hole configurations and naked singularities are critical phenomena characterized by an order parameter with critical exponent 1/2, again mimicking the behavior of more complicated (3+1)-dimensional models (although with a different critical exponent).

While pressureless dust in 2+1 dimensions necessarily collapses (provided that $\Lambda < 0$), a (2+1)-dimensional ball of fluid can be stabilized by internal pressure. The general properties of such (2+1)-dimensional "stars" have recently been investigated by Cruz and Zanelli [37]. They find that for a wide range of equations of state, a static interior solution can be joined to a BTZ exterior solution. They also find an upper limit for the mass of a stable star of fixed radius, but the result depends on an integration constant whose physical significance is not completely clear.

A BTZ black hole can also form from a collapsing pulse of radiation [38, 39, 19]. The most useful coordinates to describe this process are the Eddington-Finkelstein coordinates of (1.10), in which a suitable metric ansatz—analogous to the Vaidya metric in 3+1 dimensions—is

$$ds^2 = \left[\frac{r^2}{\ell^2} + m(v)\right] dv^2 + 2dvdr - j(v)dvd\tilde{\phi} + r^2 d\tilde{\phi}^2. \tag{3.5}$$

For the stress-energy tensor, we take that of a rotating null fluid,

$$T_{vv} = \frac{\rho(v)}{r} + \frac{j(v)\,\omega(v)}{2r^3}, \quad T_{v\tilde{\phi}} = -\frac{\omega(v)}{r}, \tag{3.6}$$

where $\rho(v)$ and $\omega(v)$ are arbitrary functions and the form of the r dependence follows from the conservation law for the stress-energy tensor. The Einstein field equations then reduce to

$$\frac{dm(v)}{dv} = 2\pi\rho(v), \qquad \frac{d\,j(v)}{dv} = 2\pi\omega(v) \tag{3.7}$$

(in BTZ units, $8G = 1$). In particular, any distribution of radiation for which $m(v) \sim M$ and $j(v) \sim J$ as v goes to infinity will approach a BTZ black hole. The pulse of radiation considered by Husain [38], for instance,

$$\rho(v) = A\,\mathrm{sech}^2\frac{v}{b}, \tag{3.8}$$

leads asymptotically to a black hole with mass $2\pi A$.

4 Interiors

One of the important open questions in black hole physics is that of the stability of the inner horizon. It is apparent from the Penrose diagram of figure 1 that in the case of a rotating black hole, an infalling observer need not hit the singularity at $r=0$, but can escape through the inner horizon $r=r_-$ to a new exterior region. On the other hand, infalling radiation is infinitely blue-shifted at $r=r_-$, suggesting that the inner horizon is not stable; and indeed, simple models in 3+1 dimensions indicate that this horizon may be destroyed by the back reaction of ingoing and back-scattered outgoing radiation. This phenomenon has been an important focus of research in (3+1)-dimensional general relativity [40, 41], where it has been shown that the internal mass function of a Reissner-Nordstrom black hole—or equivalently, the "Coulomb" component $|\Psi_2|$ of the Weyl tensor—diverges at the Cauchy horizon ("mass inflation"), but that tidal forces may nevertheless remain weak enough that physical objects can survive passage through the horizon. (For a nice review, see Ref. 42.)

In 2+1 dimensions, an exact computation analogous to that of Refs. 40–41 may be performed for the rotating BTZ black hole [19, 38]. Since the resulting phenomenon of mass inflation is discussed in some detail in Ref. 9, I will only briefly summarize the results.

The starting point is again the metric (3.5). More precisely, let us model a thin shell of *outgoing* radiation by joining an "exterior" metric of the form (3.5), with mass function $m_1(v_1)$, to an "interior" metric, also of the form (3.5), with a mass function $m_2(v_2)$. By choosing appropriate matching conditions for the interior and exterior regions, we can model the interaction of infalling radiation—$\rho(v)$ in eqn. (3.6)—with this shell of outgoing radiation. A careful analysis of these matching conditions then shows that m_2 necessarily diverges at the Cauchy horizon. (For the spinning case, it is actually the quantity

$$E_2(v) = m_2(v) - {j_2}^2(v)/4r^2 \tag{4.1}$$

that diverges.) The (2+1)-dimensional black hole thus exhibits mass inflation. On the other hand, as in 3+1 dimensions, tidal forces lead to only a finite distortion at the Cauchy horizon, so it is not clear that passage through the horizon is forbidden. Moreover, in contrast to the (3+1)-dimensional case, the Kretschmann scalar $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ remains finite. Further investigation of more complicated solutions—in particular, solutions with realistic outgoing radiation—seems feasible, and could provide valuable information on the question of stability.

5 Quantum Field Theory in a Black Hole Background

While the BTZ black hole is useful as a comparatively simple model for classical black hole physics, its real power appears when we turn to quantum theory. General relativity in 2+1 dimensions has proven to be an instructive model for realistic (3+1)-dimensional quantum gravity in a number of settings [6], and the black hole is no exception.

As in 3+1 dimensions, it is useful to warm up by considering quantum field theory in a classical black hole background. For a free field, the starting point for such a theory is an appropriate two-point function $G(x,x') = \langle 0|\phi(x)\phi(x')|0\rangle$, from which such quantities as the expectation values $\langle 0|T_{\mu\nu}|0\rangle$ can be derived. The key simplification in 2+1 dimensions comes from the quotient space picture of the BTZ black hole described in section 2, that is, its representation as a region of the universal covering space of anti-de Sitter space with appropriate identifications. This construction allows us to write the two-point function for the black hole in terms of the corresponding $\widetilde{\text{adS}}$ two-point function by means of the method of images. Specifically, if $G_A(x,x')$ is a two-point function in $\widetilde{\text{adS}}$, the corresponding function for the BTZ black hole is

$$G_{BTZ}(x,x') = \sum_n e^{-i\delta n} G_A(x, \Lambda^n x'), \tag{5.1}$$

where $\Lambda x'$ denotes the action of the group element (2.8) on x'. The phase δ is zero for ordinary ("untwisted") fields, but may in principle be arbitrary, corresponding to boundary conditions $\phi(\Lambda x) = e^{-i\delta}\phi(x)$; the choice $\delta = \pi$ leads to conventional "twisted" fields. Our problem has thus been effectively reduced to the comparatively simple problem of understanding quantum field theory on $\widetilde{\text{adS}}$.

While quantum field theory on anti-de Sitter space is fairly simple, it is by no means trivial. The main difficulty comes from the fact that neither anti-de Sitter space nor its universal covering space are globally hyperbolic. As is evident from the Penrose diagrams of figure 1, spatial infinity is timelike, and information may enter or exit from the "boundary" at infinity. One must consequently impose boundary conditions at infinity to formulate a sensible field theory.

This problem has been analyzed carefully in 3+1 dimensions by Avis, Isham, and Storey [43], who show that there are three reasonable boundary conditions for a scalar field at spatial infinity: Dirichlet (D), Neumann (N), and "transparent" (T) boundary conditions. The latter—essentially a linear combination of Dirichlet and Neumann conditions—are most easily obtained by viewing $\widetilde{\text{adS}}$ as half of an Einstein static universe; physically they correspond to a particular recirculation of momentum and angular momentum at spatial infinity. The same choices exist in 2+1 dimensions [44]. In particular, for a massless, conformally coupled scalar field, described by an action

$$I = -\int d^3x\,\sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{1}{16}R\phi^2\right), \tag{5.2}$$

the adS Greens functions are

$$\begin{aligned} G_A^{(D)} &= \frac{1}{4\pi\sqrt{2}}\left\{\sigma^{-1/2} - [\sigma + 2\ell^2]^{-1/2}\right\} \\ G_A^{(N)} &= \frac{1}{4\pi\sqrt{2}}\left\{\sigma^{-1/2} + [\sigma + 2\ell^2]^{-1/2}\right\} \\ G_A^{(T)} &= \frac{1}{4\pi\sqrt{2}}\,\sigma^{-1/2}. \end{aligned} \tag{5.3}$$

Here $\sigma(x,x')$ is the square of the distance between x and x' in the embedding space described in section 2; in the coordinates of eqn. (2.4),

$$\sigma(x,x') = \frac{1}{2}\left[(X_1 - X_1')^2 - (T_1 - T_1')^2 + (X_2 - X_2')^2 - (T_2 - T_2')^2\right]. \tag{5.4}$$

A more general Greens function with "mixed" boundary conditions,

$$G_A^{(\alpha)} = \frac{1}{4\pi\sqrt{2}}\left\{\sigma^{-1/2} - \alpha[\sigma + 2\ell^2]^{-1/2}\right\}, \tag{5.5}$$

may also be considered [45]. The corresponding BTZ Greens functions are then obtained from the sum (5.1).

For the static ($J=0$) BTZ black hole, Lifschytz and Ortiz [44] and Shiraishi and Maki [45] have analyzed the quantum behavior of a massless, conformally coupled scalar field, using the full range of boundary conditions discussed above. For each of these boundary conditions, the Greens function obtained from the sum (5.1) can be shown to be periodic in imaginary time, with a period β given by

$$\beta^{-1} = (N^\perp)^{-1}T_0, \qquad T_0 = \frac{r_+}{2\pi\ell^2} = \frac{\kappa}{2\pi}, \tag{5.6}$$

where κ is the surface gravity (1.15). As usual, β may be interpreted as a local inverse temperature, corresponding to a temperature T_0 of the black hole corrected by a red shift factor $(g_{00})^{-1/2}$. The relationship of T_0 and κ is then exactly the same as in 3+1 dimensions.* By studying the analyticity properties of the Greens functions (5.3), Lifschytz and Ortiz argue that the relevant vacuum state is the Hartle-Hawking vacuum. Similar computations of the propagator and the expectation value $\langle\phi^2\rangle$ for transparent boundary conditions have also been performed in Ref. 47.

For the action (5.2), the stress-energy tensor is

$$T_{\mu\nu} = \frac{3}{4}\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{4}g_{\mu\nu}g^{\rho\sigma}\nabla_\rho\phi\nabla_\sigma\phi - \frac{1}{4}\phi\nabla_\mu\nabla_\nu\phi + \frac{1}{4}g_{\mu\nu}g^{\rho\sigma}\phi\nabla_\rho\nabla_\sigma\phi - \frac{1}{16\ell^2}g_{\mu\nu}\phi^2, \tag{5.7}$$

*Reznik had already found the corresponding result for (2+1)-dimensional gravity with a positive cosmological constant in 1992 [46].

and the expectation value $\langle 0|T_{\mu\nu}|0\rangle$ can be obtained by differentiating a two-point function and taking coincidence limits: for example, the expectation value of the first term in (5.7) is

$$\lim_{x\to x'} \frac{3}{4}\nabla_\mu\nabla_{\nu'}G(x,x').$$

Only the $n=0$ term in the sum (5.1) diverges as $x\to x'$, and the stress-energy tensor may be renormalized by subtracting off this term, in effect setting the energy in a pure anti-de Sitter universe to zero. The resulting expectation values are discussed in Ref. 44 (for $J=0$, Neumann and Dirichlet boundary conditions, and untwisted fields), Ref. 45 (for $J=0$, arbitrary mixed boundary conditions, and twisted and untwisted fields), and Ref. 48 (for arbitrary J, transparent boundary conditions, and untwisted fields). For the static black hole, $\langle 0|T_{\mu\nu}|0\rangle$ is regular on the horizon $r = r_+$, but diverges as $1/r^3$ at $r=0$, indicating a breakdown of the semiclassical approximation and the possible emergence of a genuine curvature singularity. The back reaction on the metric may be estimated away from $r = 0$ [44, 45]; its effect is to increase the horizon size and to slightly change the radial acceleration felt by a test particle, but for large black hole masses the corrections are exponentially suppressed.

For the rotating black hole, an interesting new phenomenon arises at the inner horizon [48]. The expectation value of the stress-energy tensor (at least for transparent boundary conditions) now involves terms of the form

$$\sum \frac{a_n}{|d_n|^{5/2}}$$

where

$$d_n = \sigma(x, \Lambda^n x). \tag{5.8}$$

As a consequence, $\langle 0|T_{\mu\nu}|0\rangle$ blows up on an infinite sequence of timelike "polarized hypersurfaces" composed of points x connected to their images $\Lambda^n x$ by null geodesics. In the limit $n\to\infty$, these surfaces approach the inner horizon $r = r_-$. The back reaction of the quantum stress-energy tensor on the metric may now be estimated; one finds that metric perturbations at a geodesic distance s from a polarized hypersurface grow as $s^{-1/2}$. This result, which is strongly reminiscent of both the classical instability of section 4 and of "chronology protection" in (3+1)-dimensional models [49], strongly suggests that the inner horizon is quantum mechanically unstable.

We may obtain further information about scalar fields in a black hole background by computing the Bogoliubov coefficients between anti-de Sitter "in" states in the far past and BTZ "out" states. The relevant transformations have been analyzed by Hyun, Lee, and Yee [50] for the rotating black hole with transparent boundary conditions. A calculation of the expectation value of the "out" number operator in the "in" vacuum state yields

$$_{\rm in}\langle 0|N^{\rm out}_{\omega m}|0\rangle_{\rm in} = \frac{1}{\exp\left[2\pi(\omega - m\Omega_H)\beta_0\right] - 1}, \tag{5.9}$$

where $\Omega_H = -N^\phi(r_+)$ is the angular velocity of the horizon[†] and

$$\beta_0^{-1} = \frac{r_+^2 - r_-^2}{2\pi r_+ \ell^2} = \frac{\kappa}{2\pi}, \tag{5.10}$$

reducing to (5.6) when $J = 0$. The thermal form of (5.9) is an indication of Hawking radiation, while the dependence on $\omega - m\Omega_H$ rather than ω is a sign that the rotating BTZ black hole, like the Kerr black hole, has super-radiant modes.

The quantum field theory of a massless fermion in a static BTZ black hole background has been studied as well [51]. Hyun, Song, and Lee compute the two-point function for Dirichlet and Neumann boundary conditions at spatial infinity, again using the method of images (5.1), and study the response of a particle detector. As in the case of the conformally coupled massless scalar, the response is thermal, with a temperature given by (5.6). Curiously, the response function for fermions exhibits a Planck distribution, while the response function for the scalar field exhibits a Fermi-Dirac distribution. This "statistics flip" does not occur in the expectation value $\langle N \rangle$ of the number operator [50], and the phenomenon is not understood.

In a recent preprint [52], Ichinose and Satoh have also studied massive, non-conformally coupled scalar fields in a BTZ background. They show that the two-point functions for the Hartle-Hawking vacuum are again thermal, with a periodicity in imaginary time given by (5.10). They also compute thermodynamic quantities such as the free energy and entropy of the scalar field, but find that the results are generally divergent and regularization-dependent.

6 Thermodynamics

The results of the last section give a strong indication that the BTZ black hole, like the Kerr black hole, is a thermodynamic object with a temperature $T_0 = \kappa/2\pi$. In particular, the appearance of thermal Greens functions and the particle distribution (5.9) both point to the presence of thermal Hawking radiation. These results rely, however, on an unphysical splitting of the system into a classical gravitational background and quantum matter fields. In this section, I will briefly review four approaches to black hole thermodynamics that depend on the gravitational configuration alone: the Euclidean path integral, the microcanonical ensemble of Brown and York, the method of Noether charges, and the quantum tunneling approach. (For a comparison of several approaches to black hole entropy in a general setting, see [53].)

The most straightforward approach to black hole thermodynamics is the Euclidean path integral of Gibbons and Hawking [54], in which the black hole partition function is expressed as a path integral periodic in imaginary time. In 3+1 dimensions, it is well known that the Euclidean black hole solution exists globally only when a

[†]The ω and m dependence in (5.9) reflects the structure (1.13) of the Killing vector normal to the horizon.

particular periodicity is imposed, and that this periodicity determines a unique inverse temperature β_0. To see that the same is true in 2+1 dimensions, we must study the Riemannian continuation of the BTZ metric (1.1), obtained by letting $t = i\tau$ and $J = iJ_E$:

$$ds_E^2 = (N_E^\perp)^2 d\tau^2 + f_E^{-2} dr^2 + r^2 \left(d\phi + N_E^\phi d\tau\right)^2 \tag{6.1}$$

with

$$N_E^\perp = f_E = \left(-M + \frac{r^2}{\ell^2} - \frac{J_E^2}{4r^2}\right)^{1/2}, \qquad N_E^\phi = -iN^\phi = -\frac{J_E}{2r^2}. \tag{6.2}$$

The Euclidean lapse function now has roots

$$\begin{aligned} r_+ &= \left\{\frac{M\ell^2}{2}\left[1 + \left(1 + \frac{J_E^2}{M^2\ell^2}\right)^{1/2}\right]\right\}^{1/2}, \\ r_- &= -i|r_-| = \left\{\frac{M\ell^2}{2}\left[1 - \left(1 + \frac{J_E^2}{M^2\ell^2}\right)^{1/2}\right]\right\}^{1/2}. \end{aligned} \tag{6.3}$$

Note that the continuation of J to imaginary values, necessary for the metric (6.1) to be Riemannian, is physically sensible, since the angular velocity is now a rate of change of a real angle with respect to imaginary time.

The metric (6.1)–(6.2) is a positive definite metric of constant negative curvature, and the spacetime is therefore locally isometric to hyperbolic three-space $\mathbb{H}^3$. This isometry is most easily exhibited by means of the Euclidean analogue of the coordinate transformation (2.9), which is now globally valid [55],

$$\begin{aligned} x &= \left(\frac{r^2 - r_+^2}{r^2 - r_-^2}\right)^{1/2} \cos\left(\frac{r_+}{\ell^2}\tau + \frac{|r_-|}{\ell}\phi\right) \exp\left\{\frac{r_+}{\ell}\phi - \frac{|r_-|}{\ell^2}\tau\right\} \\ y &= \left(\frac{r^2 - r_+^2}{r^2 - r_-^2}\right)^{1/2} \sin\left(\frac{r_+}{\ell^2}\tau + \frac{|r_-|}{\ell}\phi\right) \exp\left\{\frac{r_+}{\ell}\phi - \frac{|r_-|}{\ell^2}\tau\right\} \\ z &= \left(\frac{r_+^2 - r_-^2}{r^2 - r_-^2}\right)^{1/2} \exp\left\{\frac{r_+}{\ell}\phi - \frac{|r_-|}{\ell^2}\tau\right\}. \end{aligned} \tag{6.4}$$

In these coordinates, the metric becomes that of the standard upper half-space representation of hyperbolic three-space,

$$ds_E^2 = \frac{\ell^2}{z^2}(dx^2 + dy^2 + dz^2) = \frac{\ell^2}{\sin^2\chi}\left(\frac{dR^2}{R^2} + d\chi^2 + \cos^2\chi\, d\theta^2\right), \qquad (z > 0). \tag{6.5}$$

Here (R, θ, χ) are standard spherical coordinates for the upper half-space $z > 0$,

$$(x, y, z) = (R\cos\theta\cos\chi, R\sin\theta\cos\chi, R\sin\chi), \tag{6.6}$$

and periodicity in the Schwarzschild angular coordinate ϕ requires that we identify

$$(R,\theta,\chi) \sim (Re^{2\pi r_+/\ell},\theta + \frac{2\pi|r_-|}{\ell},\chi). \tag{6.7}$$

Eqn. (6.7) is the Euclidean version of the identifications $\langle(\rho_R,\rho_L)\rangle$ of section 2. A fundamental region is the space between two hemispheres $R=1$ and $R=e^{2\pi r_+/\ell}$, with the inside and outside boundaries identified by a translation along a radial line followed by a $2\pi|r_-|/\ell$ rotation around the z axis (see figure 3). The topology is thus $\mathbb{R}^2\times S^1$, as expected.

Now, for the coordinate transformation (6.4) to be nonsingular at the z axis $r=r_+$, we must require periodicity in the arguments of the trigonometric functions; that is, we must identify

$$(\phi,\tau) \sim (\phi+\Phi,\tau+\beta_0) \tag{6.8}$$

where

$$\beta_0 = \frac{2\pi r_+\ell^2}{r_+^2-r_-^2}, \qquad \Phi = \frac{2\pi|r_-|\ell}{r_+^2-r_-^2}\,. \tag{6.9}$$

Equivalently, if we define a shifted angular coordinate $\phi'=\phi-(\Phi/\beta_0)\tau$, the identifications become $(\phi',\tau)\sim(\phi',\tau+\beta_0)$. We thus obtain exactly the same temperature (5.10) that appeared in the analysis of quantum fields. If this requirement of periodicity is lifted, the Euclidean black hole acquires a conical singularity at the horizon, whose nature I will discuss further in section 7.

To obtain the entropy from the Euclidean path integral, we must now evaluate the grand canonical partition function

$$Z = \int [dg]\, e^{I_E[g]}, \tag{6.10}$$

where I_E is the Euclidean action. Normally, only the classical approximation $Z\sim\exp\{I_E[\bar g]\}$ is considered, where $\bar g$ is the extremal metric (6.1). The action I includes boundary terms at $r=r_+$ and $r=\infty$, which are analyzed carefully in Ref. 55 for the (2+1)-dimensional black hole (see also [56] for a more general discussion). For a temperature β_0 and a rotational chemical potential Ω, one finds an action

$$I_E[\bar g] = 4\pi r_+ - \beta_0(M-\Omega J), \tag{6.11}$$

corresponding to an entropy $4\pi r_+$, or with factors of $\hbar$ and G restored,

$$S = \frac{2\pi r_+}{4\hbar G}. \tag{6.12}$$

Note the similarity with the standard Bekenstein entropy, which like (6.12) is one-fourth of the horizon size in Planck units.

In contrast to the (3+1)-dimensional theory, one can also compute the first quantum correction to the path integral (6.10)—the Van Vleck-Morette determinant—by taking advantage of the Chern-Simons formalism. This calculation was performed, in a different context, in Ref. 57; to the next order, the partition function becomes [55]

$$Z \sim \exp\left\{I[\bar{g}] + \frac{2\pi r_+}{\ell}\right\}, \tag{6.13}$$

leading naively to an entropy

$$S = \frac{2\pi r_+}{4\hbar G}\left(1 + \frac{4\hbar G}{\ell}\right). \tag{6.14}$$

Ghosh and Mitra have argued that this expression does not properly account for quantum corrections to β_0, and that the true one-loop-corrected temperature and entropy should be [58]

$$\beta_0 = \frac{1}{8\hbar G}\frac{2\pi r_+ \ell^2}{r_+^2 - r_-^2}\left(1 + \frac{8\hbar G}{\ell}\right), \qquad S = \frac{2\pi r_+}{4\hbar G}\left(1 + \frac{8\hbar G}{\ell}\right). \tag{6.15}$$

The Euclidean path integral offers a simple and attractive approach to black hole thermodynamics, but it has important limitations. In particular, the canonical and grand canonical ensembles do not always exist for arbitrarily large gravitating systems. Brown and York have recently developed a more sophisticated path integral approach [59], based on the microcanonical ensemble for a black hole in a cavity. The resulting microcanonical partition function can again be written in the form (6.10), where the action I has the same "bulk" terms as in the Gibbons-Hawking approach, but different boundary terms.*

This approach to the BTZ black hole is discussed in detail by Brown, Creighton, and Mann in Ref. 17. They find that the temperature at the boundary of a cavity of radius R is

$$T = \frac{1}{N^\perp(R)}\frac{\kappa}{2\pi}, \tag{6.16}$$

the correct red-shifted form of the temperature (5.10), and that the thermodynamic chemical potential conjugate to J is

$$\Omega = \frac{1}{N^\perp(R)}\left(N^\phi(R) - N^\phi(r_+)\right), \tag{6.17}$$

in agreement with the results of [55]. Moreover, with the entropy (6.12), the (2+1)-dimensional black hole obeys the standard first law of thermodynamics,

$$dE = TdS + \Omega dJ - \mathcal{P}d(2\pi R), \tag{6.18}$$

*A fundamental feature of general relativity is that the energy and angular momentum in a finite spatial region U may be expressed in terms of integrals over the boundary ∂U; this is the underlying reason that the microcanonical ensemble can be obtained from the canonical ensemble by adjusting only the boundary terms in the path integral.

where $\mathcal{P}$ is the surface pressure at R.

A third approach to black hole thermodynamics is that of Wald [60], who defines the entropy as the Noether charge associated with the Killing vector ξ that is normal to a bifurcate event horizon and is normalized to unit surface gravity. The argument for this definition, based on the application of the first law of thermodynamics to diffeomorphism-invariant systems, is rather long, and I will not try to reproduce it here. For (2+1)-dimensional gravity, the relevant Noether charge one-form is (see [61])

$$Q = -\frac{1}{16\pi G}\epsilon_{abc}\nabla^b\xi^c dx^a, \tag{6.19}$$

where the Killing vector ξ is determined from (1.13) by $\xi^a = \kappa^{-1}\chi^a$. The one-form (6.19) is to be integrated over the bifurcation circle $r = r_+$, with the volume element induced by the metric (1.10). It is straightforward to check that up to slight differences in normalization, the entropy (6.12) is again reproduced. The same computation has been carried out in the first-order formalism in Ref. 13.

Finally, let me briefly mention a fourth approach to black hole thermodynamics, developed by Casher and Englert [62]. It is well known that the Wheeler-DeWitt equation in quantum gravity has no preferred time parameter, but that in a region in which the gravitational field is nearly classical, the WKB approximation leads to an effective "time" for matter determined by the classical gravitational trajectories [63]. Casher and Englert argue that the thermodynamic properties of black holes reflect the breakdown of this approximation, and the consequent failure of semiclassical unitarity, in regions of superspace in which tunneling processes are important. Applying this argument to the BTZ black hole, Englert and Reznik find that the relevant tunneling process in 2+1 dimensions is one between a thick spherical shell of matter (held in equilibrium by a uniform pressure $|p_r| = |p_\theta| = \rho$) and a black hole of the same mass [64]. They compute a tunneling entropy and find an expression that is once again equal to that of eqn. (6.12).

These various results may now be used to determine thermodynamic quantities such as heat capacities [17, 65]. The microcanonical approach is especially useful for this purpose. In contrast to the (3+1)-dimensional black hole, it may be shown that the BTZ black hole has a strictly positive heat capacity $C_{R,J}$ at fixed R and J, and that the temperature T is a monotonically increasing function of r_+. Consequently, there is a unique solution $M(\beta, R, J)$ for a black hole in a cavity at fixed surface temperature and angular momentum, and the resulting black hole is thermodynamically stable—there are no negative-heat-capacity instantons like those that occur in 3+1 dimensions. The grand canonical ensemble (Ω and β fixed) is similarly stable. On the other hand, pressure ensembles ($\mathcal{P}$ fixed, R allowed to vary) are thermodynamically unstable, with negative heat capacities [65]. One may also analyze the thermodynamics of a gas of noninteracting (2+1)-dimensional black holes; if one interprets the entropy (6.12) as the logarithm of the number of states of a single black hole, Cai et al. have shown that it is possible to construct microcanonical and canonical partition functions for such a system [66].

A natural question, of fundamental importance in 3+1 dimensions, is whether the black hole evaporates completely, and if not, what final states are possible. Here, unfortunately, the (2+1)-dimensional model may not be too helpful. Note first that the temperature of a BTZ black hole is $T \sim M^{1/2}$, which, in contrast to the (3+1)-dimensional case, goes to zero as M decreases. Naively, complete evaporation would take an infinite amount of time; indeed, the energy flux in thermal radiation in 2+1 dimensions is proportional to the cube of the temperature, and Stefan's law gives [67]

$$\frac{dM}{dt} \sim M^2, \tag{6.20}$$

or $M \sim 1/t$. Moreover, the unusual boundary conditions for radiation in an asymptotically anti-de Sitter space, discussed in section 5, affect the thermodynamics; Reznik has argued that for a space with such asymptotic behavior, a black hole with a large enough initial mass will come to equilibrium with thermal radiation even in an infinitely large universe [68]. Further work on this problem—including, ideally, a detailed semiclassical computation of the interaction of a BTZ black hole with Hawking radiation with various boundary conditions at infinity—may be feasible, and would be of great interest.

7 Quantization and Statistical Mechanics

Several of the derivations of black hole entropy described in the preceding section were quantum mechanical, in the sense that they were based on the (Euclidean) path integral. They made only limited use of the machinery of quantum mechanics, however, relying instead on classical (and, in one case, one-loop) approximations. It is natural to ask whether a full quantum mechanical treatment of the (2+1)-dimensional black hole can provide us with further insight. In particular, we might hope that such a treatment could give us a "statistical mechanical" explanation of black hole entropy in terms of the microscopic physics of quantum gravitational states. We are still far from having a complete answer to this question, but several interesting first steps have been taken.

Perhaps the simplest starting point is a minisuperspace quantization of metrics of the form (1.1) or (6.1), with $N^\perp$, f, and N^ϕ treated as arbitrary functions of r. This model has been analyzed in Ref. 55 in the Euclidean setting. The natural approach to the quantum theory is that of radial quantization, in which the radial coordinate r is used as a "time" variable and canonical commutation relations are imposed on surfaces of constant r. In terms of the parameters of (6.1), one finds two pairs of conjugate variables, (β, f_E^2) and (N^ϕ, p), where

$$\beta(r) = f_E^{-1}(r) N_E^\perp(r) \tag{7.1}$$

and $p(r)$ is the r-ϕ component of the gravitational momentum. The quantum theory may be obtained by imposing equal r commutation relations at $r=r_+$,

$$[\beta(r_+), f^2(r_+)] = [N^\phi(r_+), p(r_+)] = i\hbar, \tag{7.2}$$

and describing the evolution by a "time"-dependent radial Hamiltonian that may be shown to have the form

$$H_r(r) = -\beta(r)\left[\frac{p^2(r)}{2r^3} + \frac{2r}{\ell^2}\right]. \tag{7.3}$$

In principle, we should now be able to compute the partition function (6.10) by taking the trace of the propagator $K[{}^{(2)}\mathcal{G}_2, {}^{(2)}\mathcal{G}_1; r_+, p_+; \beta(\infty), \tilde{N}^{\phi\prime}(\infty)]$ in this quantum theory. The exact calculation has not been performed, but it is argued in [55] that the classical approximation duplicates the result found in section 6 for the entropy.

To understand the canonical variables in this minisuperspace, it is useful to turn to the Chern-Simons description of (2+1)-dimensional gravity, as described in section 2. For a Euclidean black hole, the holonomy (2.8) may be analytically continued to become an SL(2, $\mathbb{C}$) holonomy

$$\rho[\gamma] = \begin{pmatrix} e^{\pi(r_+ + i|r_-|)/\ell} & 0 \\ 0 & e^{-\pi(r_+ + i|r_-|)/\ell} \end{pmatrix}, \tag{7.4}$$

where γ is the closed curve (2.16); in figure 3, such a curve is represented by the portion of the line segment L lying between the two (identified) hemispheres. At first sight, there appears to be no other closed curve around which to define a holonomy; the topology of the Euclidean black hole is $\mathbb{R}^2 \times S^1$, a manifold whose fundamental group has only a single generator. Were this the case, the Chern-Simons quantum theory would be trivial: r_+ and r_- could be treated as operators, but they would have no canonical conjugates.

For the computation of transition amplitudes and propagators, however, the relevant spacetime is not the complete Euclidean black hole, but rather the wedge $\tau_1 \le \tau \le \tau_2$, whose geometry in the upper half-space representation is depicted in figure 4. We can now consider a line segment δ connecting the points (R_1, θ_1) and $(R_2 = e^\Sigma R_1, \theta_2 = \theta_1 + \Theta)$ along a surface of constant "radius" χ. (It may be seen from (6.4) and (6.6) that constant χ does in fact correspond to constant radius r.) Such a curve is exemplified in figure 4 by the portion of the line segment K lying between the edges $\tau = \tau_1$ and $\tau = \tau_2$. A simple computation then gives an SL(2, $\mathbb{C}$) holonomy

$$\rho[\delta] = \begin{pmatrix} e^{\pi(\Sigma + i\Theta)/\ell} & 0 \\ 0 & e^{-\pi(\Sigma + i\Theta)/\ell} \end{pmatrix}. \tag{7.5}$$

For the complete spacetime of figure 3, values of Σ and Θ other than $\Sigma = 0$, $\Theta = 2\pi$ signal the presence of a conical singularity with a "helical twist" at $\chi = \pi/2$, that is, at the horizon $r = r_+$.

The minisuperspace variables f_E, p, β, and N^ϕ may now be written in terms of the constants of motion $r_\pm$, Θ, and Σ that describe the holonomies (see Ref. 55 for the complete expressions). The commutators (7.2) then induce commutators

$$\begin{aligned} [r_+, \Theta] &= [|r_-|, \Sigma] = 4i\hbar G \\ [|r_-|, \Theta] &= [r_+, \Sigma] = 0, \end{aligned} \tag{7.6}$$

providing a starting point for quantization of the Chern-Simons theory. The same commutators may be obtained directly from the holonomies, using the results of Nelson and Regge for quantization of holonomies in (2+1)-dimensional gravity [69]. Observe in particular that the horizon radius r_+ and the deficit angle Θ at the horizon are conjugate operators, implying that it is inconsistent to simply set Θ to 2π in the quantum theory. This role of the horizon deficit angle as a canonical variable is the starting point for the approach of Ref. 70 to an extended Wheeler-DeWitt equation for black holes in arbitrary dimensions.

Note that for a Chern-Simons theory—at least on a manifold without boundary—the holonomies of the connection provide a complete set of physical observables. For the Euclidean BTZ black hole topology, this implies that the observables (r_+, Θ) and (r_-, Σ), which parametrize the geometry of the horizon, should be sufficient for a complete quantum theory; the Chern-Simons formulation is *not* a minisuperspace model. In particular, if the entropy of the black hole can be explained in terms of microscopic quantum gravitational states, the relevant physics should involve states at the horizon depending on these canonical pairs of variables alone.

It is possible, however, that this conclusion is an artifact of Euclideanization, since the continuation of the BTZ metric to Riemannian signature causes the two-dimensional Lorentzian horizon to collapse to a one-dimensional circle, "compressing" the horizon dynamics. To see whether this is a problem, we must investigate the dynamics of the horizon of the Lorentzian black hole. Here, the Chern-Simons formalism proves to be a powerful tool.

Consider the gravitational action (2.13) on a spacetime consisting of the exterior of a (2+1)-dimensional black hole, with the event horizon acting as a boundary ∂M. It is well known that a Chern-Simons theory on a manifold with boundary is no longer a theory of a finite number of holonomies alone; rather, the Chern-Simons action induces a two-dimensional Wess-Zumino-Witten (WZW) action on ∂M [71,72]. This phenomenon occurs because the presence of a boundary partially breaks the gauge symmetry, allowing would-be "pure gauge" degrees of freedom to become dynamical.

For the case of (2+1)-dimensional gravity with a negative cosmological constant, this phenomenon is discussed in Ref. 73. The requirement that ∂M be an apparent horizon imposes boundary conditions

$$A^+_\phi = A^+_v = \tilde{A}^+_\phi = \tilde{A}^+_v = 0, \quad A^2_\phi = \bar{\omega} + \frac{r_+}{\sqrt{2}\ell}, \quad \tilde{A}^2_\phi = \bar{\omega} - \frac{r_+}{\sqrt{2}\ell} \tag{7.7}$$

on the gauge fields (2.12). Here r_+ is the horizon radius, while the interpretation of $\bar{\omega}$ is less clear; it is argued in Ref. 73 that one should sum over constant values of $\bar{\omega}$ to count macroscopically indistinguishable states. With these boundary conditions, the induced action on ∂M may be shown to take the form (up to possible finite renormalizations)

$$I[g,\tilde{g}] = -kI^+_{\mathrm{WZW}}[g,A] + kI^+_{\mathrm{WZW}}[\tilde{g},\tilde{A}], \tag{7.8}$$

where $I^+_{\mathrm{WZW}}[g,A]$ is the $\mathrm{SO}(2,1)$ chiral Wess-Zumino-Witten action,

$$\begin{aligned} I^+_{\mathrm{WZW}}[g,A] = \frac{1}{4\pi}\int_{\partial M} \mathrm{Tr}\left(g^{-1}\partial_\phi g\right)\left(g^{-1}\partial_v g\right) \\ + \frac{1}{2\pi}\int_{\partial M} \mathrm{Tr}\left(g^{-1}\partial_v g\right)\left(g^{-1}A_\phi g\right) + \frac{1}{12\pi}\int_M \mathrm{Tr}\left(g^{-1}dg\right)^3 \end{aligned} \tag{7.9}$$

and the constant k is

$$k = \frac{\ell\sqrt{2}}{8G}\,. \tag{7.10}$$

The effective quantum theory of the horizon is thus described by a pair of $\mathrm{SO}(2,1)$ WZW actions, and one may attempt to find the entropy by counting the resulting states.

Such $\mathrm{SO}(2,1)$ WZW models are not yet completely understood. In the limit of large k (or small Λ), however, the action (7.8) may be approximated by a system of six independent bosonic string oscillators. Such a system has an infinite number of states, but most of these are eliminated by a remaining gauge symmetry—a remnant of the Wheeler-DeWitt equation—that expresses invariance under time-dependent shifts of the angular coordinate ϕ. These transformations are generated by the Virasoro operator L_0. Acting on a naturally defined Hilbert space (built by a "highest weight" construction), L_0 takes the form [73]

$$L_0 = N + \frac{4k^2}{4k^2-1}\left(\bar{\omega} - \frac{\sqrt{2}kr_+}{\ell}\right)^2 - \frac{2k^2r_+^2}{\ell^2}, \tag{7.11}$$

where $N = \sum_{i=1}^{6} N_i$ is a number operator for the six "stringy" oscillators. Requiring that L_0 annihilate physical states thus determines N in terms of r_+.

It is now a standard result of number theory [74] and string theory [75] that the number of states of such a system behaves asymptotically as

$$n(N) \sim \exp\left\{\pi\sqrt{6\cdot\frac{2N}{3}}\right\}. \tag{7.12}$$

Using (7.11) to determine N, integrating over $\bar{\omega}$, and inserting (7.10) for k, we obtain

$$\log n(r_+) \sim \frac{2\pi r_+}{4\hbar G}, \tag{7.13}$$

which is precisely the expression for the entropy of the (2+1)-dimensional black hole derived in the preceding section.

Unfortunately, this argument relies heavily on the particular characteristics of general relativity in 2+1 dimensions, and does not directly generalize to higher dimensions. A possible guess for a (3+1)-dimensional analogue of the action (7.8) would be an induced action on the horizon for "would-be diffeomorphisms" generated by vector fields with components normal to the horizon. The resulting picture is reminiscent of the membrane model of Maggiore [76, 77] and the "quantized normal mode" picture of York [78], but the analogies are so far only suggestive.

Given this picture of black hole thermodynamics, we may now ask whether it is possible to obtain further information about the quantum mechanics of the horizon degrees of freedom. Little is known about this question, but there have been some interesting speculations. Maggiore has recently proposed an effective action for the horizon, using the position $\zeta^\mu(x)$ of the horizon as a set of collective coordinates, and has obtained a minisuperspace Schrödinger equation for spherical fluctuations [77]. By considering a related string model, Kogan has argued that the quantum black hole should have a discrete mass spectrum, with r_+^2 quantized in integral multiples of the Planck length [79].

8 Generalizations

The focus of this paper has been on the simplest form of the (2+1)-dimensional black hole, that of Bañados, Teitelboim, and Zanelli. If one introduces additional fields, however, a number of interesting generalizations become possible. In this section, I will briefly review a few of these generalized black holes: electrically charged black holes, dilatonic black holes, black holes arising in string theory, black holes in topologically massive gravity, and black holes formed from "topological" matter.

The simplest extension of the BTZ black hole may be found by coupling an electromagnetic field to obtain a (2+1)-dimensional "Reissner-Nördstrom" solution. For the static case ($N^\phi = 0$), we can take as the electromagnetic potential the one-form

$$A = -Q\ln(r/r_0)dt, \tag{8.1}$$

and modify the metric (1.1)–(1.2) by setting

$$N^\perp = f = \left(-M + \frac{r^2}{\ell^2} + \frac{J^2}{4r^2} - \frac{1}{2}Q^2\ln(r/r_0)\right)^{1/2} \tag{8.2}$$

to obtain a solution of the (2+1)-dimensional Einstein-Maxwell equations. Unlike the ordinary BTZ black hole, this solution exists even when the cosmological constant vanishes, reducing to the metric originally discovered independently by Deser and Mazur [80], Gott, Simon, and Alpert [81], and Melvin [82].

Not surprisingly, this construction fails when $J\neq 0$: a rotating black hole should have a nonvanishing magnetic field, and the form (8.1) of the vector potential must be generalized. As far as I know, the general solution for a charged rotating black hole in 2+1 dimensions is not known. However, the particular case of an extreme ($J=\pm M\ell$) black hole with a mass $M=8\pi GQ^2$ and a self dual or anti-self dual electromagnetic field ($E=\pm B$) has recently been investigated by Kamata and Koikawa [83].

The stress-energy tensor for the charged black hole is, of course, nonzero, so by (2.1), the spacetime is no longer one of constant curvature. In particular, it is not possible to express the charged black hole as a quotient of anti-de Sitter space by a group of isometries. Nevertheless, the spacetime remains simple enough that explicit quantum field theory calculations of the type described in section 5 may still be possible; work on this problem is in progress [84].

A further extension of the BTZ black hole may be obtained by introducing a dilaton coupling. Chan and Mann [85] have investigated black hole solutions for an action of the form

$$I = \int d^3x\,\sqrt{-g}\left(R - \frac{B}{2}\nabla_\mu\phi\nabla^\mu\phi - e^{-4a\phi}F_{\mu\nu}F^{\mu\nu} + 2e^{b\phi}\Lambda\right), \tag{8.3}$$

where ϕ is the dilaton field, $F_{\mu\nu}$ is the ordinary electromagnetic field, and the coupling strengths a, b, and B are arbitrary. By adjusting the couplings, they find a one-parameter family of static black holes with dilaton fields of the form $\phi=k\ln(r/r_0)$, exhibiting a wide variety of horizon structures. The temperatures, quasilocal energies, and entropies of these solutions are fairly easy to compute, and in contrast to the BTZ black hole—but in analogy to the (3+1)-dimensional black hole—one can find solutions with negative heat capacities. An action equivalent to (8.3) with $F_{\mu\nu}=0$ and $b=4$ has also been studied by Sá et al. [86], who examine the structure of horizons and geodesics for a range of values of the coupling constant B. (The action of Ref. 86 differs from that of Chan and Mann by a rescaling $g_{\mu\nu}\rightarrow e^{4\phi}g_{\mu\nu}$ of the metric.)

Perhaps the most interesting generalization of the BTZ solution comes from its connection with string theory [67, 87]. The low energy string effective action is

$$I = \int d^3x\,\sqrt{-g}e^{-2\phi}\left(\frac{4}{k} + R + 4\nabla_\mu\phi\nabla^\mu\phi - \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho}\right), \tag{8.4}$$

where ϕ is the dilaton and $H_{\mu\nu\rho}$ is an antisymmetric Kalb-Ramond field, which in three dimensions must be proportional to the volume form $\epsilon_{\mu\nu\rho}$. Horowitz and Welch [67] point out that in three dimensions, the ansatz

$$H_{\mu\nu\rho} = \frac{2}{\ell}\epsilon_{\mu\nu\rho}, \quad \phi = 0, \quad k = \ell^2 \tag{8.5}$$

reduces the equations of motion coming from (8.4) to the Einstein field equations (1.3). The BTZ black hole metric is thus a part of a solution of low energy string theory.

In fact, there is a corresponding *exact* solution of string theory. As we saw in section 2, the (2+1)-dimensional black hole can be represented as a quotient of the group manifold $\mathrm{SL}(2,\mathbb{R})$ by a discrete group of isomorphisms. On the other hand, an $\mathrm{SL}(2,\mathbb{R})$ WZW model with an appropriately chosen central charge is an exact string theory vacuum, describing the propagation of strings on this same group manifold. By quotienting out the discrete group $\langle(\rho_L,\rho_R)\rangle$ of section 2 by means of an orbifold construction, one obtains a theory that may be shown to be an exact string theoretical representation of the BTZ black hole [67, 87, 88].

This stringy BTZ black hole provides an interesting model for studying target space duality in string theory, that is, the existence of physically equivalent string theories whose low energy limits may appear to be highly inequivalent [67, 89, 90]. The (2+1)-dimensional asymptotically anti-de Sitter black hole is dual to an asymptotically flat "black string" [91] with an equal horizon circumference, and it has been suggested that this may indicate that string theory is, in some sense, insensitive to the presence of a cosmological constant [67]. Similar duality transformations have been used to construct a large family of three-dimensional stationary string solutions with horizons [92]. Ghoruku and Larsen have also studied tachyon scattering in a stringy BTZ black hole background, reproducing the standard Hawking temperature of section 5 in a suitable limit [93].

As yet another extension of the BTZ solution, one may consider black holes in (2+1)-dimensional topologically massive gravity [94], that is, Einstein gravity with a gravitational Chern-Simons term,

$$I_{\mathrm{GCS}} = \frac{k'}{4\pi}\int d^3x\, \epsilon^{\lambda\mu\nu}\Gamma^{\rho}_{\lambda\sigma}\left(\partial_\mu\Gamma^{\sigma}_{\rho\nu} + \frac{2}{3}\Gamma^{\sigma}_{\mu\tau}\Gamma^{\tau}_{\nu\rho}\right), \tag{8.6}$$

added to the action. Like the gauge Chern-Simons term (2.14), the action (8.6) is invariant—in this case under diffeomorphisms—even though it depends explicitly on the connection. The effect of such a term is to reintroduce a local, propagating degree of freedom in (2+1)-dimensional gravity. As Kaloper points out [87], the BTZ solution is also a solution of the field equations of topologically massive gravity, since its Cotton tensor

$$C^{\mu\nu} = \epsilon^{\mu\rho\sigma}\nabla_\rho(R^\nu_\sigma - \frac{1}{4}\delta^\nu_\sigma R) \tag{8.7}$$

vanishes identically. Nutku has found an additional class of solutions of topologically massive gravity with black-hole-like event horizons [95]; these are not asymptotically anti-de Sitter, however, and it is not clear that one can define such quantities as mass and angular momentum. A further class of solutions is discussed by Clément [96]. These asymptotically approach extreme BTZ black holes, but are geodesically complete and have no event horizons. Their physical interpretation is unclear, but it would be interesting to expore their possible role as stable end points of black hole evaporation.

As a final generalization of the BTZ solution, one may consider black holes in a model of (2+1)-dimensional gravity interacting with "topological" matter, that is, fields with finitely many degrees of freedom that do not couple directly to the metric in the Lagrangian. In Ref. 13, black holes in one such theory, with an action

$$I = \int_M \left(e^a \wedge R_a[\omega] + B^a \wedge D_\omega C_a \right), \tag{8.8}$$

are described. Here R_a is the curvature of the spin connection ω, and the first term is the standard first-order action for Einstein gravity, but the cosmological constant has been replaced by a pair of SO(2,1)-valued "matter" fields. (D_ω is the covariant exterior derivative.) The configuration space of this model is finite-dimensional—in fact, it is parametrized by a set of $\mathrm{ISO}(2,1)$-valued holonomies—and there are peculiar gauge transformations that mix the gravitational and matter degrees of freedom [97]. It is shown in Ref. 13 that this model admits a solution whose geometry is that of the BTZ black hole. The mass, charge, and thermodynamic properties, computed by the method of Noether charges, are rather strange, however, presumably because of contributions of the B and C fields; the physics of the model is not yet very well understood.

9 Conclusion

The universe is not three-dimensional, and one must be cautious about conclusions drawn from (2+1)-dimensional models. Nevertheless, the BTZ black hole is similar enough to the realistic Kerr solution that its properties deserve to be taken seriously. In the classical realm, this model is perhaps most useful as a pedagogical tool—it allows us to explore many of the general characteristics of black hole dynamics in a framework in which we are not swamped by mathematical complications. Thus, for example, we can investigate detailed models of collapsing matter, mass inflation, and similar phenomena without having to resort to numerical simulations.

It is in the quantum realm, however, that the power of the BTZ model truly becomes evident. In 3+1 dimensions, we simply do not have a working theory of quantum gravity, and the study of black hole quantum mechanics is necessarily approximate and speculative. In 2+1 dimensions, on the other hand, most of the obstructions to the quantization of general relativity disappear, and we can hope to reach reliable conclusions about quantum mechanical systems. The differences between the (2+1)-dimensional black hole and its (3+1)-dimensional counterpart—for example, the positive specific heat of the BTZ solution—cannot be neglected, of course, but the developing work on the BTZ black hole in quantum gravity has the potential to have far-reaching impact.

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Figure Captions

1. The Penrose diagrams for (a) the generic BTZ black hole; (b) the static ($J = 0$) black hole; and (c) the extreme ($J = \pm M\ell$) black hole.

2. The phase diagram for a collapsing dust shell with initial mass μ, initial radius r_0, and initial velocity $\dot{r}_0$. The end point in region I is open conical adS space; in region II it is the BTZ exterior metric; in region III it is the BTZ interior metric; and in region 4 it is closed conical adS space.

3. The upper half-space representation of the Euclidean black hole. The two hemispheres are to be identified along lines such as L. (The outer hemisphere has been cut open in this figure to show the inner hemisphere.)

4. A region of the Euclidean black hole between $\tau = \tau_1$ and $\tau = \tau_2$. The two nontrivial holonomies correspond to the section of the line segment K lying between the extreme values of τ, and the portion of the line segment L lying between the inner and outer hemispheres. (The latter is a closed curve, since the hemispheres are identified.)